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A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION

R. SHIMIZU

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Summary

A necessary and sufficient condition is given under which the minimum of X_k/a_k, $k=1, 2, \cdots, n$ has the same distribution as X_1, where X's are i.i.d. positive random variables and a's are given positive constants.

1. Introduction

It is well-known (Desu [2]) that if $X_{1,n}$ is the minimum in a sample $X_1, X_2, \cdots, X_n$ of size n from a distribution F on the half interval $(0, \infty)$, then $nX_{1,n}$ has the same distribution as X_1 for all n if and only if F is the exponential distribution. Some extensions of this characterization and related theorems are also known (Sethuraman [7], Gupta [4] and Huang [5]. See also Galambos-Kotz [3]). In the present paper we are concerned with the following problem: What can be said about the distribution F if for some n and for some positive constants $a_1, a_2, \cdots, a_n$, the random variable $Z=\min\{X_k/a_k\}$ has the same distribution as X_1? We shall given in the next section a complete solution to a more general problem. Throughout this paper we invariably assume that the distribution F is non-degenerate and is concentrated on the open interval $(0, \infty)$. In other words, we assume that $F(0)=0<F(x_1)<1$ for some positive x_1.

2. Results

Let $m_1, m_2, \cdots, m_n$ $(n\geqq 1)$ be positive integers and c, $a_1, a_2, \cdots, a_n$ be positive numbers such that $c>\max\{a_k\}$ if $n>1$ and $c=a_1$ if $n=1$. Let α be the unique positive number satisfying $a_1^\alpha+a_2^\alpha+\cdots+a_n^\alpha=c^\alpha$ if $n>1$ and let it be an arbitrary positive number if $n=1$. Let $X_{j,k}$, $j=1, 2, \cdots, m_k$; $k=1, 2, \cdots, n$ be i.i.d. random variables with the common distribution F. Write

$$Z_{(k)}=\min_{1\leq j\leq m_k}\{X_{j,k}\}\ ,\qquad k=1,2,\cdots,n$$

and

$$Z=\min_{1\leq k\leq n}\{(cm_k^{1/\alpha}/a_k)\cdot Z_{(k)}\}\ .$$

We shall exclude the trivial case $n=a_1=m_1=1$, where Z reduces to $X_{1,1}$.

THEOREM 1. *In order that Z has the distribution $F(x)$, it is necessary and sufficient that there exists a positive, bounded and periodic function $H(x)$ with periods $A_k=\log cm_k^{1/\alpha}/a_k$, $k=1,2,\cdots,n$ and F can be put in the form*

$$F(x)=\begin{cases}0 & x<0\\ 1-\exp\{-H(-\log x)x^\alpha\}\ , & x\geqq 0\ .\end{cases}\tag{1}$$

PROOF. By the monotone transformation $X_{j,k}\to(cX_{j,k})^\alpha$, the problem reduces to the case $\alpha=c=1$. Therefore we can and do assume this without loss of generality. Then a's satisfy

$$a_1+a_2+\cdots+a_n=1\ .\tag{2}$$

For any $x>0$, the definition of Z gives

$$\begin{aligned}\Pr\{Z>x\}&=\Pr\{(m_k/a_k)Z_{(k)}>x,\ k=1,2,\cdots,n\}\\ &=\prod_{k=1}^n\Pr\{Z_{(k)}>a_kx/m_k\}=\prod_{k=1}^n(1-F(a_kx/m_k))^{m_k}\ .\end{aligned}\tag{3}$$

If F is of the form (1) then (3) becomes

$$\begin{aligned}\Pr\{Z>x\}&=\prod_{k=1}^n\exp\{-m_kH(-\log a_kx/m_k)a_kx/m_k\}\\ &=\exp\{-H(-\log x)x\}=1-F(x)\end{aligned}$$

as was to be proved. Suppose conversely that Z has the distribution F. Then it follows from (3)

$$1-F(x)=\prod_{k=1}^n(1-F(a_kx/m_k))^{m_k}\ ,\qquad x\geqq 0\ .\tag{4}$$

As $a_k/m_k<1$ for $k=1,2,\cdots,n$ (4) implies that $F(x)<1$ for all $x>0$. Then the function

$$H(x)=-e^x\log(1-F(e^{-x}))$$

is defined for all real x and satisfies the functional equation

$$H(x)=\sum_{k=1}^n a_kH(x+A_k)\ ,\qquad -\infty<x<\infty\ .\tag{5}$$

Let x_0 be an arbitrary real number. We shall show that (5) yields

$$C \equiv \sup_{x \geqq x_0} H(x) < \infty \tag{6}$$

and

$$H(x+A_k) = H(x) , \qquad k=1, 2, \cdots, n, \ x \geqq x_0 . \tag{7}$$

These are consequences of Theorems 1–2 of Shimizu [8]. If (6) is established, we can also use Choquet-Deny's theorem (see Meyer [6]) to derive (7). But as we are dealing with the special form (5) of the equation treated in Shimizu [8], the proofs of Theorems 1–2 can be substantially simplified. We shall give here an elementary proof of (6) and (7). The method of proof used here is essentially the same as the one used in Davies-Shimizu [1]. For the proof of (6) it suffices to show that the inequality

$$H(x) \leqq e^A H(x_0) \tag{8}$$

holds for all $x \geqq x_0$, where $A = \max\{A_1, A_2, \cdots, A_n\}$. It follows from the equation (5) that there exists a k_1 such that $H(x_0) \geqq H(x_0 + A_{k_1})$. Similarly there exists a k_2 such that $H(x_0) \geqq H(x_0 + A_{k_1}) \geqq H(x_0 + A_{k_1} + A_{k_2})$. In this way we obtain a sequence $k_1, k_2, \cdots$ of positive integers such that $H(x_0) \geqq H(x_0 + A_{k_1} + A_{k_2} + \cdots + A_{k_m})$, $m = 1, 2, \cdots$. If $x_0 \leqq x \leqq x_0 + A$, (8) follows from the inequality

$$H(x+y) \leqq e^y H(x) , \qquad x \geqq x_0, \ y \geqq 0 , \tag{9}$$

which can easily be verified from the definition of $H(x)$. Suppose $x > x_0 + A$. As $\min\{A_1, A_2, \cdots, A_n\} > 0$, we can find an m such that

$$x_0 + A_{k_1} + \cdots + A_{k_m} \leqq x < x_0 + A_{k_1} + \cdots + A_{k_{m+1}} .$$

Writing $\delta \equiv x - (x_0 + A_{k_1} + \cdots + A_{k_m})$ we obtain from (9), $H(x) = H(x_0 + A_{k_1} + \cdots + A_{k_m} + \delta) \leqq e^\delta H(x_0 + A_{k_1} + \cdots + A_{k_m}) \leqq e^A H(x_0)$ as was to be proved. To complete the proof of Theorem 1 we have only to prove (7) for $k = 1$. To this end we introduce the bounded function $K(x) \equiv H(x + A_1) - H(x)$. It is easy to verify that K satisfies the functional equation

$$K(x) = \sum_{k=1}^{n} a_k K(x + A_k) , \qquad x \geqq x_0 . \tag{10}$$

Repeated application of (10) yields

$$K(x) = \sum_{k_1 + \cdots + k_n = m} \frac{m!}{k_1! \cdots k_n!} a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n} K(x + k_1 A_1 + \cdots + k_n A_n) , \tag{11}$$

where m is an arbitrary positive integer and the summation extends over all non-negative k's such that $k_1 + k_2 + \cdots + k_n = m$. Note that each of the $k_1A_1+\cdots+k_nA_n$ in the right-hand side of (11) is not less than $m\times\min\{A_1, A_2,\cdots, A_n\}$, which tends to infinity as m. In particular, for any $\varepsilon>0$, we can take m sufficiently large so that

$$\begin{aligned}|K(x)|&\leqq\sum\frac{m!}{k_1!k_2!\cdots k_n!}a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}|K(x+k_1A_1+\cdots+k_nA_n)|\\&\leqq\sum\frac{m!}{k_1!k_2!\cdots k_n!}a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}(\varepsilon+\overline{\lim_{\xi\to\infty}}|K(\xi)|)\\&\leqq(a_1+a_2+\cdots+a_n)^m(\varepsilon+a)=\varepsilon+a\ ,\qquad\text{where } a\equiv\overline{\lim_{\xi\to\infty}}|K(\xi)|\,.\end{aligned}$$

As $\varepsilon>0$ is arbitrary we conclude that

$$|K(x)|\leqq a\ ,\qquad x\geqq x_0\,. \tag{12}$$

We shall prove (7) by showing that $a=0$. We can assume without loss of generality that $a=\overline{\lim}_{\xi\to\infty}K(\xi)\geqq-\underline{\lim}_{\xi\to\infty}K(\xi)$. It follows from (10) and (12) that

$$K(x)\leqq a_1K(x+A_1)+(1-a_1)a\ ,\qquad x\geqq x_0\,.$$

Repeated substitution in this inequality gives

$$K(x)\leqq a_1^kK(x+kA_1)+(1-a_1^k)a\ ,\qquad x\geqq x_0,\ k=0, 1, 2,\cdots\,. \tag{13}$$

By the definition of a there exists, for any $\varepsilon>0$, and for any positive integer L, an x_1 ($>x_0$) such that $a-\varepsilon a_1^L\leqq K(x_1)$. It follows from (13)

$$a-\varepsilon a_1^L\leqq K(x_1)\leqq a_1^kK(x_1+kA_1)+(1-a_1^k)a\qquad k=0, 1, 2,\cdots$$

which in turn implies

$$a-\varepsilon a_1^{L-k}\leqq K(x_1+kA_1)\ ,\qquad k=0, 1, 2,\cdots\,. \tag{14}$$

Adding both sides of (14) for $k=0, 1,\cdots, L-1$, we obtain

$$\begin{aligned}L(a-\varepsilon)&\leqq\sum_{k=0}^{L-1}(a-\varepsilon a_1^{L-k})\leqq\sum_{k=0}^{L-1}K(x_1+kA_1)=H(x_1+LA_1)-H(x_1)\\&\leqq2\sup|H(x)|=2C\,.\end{aligned}$$

As ε and L are arbitrary, this is possible only if $a=0$. Our argument does not depend on the choice of x_0 and this completes the proof of Theorem 1.

COROLLARY 1. *Let $X_1,\cdots, X_n$ ($n>1$) be a sample from a distribution F and let $a_1, a_2,\cdots, a_n$ be positive constants subject to the condition* (2) *and such that $\log a_i/\log a_j$ is an irrational number for some pair i and j. If $Z=\min_{1\leqq k\leqq n}\{X_k/a_k\}$ has the same distribution F as X_1,* *then F is the exponential distribution.*

COROLLARY 2 (Arnold —see Huang [5] and Galambos-Kotz [3]—, Sethuraman [7]). *Let $X_{1,m}$ be the minimum in a sample $X_1, X_2, \cdots, X_m$ from a distribution F. If, for two values m_1 and m_2 of m such that* $\log m_1/\log m_2$ *is irrational, $m_1 X_{1,m_1}$ and $m_2 X_{1,m_2}$ have the same distribution as X_1, then F is exponential.*

PROOF. These two correspond to the cases $m_1 = m_2 = \cdots = m_n = 1$, and $n = a_1 = 1 < m_j$, $j = 1, 2$, respectively. We have only to note that by Theorem 1 F has the form (1) and H has period $A_j \equiv -\log a_j$, $j = 1, 2, \cdots, n$ (Cor. 1) and $A_j \equiv \log m_j$, $j = 1, 2$ (Cor. 2), respectively. As, in both cases, A_i/A_j is irrational for some i and j, H must be a constant. See also Remark of [9].

COROLLARY 3 (Gupta [4]). *Suppose the distribution function $F(x)$ is such that $\lim_{x \downarrow 0} F(x)/x = \lambda > 0$. If for some $m > 1$, $mX_{1,m}$ has the same distribution F, then F is exponential.*

PROOF. Again by Theorem 1, F is of the form (1). The condition of Corollary 3 implies $\lim_{x \to \infty} H(x) = \lambda$. Since H is periodic it must be a constant.

Remark. Theorem 1 does not insist that any distribution F of the form (1) has the property that Z is distributed as F. In fact we can construct a distribution function F which can be put in the form (1) with a bounded and periodic $H(x)$ for which the variable Z does not follow this distribution no matter how n, m's and a's are chosen. (See [9].) In this connection we can easily prove

THEOREM 2. *Let F be a distribution function of the form* (1) *with $\alpha = 1$, and suppose that $H(x)$ has the fundamental period $\rho > 0$. Let n, $m_1, \cdots, m_n$ be given positive integers. In order that there exist positive numbers $a_1, \cdots, a_n$ subject to the conditon* (2) *and such that the variable Z has the distribution F, it is necessary and sufficient that $e^{-\rho}$ is a zero of the polynomial $\sum_{k=1}^{n} m_k X^{p_k} - 1$, where p's are unspecified positive integers.*

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