Ann. Inst. Statist. Math.
31 (1979), Part A, 403-415

ASYMPTOTIC OPTIMALITY OF THE GENERALIZED BAYES ESTIMATOR IN MULTIPARAMETER CASES

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(Received July 28, 1978; revised Oct. 24, 1979)

Abstract

The higher order asymptotic efficiency of the generalized Bayes estimator is discussed in multiparameter cases.

For all symmetric loss functions, the generalized Bayes estimator is second order asymptotically efficient in the class $\boldsymbol{A}_2$ of the all second order asymptotically median unbiased (AMU) estimators and third order asymptotically efficient in the restricted class $\boldsymbol{D}$ of estimators.

1. Introduction

The expansion of a generalized Bayes estimator with respect to a loss function of the type $L(\theta)=|\theta|^a$ $(a\geqq 1)$ is obtained by Gusev [5]. His result can be extended to all symmetric loss functions. Strasser [8] also obtained asymptotic expansions of the distribution of the generalized Bayes estimator.

In one parameter case the second order (or third order) asymptotic efficiency of the generalized Bayes estimator has been discussed by Takeuchi and Akahira [12].

It is shown in this paper that in multiparameter case for all symmetric loss function the generalized Bayes estimator $\hat{\theta}$ is asymptotically expanded as

$$\sqrt{n}(\hat{\theta}-\theta)=U-\frac{1}{2\sqrt{n}}I^{-1}V+\frac{1}{\sqrt{n}}I^{-1}W+o_p\left(\frac{1}{\sqrt{n}}\right),$$

where the symbols of the right-hand side are defined in the contexts. And the asymptotic distributions of the estimators are the same up to the order n^{-1} except for constant location shift. Therefore if it is properly adjusted to be asymptotically median unbiased, it is third order asymptotically efficient among the estimators belonging to the class $\boldsymbol{D}$ ([3], [4], [11]) whose element $\hat{\theta}$ is third order AMU and is asymptotically expanded as

$$\sqrt{n}(\hat{\theta}-\theta)=U+\frac{1}{\sqrt{n}}Q+o_p\left(\frac{1}{\sqrt{n}}\right)$$

and $Q_\alpha=O_p(1)$ $(\alpha=1,\cdots,p)$ and $\mathrm{E}[U_\alpha Q_\beta^k]=o(1)$ $(k=1,2)$ for all $\alpha,\beta=1,\cdots,p$, where E denotes asymptotic expectation and $U=(U_1,\cdots,U_p)'$ and $Q=(Q_1,\cdots,Q_p)'$.

2. Results

Let $(\mathcal{X},\mathcal{B})$ be a sample space. We consider a family of probability measures on $\mathcal{B}$, $\mathcal{P}=\{P_\theta:\ \theta\in\Theta\}$, where the index set Θ is called a parameter space. We assume that Θ is an open set in a Euclidean p-space R^p with a norm denoted by $\|\cdot\|$. Then an element θ of Θ may be denoted by $(\theta_1,\cdots,\theta_p)$. Consider n-fold direct products $(\mathcal{X}^{(n)},\mathcal{B}^{(n)})$ of $(\mathcal{X},\mathcal{B})$ and the corresponding product measures $P_\theta^{(n)}$ of P_θ. An estimator of θ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^{(n)}$-measurable functions $\hat{\theta}_n$ on $\mathcal{X}^{(n)}$ into Θ $(n=1,2,\cdots)$. For simplicity we denote an estimator as $\hat{\theta}$ instead of $\{\hat{\theta}_n\}$. Then $\hat{\theta}$ may be denoted by $(\hat{\theta}_1,\cdots,\hat{\theta}_p)$. For an increasing sequence of positive numbers $\{c_n\}$ (c_n tending to infinity) an estimator is called consistent with order $\{c_n\}$ (or $\{c_n\}$-consistent for short) if for every $\varepsilon>0$ any every $\vartheta\in\Theta$ there exist a sufficiently small positive number δ and a sufficiently large number L satisfying the following:

$$\varlimsup_{n\to\infty}\ \sup_{\theta:\|\theta-\vartheta\|<\delta} P_\theta^{(n)}\{c_n\|\hat{\theta}-\theta\|\geqq L\}<\varepsilon \qquad ([1]).$$

For each $k=1,2,\cdots$, a $\{c_n\}$-consistent estimator $\hat{\theta}$ is kth order asymptotically median unbiased (or kth order AMU) estimator if for each $\vartheta\in\Theta$ and each $\alpha=1,\cdots,p$, there exists a positive number δ such that

$$\lim_{n\to\infty}\ \sup_{\theta:\|\theta-\vartheta\|<\delta} c_n^{k-1}\left|P_\theta^{(n)}\{\hat{\theta}_\alpha\leqq\theta_\alpha\}-\frac{1}{2}\right|=0\ ;$$

$$\lim_{n\to\infty}\ \sup_{\theta:\|\theta-\vartheta\|<\delta} c_n^{k-1}\left|P_\theta^{(n)}\{\hat{\theta}_\alpha\geqq\theta_\alpha\}-\frac{1}{2}\right|=0\ .$$

For $\hat{\theta}$ kth order AMU, $G_0(t,\theta_\alpha)+c_n^{-1}G_1(t,\theta_\alpha)+\cdots+c_n^{-(k-1)}G_{k-1}(t,\theta_\alpha)$ $(\alpha=1,\cdots,p)$ is called to be the kth order asymptotic marginal distribution of $c_n(\hat{\theta}-\theta)$ (or $\hat{\theta}$ for short) if

$$\lim_{n\to\infty} c_n^{k-1}|P_\theta^{(n)}\{c_n(\hat{\theta}_\alpha-\theta_\alpha)<t\}-G_0(t,\theta_\alpha)$$
$$-c_n^{-1}G_1(t,\theta_\alpha)-\cdots-c_n^{-(k-1)}G_{k-1}(t,\theta_\alpha)|=0\ .$$

We note that $G_i(t,\theta_\alpha)$ $(i=1,\cdots,k-1;\ \alpha=1,\cdots,p)$ may be generally absolutely continuous functions, hence the asymptotic marginal distributions for any fixed n may not be a distribution function.

Suppose that $\hat{\theta}$ is kth order AMU and has the kth order marginal asymptotic distribution $G_0(t, \theta_\alpha)+c_n^{-1}G_1(t, \theta_\alpha)+\cdots+c_n^{-(k-1)}G_{k-1}(t, \theta_\alpha)$ $(\alpha=1, \cdots, p)$ and the joint distribution of $\hat{\theta}$ admits asymptotic expansion up to kth order, i.e., the order of $c_n^{-(k-1)}$. Letting θ_0 $(\in \Theta)$ be arbitrary but fixed. Denote θ_0 by $(\theta_{01}, \cdots, \theta_{0p})$. Let α be arbitrary but fixed in $1, \cdots, p$. We consider the problem of testing hypothesis $H^+: \theta_\alpha=\theta_{0\alpha}+tc_n^{-1}$ $(t>0)$ against $K: \theta_\alpha=\theta_{0\alpha}$. Put $\Phi_{1/2}=\{\{\phi_n\};\ \mathrm{E}^{(n)}_{\theta_{0\alpha}+tc_n^{-1}}(\phi_n)=1/2+o(c_n^{-(k-1)}),\ 0\leqq \phi_n(\tilde{x}_n)\leqq 1$ for all $\tilde{x}_n \in \mathcal{X}^{(n)}$ $(n=1, 2, \cdots)\}$. Putting $A_{\hat{\theta}_\alpha, \theta_{0\alpha}}=\{c_n(\hat{\theta}_\alpha-\theta_{0\alpha})\leqq t\}$, we have

$$\lim_{n\to\infty} P^{(n)}_{\theta_{0\alpha}+tc_n^{-1}}(A_{\hat{\theta}_\alpha, \theta_{0\alpha}})=\lim_{n\to\infty} P^{(n)}_{\theta_{0\alpha}+tc_n^{-1}}\{\hat{\theta}_\alpha\leqq\theta_{0\alpha}+tc_n^{-1}\}=\frac{1}{2} .$$

Hence it is seen that a sequence $\{\chi_{A_{\hat{\theta}_\alpha, \theta_{0\alpha}}}\}$ of the indicators (or characteristic functions) of $A_{\hat{\theta}_\alpha, \theta_{0\alpha}}$ $(n=1, 2, \cdots)$ belongs to $\Phi_{1/2}$. If

$$\sup_{\{\phi_n\}\in\Phi_{1/2}} \overline{\lim_{n\to\infty}}\ c_n^{k-1}\{\mathrm{E}^{(n)}_{\theta_{0\alpha}}(\phi_n)-H_0^+(t, \theta_{0\alpha})-c_n^{-1}H_1^+(t, \theta_{0\alpha})-\cdots -c_n^{-(k-1)}H_{k-1}^+(t, \theta_{0\alpha})\}=0 ,$$

then we have

$$G_0(t, \theta_{0\alpha})\leqq H_0^+(t, \theta_{0\alpha}) ;$$

and for any positive integer j $(\leqq k)$ if $G_i(t, \theta_{0\alpha})=H_i^+(t, \theta_{0\alpha})$ $(i=1, \cdots, j-1)$ then

$$G_j(t, \theta_{0\alpha})=H_j^+(t, \theta_{0\alpha}) .$$

Consider next the problem of testing hypothesis $H^-: \theta_\alpha=\theta_{0\alpha}+tc_n^{-1}$ $(t<0)$ against $K: \theta_\alpha=\theta_{0\alpha}$. If

$$\sup_{\{\phi_n\}\in\Phi_{1/2}} \underline{\lim}_{n\to\infty}\ c_n^{k-1}\{\mathrm{E}^{(n)}_{\theta_{0\alpha}}(\phi_n)-H_0^-(t, \theta_{0\alpha})-c_n^{-1}H_1^-(t, \theta_{0\alpha})-\cdots -c_n^{-(k-1)}H_{k-1}^-(t, \theta_{0\alpha})\}=0 ,$$

then we have

$$G_0(t, \theta_{0\alpha})\geqq H_0^-(t, \theta_{0\alpha}) ;$$

and for any positive integer j $(\leqq k)$ if $G_i(t, \theta_{0\alpha})=H_i^-(t, \theta_{0\alpha})$ $(i=1, \cdots, j-1)$, then $G_j(t, \theta_{0\alpha})\geqq H_j^-(t, \theta_{0\alpha})$.

$\hat{\theta}$ is called to be kth order asymptotically efficient in the class $\boldsymbol{A}_k$ of the all kth order AMU estimators if the kth order asymptotic marginal distribution of it attains uniformly the bound of the kth order asymptotic marginal distributions of kth order AMU estimators, that is, for each $\alpha=1,\cdots,p$

$$G_i(t,\theta_\alpha)=\begin{cases} H_i^+(t,\theta_\alpha) & \text{for } t>0\,, \\ H_i^-(t,\theta_\alpha) & \text{for } t<0\,, \end{cases}$$

$i=0,\cdots,k-1$ ([2], [4], [9]). [Note that for $t=0$ and each $\alpha=1,\cdots,p$ we have $G_i(0,\theta_\alpha)=H_i^+(0,\theta_\alpha)=H_i^-(0,\theta_\alpha)$ $(i=0,\cdots,k-1)$ from the condition of kth order asymptotically median unbiasedness.]

$\hat{\theta}$ is called to be third order asymptotically efficient in the class $\boldsymbol{D}$ if the third order asymptotic marginal distribution of it attains uniformly the bound of the third order asymptotic marginal distributions of estimators in $\boldsymbol{D}$. It is generally shown by Pfanzagl and Wefelmeyer [6], [7] and Akahira and Takeuchi [3], [4], [10], [11], [14] that there exist second order asymptotically efficient estimators but not third order asymptotically efficient estimators in the class $\boldsymbol{A}_3$. But it was also shown in [4], [10] and [11] that if we restrict the class of estimators appropriately, we have asymptotically efficient estimators among the restricted class of estimators and that the maximum likelihood estimator belongs to the class of higher order asymptotically efficient estimators.

We assume that for each $\theta\in\Theta$ P_θ is absolutely continuous with respect to σ-finite measure μ. We denote a density $dP_\theta/d\mu$ by $f(x,\theta)$. Then the joint density is given by $\prod_{i=1}^{n} f(x_i,\theta)$.

In the subsequent discussion we shall deal with the case when $c_n=\sqrt{n}$. Let $\Theta=R^p$. Let $L_n(u)$ $(u=(u_1,\cdots,u_p)\in R^p)$ be a bounded non-negative and quasi-convex function around the origin, i.e. for any c the set $\{u\colon L(u)\leqq c\}$ $(\subset R^p)$ is convex and contains the origin and $\pi(\theta)$ be a non-negative function. Define a posterior density $p_n(\theta\,|\,\tilde{x}_n)$ and a posterior risk $r_n(d\,|\,\tilde{x}_n)$ by

$$p_n(\theta\,|\,\tilde{x}_n)=\left\{\prod_{i=1}^{n} f(x_i,\theta)\right\}\pi(\theta)\left[\int_\Theta\left(\prod_{i=1}^{n} f(x_i,\theta)\right)\pi(\theta)d\theta\right]^{-1}$$

and

$$r_n(d\,|\,\tilde{x}_n)=\int_\Theta L_n(d-\theta)p_n(\theta\,|\,\tilde{x}_n)d\theta\,,$$

respectively, where $\tilde{x}_n=(x_1,\cdots,x_n)$. Now suppose that $\lim_{n\to\infty} L_n(u/\sqrt{n})=L^*(u)$ for all $u\in R^p$. We define

$$r_n^*(d\,|\,\tilde{x}_n)=\int_\Theta L^*(\sqrt{n}(d-\theta))p_n(\theta\,|\,\tilde{x}_n)d\theta\,.$$

An estimator $\hat{\theta}$ is called a generalized Bayes estimator with respect to a loss function L^* and a prior density π if

$$r_n^*(\hat{\theta}\,|\,\tilde{x}_n)=\inf_{d\in\Theta} r_n^*(d\,|\,\tilde{x}_n)\ .$$

Then $\hat{t}=\sqrt{n}(\hat{\theta}-\theta)$ may also be called a generalized Bayes estimator w.r.t. L^* and π. Since

$$\lim_{n\to\infty}\left|\inf_{d\in\Theta}\int_\Theta L_n(d-\theta)\tilde{p}(\theta)d\theta-\inf_{d\in\Theta}\int_\Theta L^*(\sqrt{n}(d-\theta))\tilde{p}(\theta)d\theta\right|=0$$

uniformly in every posterior density $\tilde{p}(\theta)$, it follows that for a generalized Bayes estimator

$$\lim_{n\to\infty}|\inf_{d\in\Theta} r_n(d\,|\,\tilde{x}_n)-r_n^*(\hat{\theta}\,|\,\tilde{x}_n)|=0\ .$$

Suppose that $X_1, X_2,\cdots, X_n,\cdots$ is a sequence of i.i.d. random variables with a density $f(x,\theta)$ satisfying (i)–(iv).

(i) $\{x\colon f(x,\theta)>0\}$ does not depend on θ.

(ii) For almost all $x[\mu]$, $f(x,\theta)$ is three times continuously differentiable in θ_α $(\alpha=1,\cdots,p)$.

(iii) For each α, β $(\alpha,\beta=1,\cdots,p)$

$$\begin{aligned}0<I_{\alpha\beta}(\theta)&=\mathrm{E}_\theta\left[\left\{\frac{\partial}{\partial\theta_\alpha}\log f(x,\theta)\right\}\left\{\frac{\partial}{\partial\theta_\beta}\log f(x,\theta)\right\}\right]\\&=-\mathrm{E}_\theta\left[\frac{\partial^2}{\partial\theta_\alpha\partial\theta_\beta}\log f(x,\theta)\right]<\infty\ .\end{aligned}$$

(iv) There exist

$$J_{\alpha\beta\cdot\gamma}(\theta)=\mathrm{E}_\theta\left[\left\{\frac{\partial^2}{\partial\theta_\alpha\partial\theta_\beta}\log f(x,\theta)\right\}\left\{\frac{\partial}{\partial\theta_\gamma}\log f(x,\theta)\right\}\right]\ ,$$

$$\begin{aligned}K_{\alpha\beta\gamma}(\theta)=\mathrm{E}_\theta\Bigg[&\left\{\frac{\partial}{\partial\theta_\alpha}\log f(x,\theta)\right\}\left\{\frac{\partial}{\partial\theta_\beta}\log f(x,\theta)\right\}\\&\cdot\left\{\frac{\partial}{\partial\theta_\gamma}\log f(x,\theta)\right\}\Bigg]\end{aligned}$$

and

$$W_{\alpha\beta\gamma}(\theta)=\mathrm{E}_\theta\left[\frac{\partial^3}{\partial\theta_\alpha\partial\theta_\beta\partial\theta_\gamma}\log f(x,\theta)\right]$$

and the following holds:

$$\mathrm{E}_\theta\left[\frac{\partial^3}{\partial\theta_\alpha\partial\theta_\beta\partial\theta_\gamma}\log f(x,\theta)\right]=-J_{\alpha\beta\cdot\gamma}(\theta)-J_{\alpha\gamma\cdot\beta}(\theta)-J_{\beta\gamma\cdot\alpha}(\theta)-K_{\alpha\beta\gamma}(\theta)\ .$$

It was shown by the same way in [9] that a maximum likelihood estimator (MLE) is second order asymptotically efficient. Let $\hat{\theta}$ be an MLE. By Taylor expansion we have

$$0=\sum_\alpha \sum_i \frac{\partial}{\partial\theta_\alpha} \log f(X_i, \hat\theta)$$
$$=\sum_\alpha \left\{\sum_i \frac{\partial}{\partial\theta_\alpha} \log f(X_i, \theta)\right\}(\hat\theta_\alpha-\theta_\alpha)+\sum_\alpha \sum_\beta \left\{\sum_i \frac{\partial^2}{\partial\theta_\alpha\partial\theta_\beta} \log f(X_i, \theta)\right\}$$
$$\cdot (\hat\theta_\alpha-\theta_\alpha)(\hat\theta_\beta-\theta_\beta)+\frac{1}{2}\sum_\alpha\sum_\beta\sum_\gamma \left\{\sum_i \frac{\partial^3}{\partial\theta_\alpha\partial\theta_\beta\partial\theta_\gamma} \log f(X_i, \theta^*)\right\}$$
$$\cdot (\hat\theta_\alpha-\theta_\alpha)(\hat\theta_\beta-\theta_\beta)(\hat\theta_\gamma-\theta_\gamma) ,$$

where $\|\theta^*-\theta\|\leqq\|\hat\theta-\theta\|$. Putting $T=\sqrt{n}(\hat\theta-\theta)$ we obtain

$$0=\frac{1}{\sqrt{n}}\sum_\alpha \left\{\sum_i \frac{\partial}{\partial\theta_\alpha} \log f(X_i, \theta)\right\} T_\alpha+\frac{1}{n}\sum_\alpha\sum_\beta \left\{\sum_i \frac{\partial^2}{\partial\theta_\alpha\partial\theta_\beta} \log f(X_i, \theta)\right\} T_\alpha T_\beta$$
$$+\frac{1}{2n\sqrt{n}}\sum_\alpha\sum_\beta\sum_\gamma \left\{\sum_i \frac{\partial^3}{\partial\theta_\alpha\partial\theta_\beta\partial\theta_\gamma} \log f(X_i, \theta^*)\right\} T_\alpha T_\beta T_\gamma ,$$

where $T=(T_1,\cdots, T_p)'$. Set

$$Z_\alpha(\theta)=\frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{\partial}{\partial\theta_\alpha} \log f(X_i, \theta) ;$$

$$Z_{\alpha\beta}(\theta)=\frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{\partial^2}{\partial\theta_\alpha\partial\theta_\beta} \log f(X_i, \theta)+I_{\alpha\beta}(\theta) .$$

Then it follows that $W_{\alpha\beta\gamma}(\theta)$ converges in probability to $-\{J_{\alpha\beta\cdot\gamma}(\theta)+J_{\alpha\gamma\cdot\beta}(\theta)+J_{\beta\gamma\cdot\alpha}(\theta)+K_{\alpha\beta\gamma}(\theta)\}$. We put $\rho_{\alpha\beta\gamma}(\theta)=J_{\alpha\beta\cdot\gamma}(\theta)+J_{\alpha\gamma\cdot\beta}(\theta)+J_{\beta\gamma\cdot\alpha}(\theta)+K_{\alpha\beta\gamma}(\theta)$. Hence the following theorem holds:

THEOREM 1. *Under conditions* (i)-(iv)

$$\sqrt{n}(\hat\theta-\theta)=U-\frac{1}{2\sqrt{n}}I^{-1}V+\frac{1}{\sqrt{n}}I^{-1}W+o_p\left(\frac{1}{\sqrt{n}}\right) ,$$

where $I=(I_{\alpha\beta})$ *and* $P=(P_{\alpha\beta})$ *are matrices and* $V=(\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}U_\beta U_\gamma)$, $W=(\sum_\beta U_\beta Z_{\beta\gamma})$ *and* $U=(U_\alpha)$ *are p-dimensional column vectors and* $U_\alpha=\sum_\beta I^{\alpha\beta}Z_\beta$ *and* $I^{\alpha\beta}$ *denotes the* (α, β)*-element of the inverse matrix of the information matrix I.*

Since the proof of the theorem is essentially same as that of one parameter case ([9]), it is omitted.

Put

$$\hat\theta^*=\hat\theta+\frac{1}{6n}Y ,$$

where $Y=(\sum_\beta\sum_\gamma U_\beta W_{\alpha\beta\gamma})$ is a column vector. Then $\hat\theta^*$ is second order

AMU. From Theorem 1 we have established the following:

THEOREM 2. *Under conditions* (i)-(iv) $\hat{\theta}^*$ *is second order asymptotically efficient in the class* $\boldsymbol{A}_2$.

Since the proof of the theorem is essentially same as that of one parameter case ([9]), it is omitted.

It will be shown that the generalized Bayes estimator w.r.t. a loss function and a prior density is second order asymptotically efficient. Let θ_0 be a true parameter of θ ($\in \Theta$). Denote θ and θ_0 by $(\theta_1, \cdots, \theta_p)'$ and $(\theta_{01}, \cdots, \theta_{0p})'$ respectively. Further we assume the following:

(v) For each $\alpha=1, \cdots, p$, $\pi(\theta)$ is twice partially differentiable in θ_α. Then we have

$$
\begin{aligned}
&p_n(\theta \mid \tilde{x}_n)/p_n(\theta_0 \mid \tilde{x}_n) \\
&\quad = \exp\left[\log\{p_n(\theta \mid \tilde{x}_n)/p_n(\theta_0 \mid \tilde{x}_n)\}\right] \\
&\quad = \exp\left[\sum_{i=1}^n \log\{f(x_i, \theta)/f(x_i, \theta_0)\} + \log\{\pi(\theta)/\pi(\theta_0)\}\right] \\
&\quad = \exp\left[\sum_{i=1}^n \log f(x_i, \theta) - \sum_{i=1}^n \log f(x_i, \theta_0) + \log \pi(\theta) - \log \pi(\theta_0)\right] \\
&\quad = \exp\left[\sum_\alpha \left\{\sum_{i=1}^n \frac{\partial}{\partial \theta_\alpha} \log f(x_i, \theta_0)\right\}(\theta_\alpha - \theta_{0\alpha})\right. \\
&\qquad + \frac{1}{2} \sum_\alpha \sum_\beta \left\{\sum_{i=1}^n \frac{\partial^2}{\partial \theta_\alpha \partial \theta_\beta} \log f(x_i, \theta_0)\right\}(\theta_\alpha - \theta_{0\alpha})(\theta_\beta - \theta_{0\beta}) \\
&\qquad + \frac{1}{6} \sum_\alpha \sum_\beta \sum_\gamma \left\{\sum_{i=1}^n \frac{\partial^3}{\partial \theta_\alpha \partial \theta_\beta \partial \theta_\gamma} \log f(x_i, \theta^*)\right\} \\
&\qquad \left. \cdot (\theta_\alpha - \theta_{0\alpha})(\theta_\beta - \theta_{0\beta})(\theta_\gamma - \theta_{0\gamma}) + \sum_\alpha \frac{\pi'_\alpha(\theta_0)}{\pi(\theta_0)}(\theta_\alpha - \theta_{0\alpha}) + o\left(\frac{1}{\sqrt{n}}\right)\right] \\
&\quad = \exp\left[\sqrt{n} \sum_\alpha Z_\alpha(\theta_0)(\theta_\alpha - \theta_{0\alpha}) + \frac{1}{2} \sum_\alpha \sum_\beta \{\sqrt{n} Z_{\alpha\beta}(\theta_0) - n I_{\alpha\beta}(\theta_0)\}\right. \\
&\qquad \cdot (\theta_\alpha - \theta_{0\alpha})(\theta_\beta - \theta_{0\beta}) + \frac{n}{6} \sum_\alpha \sum_\beta \sum_\gamma W_{\alpha\beta\gamma}(\theta^*)(\theta_\alpha - \theta_{0\alpha})(\theta_\beta - \theta_{0\beta}) \\
&\qquad \left. \cdot (\theta_\gamma - \theta_{0\gamma}) + \sum_\alpha \frac{\pi'_\alpha(\theta_0)}{\pi(\theta_0)}(\theta_\alpha - \theta_{0\alpha}) + o\left(\frac{1}{\sqrt{n}}\right)\right],
\end{aligned}
$$

where $\pi'_\alpha(\theta) = \partial \pi(\theta)/\partial \theta_\alpha$ $(\alpha=1, \cdots, p)$. It follows that

$$
\begin{aligned}
&p_n(\theta \mid \tilde{x}_n)/p_n(\theta_0 \mid \tilde{x}_n) \\
&\quad = \exp\left[\sum_\alpha Z_\alpha(\theta_0)\{\sqrt{n}(\theta_\alpha - \theta_{0\alpha})\} + \frac{1}{2} \sum_\alpha \sum_\beta \left\{\frac{Z_{\alpha\beta}(\theta_0)}{\sqrt{n}} - I_{\alpha\beta}(\theta_0)\right\}\right. \\
&\qquad \cdot \{n(\theta_\alpha - \theta_{0\alpha})(\theta_\beta - \theta_{0\beta})\} - \frac{1}{6\sqrt{n}} \sum_\alpha \sum_\beta \sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0) \\
&\qquad \cdot \{n\sqrt{n}(\theta_\alpha - \theta_{0\alpha})(\theta_\beta - \theta_{0\beta})(\theta_\gamma - \theta_{0\gamma})\}
\end{aligned}
$$

$$+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\sum_\alpha \pi'_\alpha(\theta_0)\{\sqrt{n}(\theta_\alpha-\theta_{0\alpha})\}+o_p\left(\frac{1}{\sqrt{n}}\right)\Big] .$$

Putting $t_\alpha=\sqrt{n}(\theta_\alpha-\theta_{0\alpha})$ $(\alpha=1,\cdots,p)$ we obtain

$$(1)\qquad p_n(\theta\,|\,\tilde{x}_n)/p_n(\theta_0\,|\,\tilde{x}_n)$$

$$=\exp\left[\sum_\alpha Z_\alpha(\theta)t_\alpha+\frac{1}{2}\sum_\alpha\sum_\beta\left\{\frac{Z_{\alpha\beta}(\theta_0)}{\sqrt{n}}-I_{\alpha\beta}(\theta_0)\right\}t_\alpha t_\beta\right.$$

$$-\frac{1}{6\sqrt{n}}\sum_\alpha\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)t_\alpha t_\beta t_\gamma$$

$$\left.+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\sum_\alpha \pi'_\alpha(\theta_0)t_\alpha+o_p\left(\frac{1}{\sqrt{n}}\right)\right]$$

$$=\exp\left[-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)(t_\alpha-U_\alpha)(t_\beta-U_\beta)+\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)U_\alpha U_\beta\right.$$

$$+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\sum_\alpha \pi'_\alpha(\theta_0)t_\alpha+\frac{1}{2\sqrt{n}}\sum_\alpha\sum_\beta Z_{\alpha\beta}(\theta_0)t_\alpha t_\beta$$

$$\left.-\frac{1}{6\sqrt{n}}\sum_\alpha\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)t_\alpha t_\beta t_\gamma+o_p\left(\frac{1}{\sqrt{n}}\right)\right]$$

$$=\left(\exp\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)U_\alpha U_\beta\right)$$

$$\cdot\left[\exp\left\{-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)(t_\alpha-U_\alpha)(t_\beta-U_\beta)\right\}\right]$$

$$\cdot\exp\left\{\frac{1}{\sqrt{n}\,\pi(\theta_0)}\sum_\alpha \pi'_\alpha(\theta_0)t_\alpha+\frac{1}{2\sqrt{n}}\sum_\alpha\sum_\beta Z_{\alpha\beta}(\theta_0)t_\alpha t_\beta\right.$$

$$\left.-\frac{1}{6\sqrt{n}}\sum_\alpha\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)t_\alpha t_\beta t_\gamma+o_p\left(\frac{1}{\sqrt{n}}\right)\right\}$$

$$=\left(\exp\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)U_\alpha U_\beta\right)$$

$$\cdot\left[\exp\left\{-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)(t_\alpha-U_\alpha)(t_\beta-U_\beta)\right\}\right]$$

$$\cdot\left\{1+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\sum_\alpha \pi'_\alpha(\theta_0)t_\alpha+\frac{1}{2\sqrt{n}}\sum_\alpha\sum_\beta Z_{\alpha\beta}(\theta_0)t_\alpha t_\beta\right.$$

$$\left.-\frac{1}{6\sqrt{n}}\sum_\alpha\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)t_\alpha t_\beta t_\gamma+o_p\left(\frac{1}{\sqrt{n}}\right)\right\}$$

$$=q_n(t,\theta_0\,|\,\tilde{x}_n)\qquad\text{(say)},$$

where $t=(t_1,\cdots,t_p)'$. Let $\hat{t}=\sqrt{n}(\hat{\theta}-\theta_0)$. Then the posterior risk is given by

$$(2)\qquad r_n^*(\hat{t}\,|\,\tilde{x}_n)=\frac{1}{\sqrt{n}}p_n(\theta_0\,|\,\tilde{x}_n)\int L^*(\hat{t}-t)q_n(t,\theta_0\,|\,\tilde{x}_n)dt .$$

Further we assume the following:

(vi) $L^*(u)$ is a convex function;

(vii) For each $\alpha=1,\cdots,p$, $\int L^*(-u)q_n(u+t,\theta_0|\tilde{x}_n)du$ is continuously partially differentiable with respect to t_α under the integral sign.

By (2) and the assumption (vi) it is shown that the generalized Bayes estimator $\hat{t}$ w.r.t. $L^*(\cdot)$ and $\pi(\cdot)$ is given as a solution u of the equation

$$(d/du_\alpha)\int L^*(u-t)q_n(t,\theta_0|\tilde{x}_n)dt=0 \qquad (\alpha=1,\cdots,p).$$

Since by (vii)

$$\begin{aligned}
&\frac{d}{du_\alpha}\int L^*(u-t)q_n(t,\theta_0|\tilde{x}_n)dt\\
&=\frac{d}{du_\alpha}\int L^*(-t)q_n(u+t,\theta_0|\tilde{x}_n)dt\\
&=\frac{d}{du_\alpha}\int L^*(-u)q_n(t+u,\theta_0|\tilde{x}_n)du\\
&=\int L^*(-u)\left\{\frac{d}{dt_\alpha}q_n(t+u,\theta_0|\tilde{x}_n)\right\}du \qquad (\alpha=1,\cdots,p),
\end{aligned}$$

the generalized Bayes estimator $\hat{t}$ is obtained by a solution of the equation

$$\int L^*(-u)\left\{\frac{d}{dt_\alpha}q_n(t+u,\theta_0|\tilde{x}_n)\right\}du=0 \qquad (\alpha=1,\cdots,p). \tag{3}$$

Since $t=\sqrt{n}(\theta-\theta_0)$ and $\hat{t}=\sqrt{n}(\hat{\theta}-\theta_0)$, $\hat{t}$ may be called to be the generalized Bayes estimator. From (1) and (3) we have

$$\begin{aligned}
0=\int L^*(-u)\exp&\left[-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)(\hat{t}_\alpha+u_\alpha-U_\alpha)(\hat{t}_\beta+u_\beta-U_\beta)\right]\\
\cdot&\left[-\sum_\beta I_{\alpha\beta}(\theta_0)(\hat{t}_\beta+u_\beta-U_\beta)\left\{1+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\sum_\alpha \pi'_\alpha(\theta_0)(\hat{t}_\alpha+u_\alpha)\right.\right.\\
&+\frac{1}{2\sqrt{n}}\sum_\alpha\sum_\beta Z_{\alpha\beta}(\theta_0)(\hat{t}_\alpha+u_\alpha)(\hat{t}_\beta+u_\beta)\\
&\left.-\frac{1}{6\sqrt{n}}\sum_\alpha\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)(\hat{t}_\alpha+u_\alpha)(\hat{t}_\beta+u_\beta)(\hat{t}_\gamma+u_\gamma)\right\}\\
&+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\pi'_\alpha(\theta_0)+\frac{1}{\sqrt{n}}\sum_\beta Z_{\alpha\beta}(\theta_0)(\hat{t}_\beta+u_\beta)\\
&\left.-\frac{1}{2\sqrt{n}}\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)(\hat{t}_\beta+u_\beta)(\hat{t}_\gamma+u_\gamma)\right]du+o_p\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Putting $\hat{u}_\alpha=\hat{t}_\alpha-U_\alpha$ ($\alpha=1,\cdots,p$) we obtain

$$0=\int L^*(-u)\Big[\exp\Big\{-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)(u_\alpha+\hat{u}_\alpha)(u_\beta+\hat{u}_\beta)\Big\}\Big]$$
$$\cdot\Big[-\sum_\beta I_{\alpha\beta}(u_\beta+\hat{u}_\beta)\Big\{1+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\sum_\alpha \pi'_\alpha(\theta_0)(u_\alpha+\hat{u}_\alpha+U_\alpha)$$
$$+\frac{1}{2\sqrt{n}}\sum_\alpha\sum_\beta Z_{\alpha\beta}(\theta_0)(u_\alpha+\hat{u}_\alpha+U_\alpha)(u_\beta+\hat{u}_\beta+U_\beta)$$
$$-\frac{1}{6\sqrt{n}}\sum_\alpha\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)(u_\alpha+\hat{u}_\alpha+U_\alpha)(u_\beta+\hat{u}_\beta+U_\beta)(u_\gamma+\hat{u}_\gamma+U_\gamma)\Big\}$$
$$+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\pi'_\alpha(\theta_0)+\frac{1}{\sqrt{n}}\sum_\beta Z_{\alpha\beta}(\theta_0)(\hat{u}_\beta+u_\beta+U_\beta)$$
$$-\frac{1}{2\sqrt{n}}\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)(\hat{u}_\beta+u_\beta+U_\beta)(\hat{u}_\gamma+u_\gamma+U_\gamma)\Big]du+o_p\Big(\frac{1}{\sqrt{n}}\Big)\,.$$

Further we assume the following.

(viii) $L^*(u)$ is a symmetric loss function about the origin.

We define

$$M=\int L^*(-u)\exp\Big[-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)u_\alpha u_\beta\Big]du\ ;$$

$$P_{\alpha\beta}=\int L^*(-u)u_\alpha u_\beta\exp\Big[-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)u_\alpha u_\beta\Big]du \qquad (\alpha,\beta=1,\cdots,p)\ ;$$

$$Q_{\alpha\beta\gamma\delta}=\int L^*(-u)u_\alpha u_\beta u_\gamma u_\delta\exp\Big[-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)u_\alpha u_\beta\Big]du \qquad (\alpha,\beta,\gamma,\delta=1,\cdots,p)\,.$$

Note that by (viii)

$$\int L^*(-u)u_\alpha\exp\Big(-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)u_\alpha u_\beta\Big)du$$
$$=\int L^*(-u)u_\alpha u_\beta u_\gamma\exp\Big(-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)u_\alpha u_\beta\Big)du=0 \qquad (\alpha,\beta,\gamma=1,\cdots,p)\,.$$

Then we have

$$0=\int L^*(-u)\Big[\exp\Big\{-\frac{1}{2}\sum_\alpha\sum_\beta I_{\alpha\beta}(\theta_0)u_\alpha u_\beta\Big\}\Big]\{1-\sum_\gamma\sum_\delta I_{\gamma\delta}(\theta_0)u_\gamma u_\delta\}$$
$$\cdot\Big[-\sum_\beta I_{\alpha\beta}(\theta_0)u_\beta-\sum_\beta I_{\alpha\beta}(\theta_0)\hat{u}_\beta-\sum_\beta\sum_\gamma\frac{I_{\alpha\beta}(\theta_0)\pi'_\gamma(\theta_0)}{\sqrt{n}\,\pi(\theta_0)}$$

$$\cdot (u_\beta+\hat{u}_\beta)(u_\gamma+\hat{u}_\gamma+U_\gamma)-\frac{1}{2\sqrt{n}}\sum_\beta\sum_\gamma\sum_\delta I_{\alpha\beta}(\theta_0)Z_{\gamma\delta}(\theta_0)$$
$$\cdot (u_\beta+\hat{u}_\beta)(u_\gamma u_\delta+U_\gamma U_\delta+2u_\gamma\hat{u}_\delta+2U_\gamma\hat{u}_\delta+2U_\gamma u_\delta)$$
$$+\frac{1}{6\sqrt{n}}\sum_\beta\sum_\gamma\sum_\delta\sum_\xi I_{\alpha\beta}(\theta_0)\rho_{\gamma\delta\xi}(\theta_0)(u_\beta+\hat{u}_\beta)(u_\gamma u_\delta u_\xi+U_\gamma U_\delta U_\xi$$
$$+3u_\gamma u_\delta\hat{u}_\xi+3u_\beta u_\gamma U_\xi+3U_\gamma U_\delta u_\xi+3U_\gamma U_\delta\hat{u}_\xi+6u_\gamma\hat{u}_\delta U_\xi)$$
$$+\frac{1}{\sqrt{n}\,\pi(\theta_0)}\pi'_\alpha(\theta_0)+\frac{1}{\sqrt{n}}\sum_\beta Z_{\alpha\beta}(\theta_0)u_\beta+\frac{1}{\sqrt{n}}\sum_\beta Z_{\alpha\beta}(\theta_0)U_\beta-\frac{1}{2\sqrt{n}}$$
$$\cdot\sum_\beta\sum_\gamma \rho_{\alpha\beta\gamma}(\theta_0)(u_\beta u_\gamma+U_\beta U_\gamma+2u_\beta\hat{u}_\gamma+2U_\beta u_\gamma+2U_\beta\hat{u}_\gamma)\Big]du+o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$=-M\sum_\beta I_{\alpha\beta}(\theta_0)\hat{u}_\beta-\frac{1}{\sqrt{n}}\sum_\beta\sum_\gamma I_{\alpha\beta}(\theta_0)\frac{\pi'_\gamma(\theta_0)}{\pi(\theta_0)}P_{\beta\gamma}-\frac{1}{\sqrt{n}}\sum_\beta\sum_\gamma\sum_\delta I_{\alpha\beta}(\theta_0)$$
$$\cdot Z_{\gamma\delta}(\theta_0)U_\gamma P_{\beta\delta}+\frac{1}{6\sqrt{n}}\sum_\beta\sum_\gamma\sum_\delta\sum_\xi I_{\alpha\beta}(\theta_0)\rho_{\gamma\delta\xi}(\theta_0)(Q_{\beta\gamma\delta\xi}+3U_\gamma U_\delta P_{\beta\xi})$$
$$+\frac{\pi'_\alpha(\theta_0)}{\sqrt{n}\,\pi(\theta_0)}M+\frac{1}{\sqrt{n}}\sum_\beta U_\beta Z_{\alpha\beta}(\theta_0)M-\frac{1}{2\sqrt{n}}\sum_\beta\sum_\gamma\rho_{\alpha\beta\gamma}(\theta_0)$$
$$\cdot(P_{\beta\gamma}+U_\beta U_\gamma M)+\sum_\beta\sum_\gamma\sum_\delta I_{\gamma\beta}(\theta_0)I_{\alpha\delta}(\theta_0)P_{\delta\gamma}\hat{u}_\beta+o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$(\alpha=1,\cdots,p)\,.$$

Using a matrix representation we obtain

$$(IPI-MI)\hat{u}=\frac{1}{\sqrt{n}}(PI-ME)\pi^*+\frac{1}{\sqrt{n}}L-\frac{1}{2\sqrt{n}}(PI-ME)V$$
$$+\frac{1}{\sqrt{n}}(PI-ME)W+o_p\left(\frac{1}{\sqrt{n}}\right),$$

where E is an unit matrix, and L, V and W are column vectors with

$$\pi^*=\left(\frac{\pi'_\alpha(\theta_0)}{\pi(\theta_0)}\right);$$

$$L=\left(-\frac{1}{6}\sum_\beta\sum_\gamma\sum_\delta\sum_\xi I_{\alpha\beta}(\theta_0)\rho_{\gamma\delta\xi}(\theta_0)Q_{\beta\gamma\delta\xi}+\frac{1}{2}\sum_\beta\sum_\gamma\rho_{\alpha\beta\gamma}(\theta_0)P_{\beta\gamma}\right);$$

$$V=\left(\sum_\beta\sum_\gamma\rho_{\alpha\beta\gamma}(\theta_0)U_\beta U_\gamma\right);$$

$$W=\left(\sum_\beta U_\beta Z_{\beta\gamma}(\theta_0)\right).$$

Since it is derived from (vi) that the matrix $IPI-MI$ is positive definite, it follows that

$$\hat{u}=\frac{1}{\sqrt{n}}I^{-1}\pi^*+\frac{1}{\sqrt{n}}(IPI-MI)^{-1}L-\frac{1}{2\sqrt{n}}I^{-1}V \tag{4}$$

$$+\frac{1}{\sqrt{n}}I^{-1}W+o_p\left(\frac{1}{\sqrt{n}}\right).$$

Hence we have

$$\hat{t}_\alpha=\hat{u}_\alpha+U_\alpha \qquad (\alpha=1,\cdots,p), \tag{5}$$

where $\hat{u}=(\hat{u}_1,\cdots,\hat{u}_p)'$ is given by (4). Since $\hat{t}=\sqrt{n}(\hat{\theta}-\theta_0)$, we modify $\hat{\theta}$ to be second order AMU and denote by $\hat{\theta}^*$. From Theorem 1, (4) and (5) it follows that the MLE $\hat{\theta}^*_{ML}$ is asymptotically equivalent to the generalized Bayes estimator $\hat{\theta}^*$ up to order $n^{-1/2}$. By Theorem 2 it is seen that $\hat{\theta}^*$ is second order asymptotically efficient in the class $\boldsymbol{A}_2$.

We have defined in [4], [11] and [12] the class $\boldsymbol{D}$ as the set of the all third order AMU estimators $\hat{\theta}$ satisfying the following:

(a) $\hat{\theta}$ is asymptotically expanded as

$$\sqrt{n}(\hat{\theta}-\theta)=U+\frac{1}{\sqrt{n}}Q+o_p\left(\frac{1}{\sqrt{n}}\right)$$

and $Q_\alpha=O_p(1)$ $(\alpha=1,\cdots,p)$ and $\mathrm{E}(U_\alpha Q_\beta^k)=o(1)$ $(k=1,2)$ for all $\alpha,\beta=1,\cdots,p$, where E denotes the asymptotic expectation of $U_\alpha Q_\beta^k$ with

$$U_\alpha=\frac{1}{\sqrt{n}}\sum_\beta I^{\alpha\beta}\frac{\partial}{\partial\theta_\beta}\sum_{i=1}^n \log f(X_i,\theta) \qquad (\alpha=1,\cdots,p);$$

(b) The joint distribution of $\hat{\theta}$ admits Edgeworth expansion.

It follows from (4) and (5) that the generalized Bayes estimator $\hat{\theta}^*$ belongs to the class $\boldsymbol{D}$. Then the asymptotic marginal distribution of $\hat{\theta}^*$ is equivalent to that of the MLE $\hat{\theta}^*_{ML}$ up to order n^{-1}. Since $\hat{\theta}^*_{ML}$ is third order asymptotically efficient in the class $\boldsymbol{D}$ ([4], [10], [11]), $\hat{\theta}^*$ is also so. Hence we have established:

THEOREM 3. *Under the assumptions* (i)-(viii), *the generalized Bayes estimator* $\hat{\theta}^*$ *is second order asymptotically efficient in the class* $\boldsymbol{A}_2$ *and also third order asymptotically efficient in the class* $\boldsymbol{D}$.

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