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ON RANDOM COMPLETE PACKING BY DISCS

MASAHARU TANEMURA

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Summary

A computational algorithm for random complete packing by discs is proposed. Monte Carlo simulations using this algorithm give the value 0.5473 for random packing density of discs. It greatly improves the Solomon's result, 0.4756.

1. Introduction

Random sequential packing of non-overlapping objects in a container of finite size is a typical one among the various random packings. The centre of the non-overlapping object is assumed to be uniformly distributed over the occupiable space of the container, which we call the residual space, hereafter. The packing procedure ends when the residual space becomes null. We call the packing in which the residual space completely vanishes the random complete packing. This type of packing has applications in connection with the adsorption of molecules on a crystal surface, the structure of liquids, the spatial pattern in ecological systems and so on.

Since an exact mathematical treatment was given by Rényi [1] of the so called car parking problem, random sequential packing has attracted attention of many authors. Most of their works are, however, limited to the theoretical investigation of random packing in a one-dimensional lattice or continuous space (for example, Page [2], Mackenzie [3], Bánkövi [4], Dvoretzky and Robbins [5], and Mannion [6], [12]). In the case of higher dimensions, there seem to exist unsurmountable difficulties in a mathematically strict treatment and we are compelled to investigate by computer simulations. Some attempts on this subject up to the present time (Palásti [7], Solomon [8], Blaisdell and Solomon [9], and Akeda and Hori [10], [11]) are more or less connected with the conjecture of Palásti. She suggested that the homothetic packing density of n-dimensional unit cubes is equal to the n-th power of the packing density of unit rods on a line, where by "homothetic" is meant

the situation where the sides of cubes are parallel to the sides of n-dimensional cubic container. The main concern of these authors was the limiting packing density; i.e., the fraction of the total volume of the packed objects to the container volume in the limit of an infinite container size, and it is also our present concern. With regard to the random sequential packing of spheres, systematic studies do not seem to have been so often carried out compared with those of the homothetic packing of cubes. Solomon [8] reported packing densities of n-dimensional spheres ($2 \leqq n \leqq 5$) but the number of spheres used for each dimension was rather small. Furthermore, it is doubtful whether a random "complete" packing was attained by each of the computer simulations in continuous space so far appearing in literatures.

The purpose of this paper is twofold. The first is to provide a computational technique for a random complete packing of two-dimensional spheres or discs. And the second is to give an accurate packing density for an infinite system of discs. In the next section, a prevailing computational technique for random sequential packing, the simple rejection scheme, is examined. In Section 3, we propose an economical algorithm for constructing a complete packing of spheres. Section 4 is devoted to the case of discs, namely, the proposed algorithm is examined and the result of simulation is given for this case. Finally, in Section 5, some discussions about our method and result are given.

2. Examination of simple rejection scheme

A computational technique which corresponds to the procedure of random sequential packing of non-overlapping particles is the simple rejection scheme. In this case, the coordinates of the centre of a "test" particle are determined by a random point which is uniformly distributed in the part of the container where the centre of the first particle can be placed. If a test particle does not overlap with the particles already placed, it is fixed at its position. Otherwise, the particle is discarded and another test particle is generated. The process is continued until a prescribed stopping condition is satisfied. As a stopping rule of the scheme, two types of conditions are usually employed. The first rule prescribes that the procedure ought to stop if a successive number of failures for test particle to hit residual space exceeds a given maximum value and the second rule prescribes it should stop if the total number of test particles generated exceeds a given value.

We examine the simple rejection scheme in the case of the packing of discs. Let the disc be denoted by K and let $N_t(k)$ be the number of test particles generated up to the addition of the k-th disc K_k. Let

R_k be the residual space which is left for the addition of K_k. The probability that a test particle hits R_k is proportional to $|R_k|$, the area of R_k. Because $|R_k|$ decreases monotonously with the increase of k, we can easily see that the expectation of $N_t(k)$ will rapidly grow large with k. That means the number of test particles which would be required to add one disc increases as the packing procedure goes on.

In order to examine the behaviour of $N_t(k)$ with k, we performed a Monte Carlo simulation employing the second stopping rule. Letting the diameter of the disc be unity, we used a rectangular of size $x \times y = 31.000 \times 32.177$ as a container, where x and y represent the side lengths of the container. The container can accommodate 1129 discs in hexagonal close packing. Let N_t be the number such that the procedure

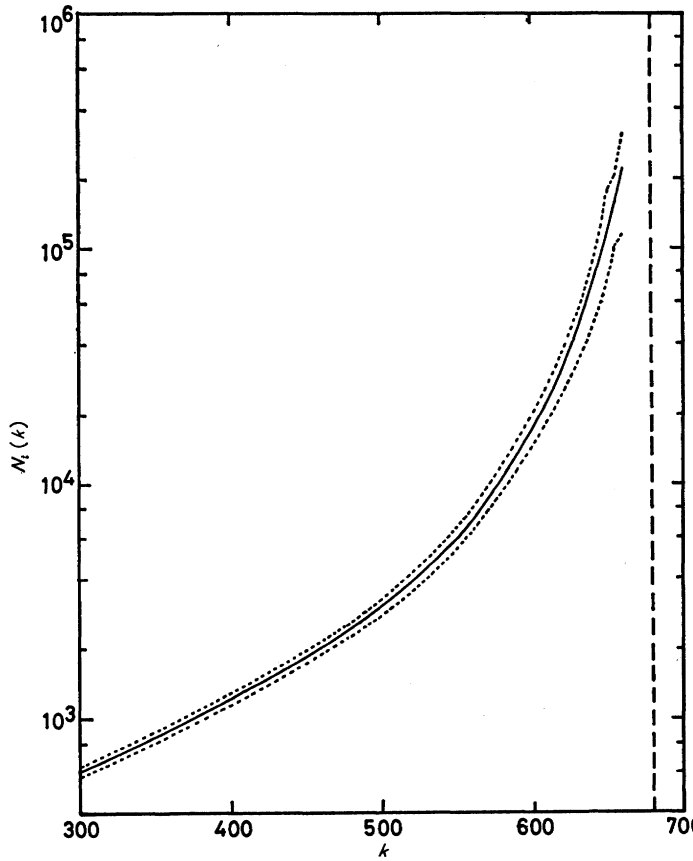

Fig. 1 Relation between the number of added discs, k, and the number of test particles generated up to the addition of the k-th disc, $N_t(k)$, in case of the rectangle $x \times y = 31.000 \times 32.177$. Solid curve is the mean of $N_t(k)$. Upper and lower dotted curves bound the standard deviation of $N_t(k)$. Vertical broken line represents the mean of the number of added discs in complete packing.

stops if the total number of trials exceeds it. Simulation was carried out for seven cases of N_t, i.e., $(2.5, 3, 4, 6, 10, 20, 40)\times 10^4$ and for each case ten trials were performed. We will mention the results of this simulation also in a later section. Figure 1 represents an empirical relation between k and $N_t(k)$ in a semi-log scale obtained by superposing the results of repeated trials. The solid curve is the sample mean of $N_t(k)$ and the dotted curves represent the standard deviation of $N_t(k)$. In this figure, the number of samples used for these estimations is 70 for $k\leqq 600$, while it gradually decreases for $k>600$. The vertical broken line at the right hand side in the figure indicates the average number of added discs obtained by an independent simulation of 50 trials employing the complete packing algorithm to which we will refer in the next section. The figure shows that $N_t(k)$ increases with k more than exponentially and that it becomes more and more difficult to add a further disc as we approach the state of complete packing. In other words, the area of the residual space becomes small very quickly as we go near the complete packing.

We performed another simulation with $N_t=2\times 10^6$ in order to see how many more discs could be added before a complete packing was attained starting from a state which had been obtained by the generation of such a large number of test particles. Let the number of added discs attained for a given N_t in a rectangular container of size $x\times y$ be $\nu_{xy}^{(1)}$, where the superscript (1) indicates the simple rejection scheme. And let ν_{xy} be the number of added discs in the container in the complete packing. In the simulation, we used the container of the same size as shown above. The results of three trials are

trial	$\nu_{xy}^{(1)}$	ν_{xy}	$\Delta=\nu_{xy}-\nu_{xy}^{(1)}$
a	669	676	7
b	678	686	8
c	673	681	8

These should be compared with the results of the simulation mentioned above in which the averages of Δ $(=\nu_{xy}-\nu_{xy}^{(1)})$ are 18.5, 25.4 and 37.9 for $N_t=4\times 10^5$, 2×10^5 and 10^5, respectively. This suggests that Δ will not vanish even if more than several millions were chosen for N_t. Therefore, we conclude that, practically, it is very difficult to obtain a random complete packing by the simple rejection scheme alone.

3. Algorithm for complete packing

We have seen in the previous section that the single use of the simple rejection scheme is actually inadequate to fill the residual space

completely, that is, to attain a complete packing. Accordingly, at a proper number of generated test particles, the simple rejection scheme must be replaced by an economical scheme which can fill the residual space entirely. In a lattice space, a test particle ranges over the lattice sites which are finite in number. In this case, therefore, economical and unambiguous techniques can be devised. Blaisdell and Solomon [9] made use of a "random occupiable site method", where indices of occupiable sites which remained after some duration of the use of the simple rejection scheme are stored and from them a site to which a new particle is to be assigned is selected at random. In this method, the number of occupiable sites vanishes definitely within some finite number of steps and a complete packing is obtained.

In case of continuous space, on the contrary, it is difficult to prepare a scheme which corresponds exactly to the method such as the "random occupiable site method" mentioned above. This is because in practice we are unable to assign a test particle at random in the residual space which may be separated into a number of simply connected regions of various shapes and sizes. Let us call each of these regions an occupiable region. From the very definition of the random sequential packing, these regions should be treated one at a time. Namely, at every step of the addition of a particle, the areas of all the occupiable regions have to be calculated in order to select one region with a probability proportional to its area. However, if all the regions are isolated from one another in the sense that the addition of particles into an occupiable region does not affect the addition into others, then the order of selection of regions will be irrelevant. Thus, if the locations of such regions are detected one by one, we can place particles in the regions by generating relatively small number of test particles and can obtain a random complete packing. We give below a method of finding the location of an occupiable region and present an algorithm for complete packing in case of random sequential addition of discs.

3.1. *Detection of an occupiable region*

We assume that in a rectangular container a certain number of discs of unit diameter are already placed by the simple rejection scheme employing the second stopping rule. Moreover, we assume from now on that the residual space thus obtained is separated to a certain number of simply connected regions, i.e., occupiable regions, whose sizes are small compared with the area of a disc. The second assumption will hold if a properly large value is chosen for N_t. Figure 2 illustrates how an occupiable region is detected. Each solid circle represents a disc K_i already placed and each dotted circle $\tilde{K}_i$ is the circle which is concentric with K_i and whose diameter is twice of K_i's. We call each

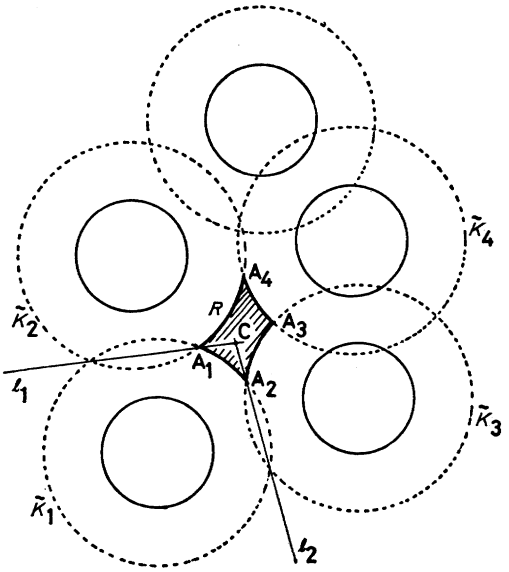

Fig. 2 Illustration for finding the location of an occupiable region. Shaded region R is an occupiable region found nearby the disc $\tilde{K}_1$.

$\tilde{K}_i$ a large disc. The regions which are not covered by large discs in the set S, the inner parallel set of the rectangle at the distance 1/2, are occupiable regions. Generally, such a region is surrounded by a certain number of arcs of large discs. Therefore, the space is a "concave polygon" whose definition is given in Appendix. In the figure, R is an occupiable region and it is surrounded by four arcs, each of which is the arc of $\tilde{K}_1$, $\tilde{K}_2$, $\tilde{K}_3$, or $\tilde{K}_4$. For each vertex of the occupiable region, we draw the common chord extended outside the discs. Next, we obtain a cross point of two such chords which are drawn for a pair of adjacent vertices and then inspect whether it is covered by at least one of large discs or not. If the cross point is not covered by any large disc, it is inside an occupiable region. According to the proposition given in Appendix, such a cross point certainly exists among the cross points obtained for all the pairs of adjacent vertices. Thus we can find out the position of an occupiable region. Two lines l_1 and l_2 in Fig. 2 are above-mentioned lines drawn for the vertices A_1 and A_2, respectively, and the cross point C of l_1 and l_2 is contained by R. In the neighbourhood of the edges of S, the occupiable regions may be truncated by the boundary of S. These regions, however, can be considered as the concave polygons in which some of the arcs are rectilinear. Therefore, the method just given is also applicable to this case.

3.2. *Complete packing algorithm (CPA)*

Under the assumptions of the previous subsection, a complete packing algorithm, abbreviated as CPA, searchs for the locations of occupiable regions one by one by the method given above and places discs there by random sampling. The CPA is composed of essentially two steps:

(I) *Picking out one of the discs already placed and the detection of occupiable regions in the neighbourhood of the disc*

Let K_1 be a disc we picked out. We register all the large discs, $\tilde{K}_i$'s, which intersect $\tilde{K}_1$. Let its number be p. If $p<3$, for any configuration of the intersecting large discs (if any), $\tilde{K}_1$ is certainly adjacent to the residual space. Then, we jump into the step (II) ($p<3$ case) and, if $p=3$ is attained there, we return here.

If $p\geqq 3$, we draw, for each $\tilde{K}_i$ which intersects $\tilde{K}_1$, a line which connects the two point common to $\tilde{K}_i$ and $\tilde{K}_1$. Then we get a cross point of two such lines which are drawn for a pair of $\tilde{K}_i$'s whose centres are adjacent to each other if we look at them from the centre of $\tilde{K}_1$. We inspect whether a cross point thus obtained is covered by at least one of the existing large discs in the container. If it is not covered by any large disc, the point is surely inside an occupiable region and we go to the step (II) ($p\geqq 3$ case). Otherwise, we inspect other cross points and jump into the step (II) if necessary. If all the cross points obtained for $\tilde{K}_1$ are covered by some large discs, then we ask if all the discs including those added in the step (II) have been picked out. If it is true, the CPA stops and otherwise, we return to the start of this step.

(II) *Addition of a disc into the occupiable region obtained in the step* (I)

i) $p<3$ *case.* In this case, contrary to our initial assumption in 3.1, the occupiable region adjacent to $\tilde{K}_1$ may be large in size compared to the size of a single disc even if the region is simply connected in accordance with its definition. Therefore, we must consider that the value of N_t was not large enough to meet the assumption, and we continue again the simple rejection procedure until $p=3$ is attained for $\tilde{K}_1$.

ii) $p\geqq 3$ *case.* In this case, test particles are sampled at random in a square whose centre is at the cross point just obtained in the first step. Let the size of the square be sufficient to cover the occupiable region entirely. The region in the square will be hit by a test particle after generating a relatively small number of test particles. Let M_t be a given number. If the region is not hit by M_t test particles successively, we reduce the size of the square by half keeping its centre

unchanged and again we sample test particles in the reduced square. The process is continued until a disc is placed. If the number of times of the reduction exceeds a given number M, we place a disc at the centre of the square. Then we return to the step (I).

It is easily seen from the argument in 3.1 that we can detect all of the occupiable regions by inspecting all the discs including those added in the step (II). Therefore, for any configuration of discs whose residual space is composed of a certain number of simply connected regions, the CPA makes the residual space strictly null. On that account, our method assures the completeness of random packing. As for the randomness of this complete packing, on the contrary, the resulting configuration of discs in the residual space may be an approximation to the very sense of the random sequential packing. Firstly, in the step (I), the occupiable regions are selected in an order irrespective of their area, while it may happen that some regions are not isolated from one another. Secondly, in the step (II), there is a possibility that, in some occupiable regions, the sampling of test particle is not performed uniformly over each region. Nevertheless, we note here that the number of discs which can be accommodated in the residual space depends less on the way of sampling than the configuration of them. Therefore, the approximation of the packing density is always better than that of configuration.

The randomness of the configuration of discs in the strict sense, however, can be attained approximated closely if a suitably large value is chosen for N_t in the simple rejection scheme and for the constants in the CPA. In practice, we have determined N_t by fixing the constants M_t and M in the CPA at 50 and 10, respectively.

4. Monte Carlo simulation for random complete packing of discs

In order to obtain the random packing density of discs, we carried out a Monte Carlo simulation using the algorithm proposed in the previous section. The programme consists of two parts, namely, the simple rejection scheme and the CPA. The rectangular container is chosen in such a way that it entirely encloses a system of hexagonally close packed discs and that at the same time its form is as close to a square as possible. Use of such a container allows one for each admissible size of container to compute a random packing fraction, i.e., the ratio of the added discs to the hexagonally close packed discs. The random packing fraction is a useful quantity as well as the random packing density.

4.1. *Estimation of N_t and the justification of the CPA*

In order to make the programme most efficient without hurting its

validity, we must choose an appropriate value for N_t, the number by which the first part of the programme is stopped. To make the residual space small enough as assumed in Section 3, N_t should be as large as possible. From an economical point of view, however, a smaller N_t is favourable. To estimate an efficient value of N_t, we performed some experiments in case of the container size $x \times y = 31.000 \times 32.177$. Figure 3 shows a relation between N_t and the number of added discs obtained

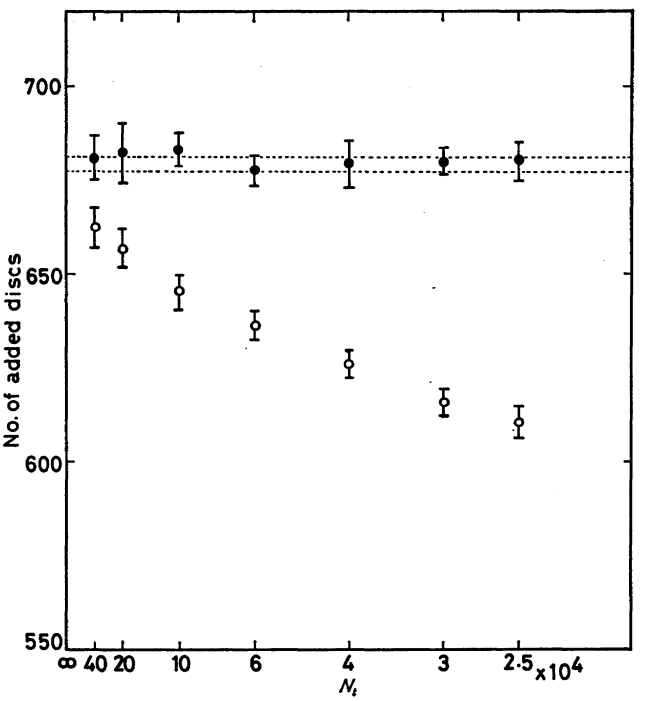

Fig. 3 Relation between the total number of test particles, N_t, and the number of added discs in case of the rectangle $x \times y = 31.000 \times 32.177$. Open circles represent $\bar{\nu}_{xy}^{(1)}$ obtained by N_t test particles and closed circles represent $\bar{\nu}_{xy}$ obtained by means of CPA. Error bars are their 95% confidence intervals. Dotted lines indicate 95% interval of $\bar{\nu}_{xy}$ obtained for 50 trials in case of $N_t = 4 \times 10^4$.

by the simulation for the seven N_t's, to which we have referred in Section 2. The abscissa is scaled as the inverse of N_t. Open and closed circles represent $\bar{\nu}_{xy}^{(1)}$ (sample mean of $\nu_{xy}^{(1)}$) and $\bar{\nu}_{xy}$ (sample mean of ν_{xy}), respectively, where $\nu_{xy}^{(1)}$ is, as before, the number of discs added in the first part of the programme and ν_{xy} is $\nu_{xy}^{(1)}$ plus the number of discs added in the second part. Error bars are 95% confidence intervals of the mean for each of ten trials. Two parallel dotted lines represent the 95% confidence interval of $\bar{\nu}_{xy}$ obtained by an independent simulation of 50 trials for $N_t = 4 \times 10^4$. The figure shows that although $\nu_{xy}^{(1)}$ increase

with N_t, ν_{xy} scarcely changes within the range of the error. Therefore, we conclude in this case that the CPA gives a good value of ν_{xy} for $N_t \gtrsim 2.5\times10^4$. Consequently, as a suitable choice of N_t we take a value such that the number of test particles generated per added disc in the first stage, i.e. $N_t/\bar{\nu}_{xy}^{(1)}$, amounts to about 40 or more. We set its value to be about 60 for the sake of reliability. In case of the container size used in this experiment, accordingly, $N_t=4\times10^4$ ($N_t/\bar{\nu}_{xy}^{(1)}\cong63.9$) would be favourable.

Next, we investigate whether such a choice of N_t works well for other container size. We examined the patterns of residual space for three trials of the simulation in which $x\times y=41.000\times42.569$ and $N_t/\bar{\nu}_{xy}^{(1)}\cong64.3$. For each pattern which was drawn by means of the XY-plotter, we estimated by hand how many more discs could be placed to entirely cover the residual space which remained after the generation of N_t test particles. Let us denote its minimum and maximum number by $\Delta_{\min}$ and $\Delta_{\max}$, respectively. The result is

trial	$\nu_{xy}^{(1)}$	$\Delta_{\min}$	$\Delta_{\max}$	Δ	ν_{xy}
a	1084	105	108	107	1191
b	1078	107	112	111	1189
c	1073	122	125	125	1198 .

Here we also listed the values of ν_{xy} and $\Delta=\nu_{xy}-\nu_{xy}^{(1)}$ obtained by employing the CPA. The result shows that by such a choice of N_t all the occupiable regions are well isolated from one another in the sense described at the beginning of Section 3. At the same time, this guarantees the validity of all the settings in the programme of the CPA which were given in the last part of the previous section. For these settings, including N_t, the case i), i.e., $p<3$ case in the step (II) of the CPA never appeared and took between one half and two thirds of the time required for the simple rejection part of the programme. Thus, it may be said that the CPA method is very accurate and its employment proves to be economical.

4.2. *Random packing density of discs*

On the basis of the discussions so far, we performed simulations for ten container sizes for the purpose of obtaining the random packing density of discs in the limit of the infinite container size. Table I presents the results of our simulations. In the table the number of hexagonally close packed discs, ν_{hex}, is also given. For the data processing, we make the following assumptions: firstly, the random packing density $\rho_{xy}=\nu_{xy}|K|/xy$, $|K|$ being the area of a packed disc, obeys a normal distribution; secondly, it can be expressed using the limiting

Table I. Results of simulations for random packing density of discs

$x \times y$	Number of trials	N_t	ν_{hex}	$\bar{\rho}_{xy}$	S.D. of ρ_{xy}
11.000× 11.392	100	5×10^3	137	0.51580	0.01439
16.000× 16.589	100	10^4	295	0.52504	0.01051
21.000× 21.785	50	2×10^4	513	0.52957	0.00761
31.000× 32.177	50	4×10^4	1129	0.53479	0.00543
41.000× 42.569	50	7×10^4	1985	0.53901	0.00385
51.000× 51.230	50	10^5	2980	0.54072	0.00278
61.000× 61.622	40	1.5×10^5	4296	0.54134	0.00253
81.000× 82.406	20	3×10^5	7648	0.54380	0.00176
101.000×101.459	20	4×10^5	11759	0.54381	0.00118
201.000×201.918	10	1.8×10^6	46717	0.54525	0.00069

packing density $\rho=\lim_{x,y\to\infty}\rho_{xy}$ in the form

$$\rho_{xy}=\rho+au^{-1}+o(u^{-1}) ,$$

where $u^{-1}=x^{-1}+y^{-1}$, and finally the standard deviation of ρ_{xy} is proportional to u^{-1}. Each of these assumptions was also made by Akeda and Hori [11] in case of homothetic packing of squares. In Fig. 4, we give

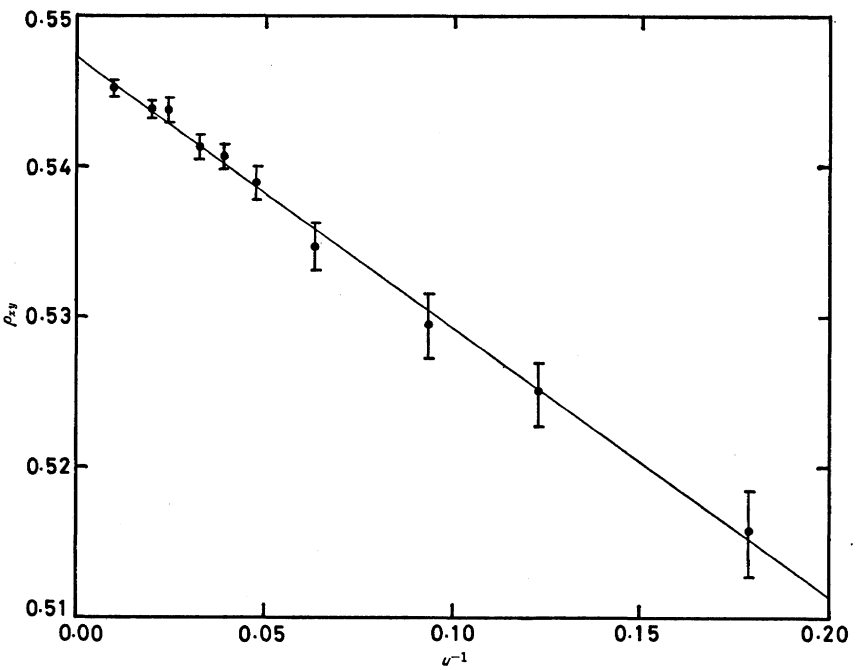

Fig. 4 Relation between u^{-1} and ρ_{xy}. Closed circles are $\bar{\rho}_{xy}$ and error bars are their 95% intervals. Solid line is obtained by a weighted regression analysis of ten data points. The analysis gives 0.5473 ± 0.0009 for ρ and -0.179 ± 0.014 for a.

the relation between ρ_{xy} and u^{-1}. Each closed circle is $\bar{\rho}_{xy}$, the mean of ρ_{xy}, and each error bar represents its 95% confidence interval. As can be seen from the figure, the linearity of ρ_{xy} with respect to u^{-1} holds well. The solid line is obtained by a weighted regression analysis of ten data points. The analysis gives for the limiting packing density $\rho=0.5473\pm0.0009$ (95% int.) and for the coefficient $a=-0.179\pm0.014$ (95% int.). This value for ρ is significantly larger than Solomon's result, $\rho=0.4756$. This difference is probably due to the smallness of the system size and to the incompleteness of packing in Solomon's simulation. As for random packing fraction, $\eta_{xy}=\nu_{xy}/\nu_{\text{hex}}$, its limiting value $\eta=\lim_{x,y\to\infty}\eta_{xy}$ can be obtained, using the value of ρ just obtained, as $\rho\cdot\sqrt{12}/\pi=0.6035$, where $\pi/\sqrt{12}$ is the regular packing density. This value should be compared with $\eta=0.6036\pm0.0010$ (95% int.) obtained by the regression analysis similar to ρ.

5. Discussion

The complete packing algorithm, CPA, proposed in this paper can be directly extended to the random sequential packing of higher dimensional spheres. Moreover, as can be inferred from the proposition in Appendix, the algorithm may be generalized to be used not only for the random packing of any identical convex objects such as cubes and ellipsoids but also for that of non-identical convex objects.

As we have mentioned in the last part of Section 3, the resulting configuration of discs in the residual space obtained by the CPA is inevitably an approximation to that for the true randomness, though it can be made almost exact. It can be seen, however, as we have also noted, that the packing density is more accurate than the configuration of discs, which confirms the validity of our result.

Recently, Noguchi and Hori [13] computed the random packing density of discs by an algorithm whose idea is on the same line as ours. Their result is $\rho=0.5474\pm0.0007$ (95% int.). It is very close to our result.

We are now carrying out simulations of the random sequential packing of spheres employing the CPA extended to three dimensions. Preliminary simulations indicate that the random packing density of spheres amounts to 0.351 or more, which should be compared with Solomon's result for spheres, $\rho=0.280$. Those results for discs and spheres strongly suggest that the random packing densities of higher dimensional spheres will be also greatly improved by employing the CPA.

Appendix

We present below a proposition which assures the validity of our method for searching occupiable regions. As a preliminary, we define the following concepts:

DEFINITION 1. A *concave n-gon* is a simply connected open set whose boundary is a simple closed curve which is composed of n arcs each of which is concave with respect to the set.

DEFINITION 2. A line which goes through a vertex of a concave polygon is said to lie inside the *concave angle* about the vertex if in every neighbourhood of the concerned vertex the line meets the interior of the polygon.

With these definitions, the proposition is stated as follows;

PROPOSITION. *Let P be an arbitrarily given concave polygon and suppose for every vertex of P a line is drawn through the vertex and inside the concave angle about the vertex. Then, among these lines, at least one pair which belong to an adjacent pair of vertices intersect each other inside P.*

PROOF. Let $A_1, A_2, \cdots, A_n$ be the vertices of concave n-gon P as shown in Fig. 5 A. We shall prove the proposition by deriving to a contradiction, assuming that the condition does not hold. Namely, we assume all of the n pairs of lines, each pair going through an adjacent pairs of vertices and being inside the respective concave angles, do not intersect inside P. Let A_1B_1 and A_2B_2 be any such a pair of lines drawn

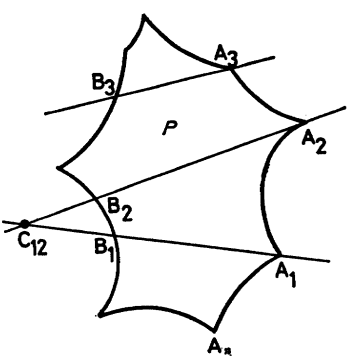

Fig. 5 A "Concave n-gon" P. $A_1, A_2, \cdots, A_n$ are its vertices. C_{12} is a cross point of of two lines A_1B_1 and A_2B_2 and is located outside P.

for an adjacent pair of vertices, A_1, and A_2, respectively. For simplicity, let B_1 and B_2 be the points at which the respective lines at first intersect with the boundary of P. In what follows we define B_i ($i=3, 4, \cdots, n$) similarly to this. The segment $\overline{A_1B_1}$ divides P into two concave subpolygons, $\overline{A_1B_1}$ itself being the arc common to both polygons. Let P_1 and $P_1'=P-P_1$ be such polygons, where P_1 includes A_2 as one of its vertices. Since, by the assumption, the cross point C_{12} of the two lines A_1B_1 and A_2B_2, if it exists, lies outside P, it is obvious that the segment $\overline{A_2B_2}$ lies in P_1 and does not meet $\overline{A_1B_1}$. Similarly, the segment $\overline{A_2B_2}$ divides P_1 into two subpolygons P_2 and $P_2'=P_1-P_2$, where P_2 is defined as to include A_3. Therefore, by the assumption, any line segment $\overline{A_3B_3}$ which goes through A_3 and is inside the concave angle about A_3 must lie in P_2 and must not meet $\overline{A_2B_2}$. The number of vertices of P_2 is less than that of P_1 by at least one. If we continue the same procedure, since the number of vertices of subpolygon P_i which contains A_{i+1} decreases by at least one with each increase of i, we shall arrive, for some k ($\leqq n$), at the situation where the subpolygon P_k including the vertex A_{k+1} and determined by a line segment $\overline{A_kB_k}$ is a concave triangle as shown in Fig. 5 B. In this situation, it is obvious

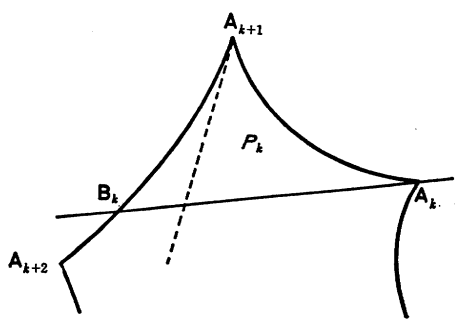

Fig. 5 B An ultimate situation in the proof of the proposition. Broken line certainly intersects the segment $\overline{A_kB_k}$.

by the concavity of arcs $\overset{\frown}{A_kA_{k+1}}$ and $\overset{\frown}{A_{k+1}B_k}$ that any line which goes through A_{k+1} and is inside the concave angle about A_{k+1} has to meet the segment $\overline{A_kB_k}$, whence the meeting point of the line with A_kB_k is inside P. This contradicts the assumption, which proves the proposition.

The conclusion of the proposition can be extended under the same conditions. We give a more general conlusion in the following;

COROLLARY. *Under the same conditions of the proposition, at least two pairs of lines which belong to two distinct adjacent pairs of vertices intersect pairwise inside P.*

PROOF. It is almost self-evident if we use the same idea in the proof of the proposition for another subpolygon P_1' determined by the line segment $\overline{A_1B_1}$.

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REFERENCES

[1] Rényi, A. (1958). On a one-dimensional problem concerning random space filling, *Publ. Math. Inst. Hung. Acad. Sci.*, **3**, 109-127.

[2] Page, E. S. (1959). The distribution of vacancies on a line, *J. R. Statist. Soc.*, A, **21**, 364-374.

[3] Mackenzie, J. K. (1962). Sequential filling of a line by intervals placed at random and its application to linear adsorption, *J. Chem. Phys.*, **37**, 723-728.

[4] Bánkövi, G. (1962). On gaps generated by a random space filling procedure, *Publ. Math. Inst. Hung. Acad. Sci.*, **7**, 395-407.

[5] Dvoretzky, A. and Robbins, H. (1964). On the "parking" problem, *Publ. Math. Inst. Hung. Acad. Sci.*, **9**, 209-225.

[6] Mannion, D. (1964). Random space filling in one dimension, *Publ. Math. Inst. Hung. Acad. Sci.*, **9**, 143-154.

[7] Palásti, I. (1960). On some random space filling problems, *Publ. Math. Inst. Hung. Acad. Sci.*, **5**, 353-360.

[8] Solomon, H. (1967). Random packing density, *Proc. Fifth Berkeley Symp. Math. Statist. Prob.*, **3**, 119-134.

[9] Blaisdell, B. E. and Solomon, H. (1970). On random sequential packing in the plane and a conjecture of Palásti, *J. Appl. Prob.*, **7**, 667-689.

[10] Akeda, Y. and Hori, M. (1975). Numerical test of Palásti's conjecture on two-dimensional random packing density, *Nature*, **254**, 318-319.

[11] Akeda, Y. and Hori, M. (1976). On random sequential packing in two and three dimensions, *Biometrika*, **63**, 361-366.

[12] Mannion, D. (1976). Random packing of an interval, *Adv. Appl. Prob.*, 8, 477-501.

[13] Noguchi, K. and Hori, M. (1979). to be published.