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A NOTE ON THE MOMENTS OF ORDER STATISTICS FROM DOUBLY TRUNCATED EXPONENTIAL DISTRIBUTION

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Summary

In a recent paper [2], the author has obtained some recurrence relations between the moments of order statistics from the exponential and right truncated exponential distributions. In this paper, similar relations are derived for a doubly truncated exponential distribution. It is shown that one can obtain all the moments by using these recurrence relations.

1. Introduction

Let X be a doubly truncated exponential random variable with distribution function $F(x)$ and density function

$$(1)\qquad f(x)=e^{-x}/(P-Q)\,, \qquad -\log(1-Q)\leqq x\leqq -\log(1-P)\,,$$

where Q is the proportion of truncation on the left and $1-P$ is the proportion of truncation on the right of the standard exponential distribution. The proportions Q and P, with $Q<P$, are assumed to be known. Let $X_{1,n}\leqq X_{2,n}\leqq\cdots\leqq X_{n,n}$ represent an ordered sample of size n from (1). Denote the ith moment of kth order statistic $X_{k,n}$ by $\alpha_{k,n}^{(i)}$, $1\leqq k\leqq n$.

For $Q=0$, this distribution reduces to the right truncated exponential distribution, which has been considered in detail by Joshi [2] and Saleh et al. [3]. Saleh et al. provide exact finite series expressions for first and second order moments of $X_{k,n}$ by direct integration, while Joshi has obtained some recurrence relations for $\alpha_{k,n}^{(i)}$, and has tabulated the third and fourth order moments. If P is also equal to 1 then X follows the standard exponential distribution, which has been studied in considerable detail. These truncated and untruncated exponential distributions have been used as a model in the life testing data. For these

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distributions, order statistics and their moments play an important role in estimation, testing of hypothesis and related problems.

Although it is possible to proceed along the lines of Saleh et al. [3] for the double truncated case also, yet the expressions for moments become extremely complicated. On the other hand, the results given by Joshi [2] can be immediately extended. These relations link the higher order moments in terms of lower order moments and also give the results for left truncation by setting $P=1$.

2. Recurrence relations

Throughout this section, we shall follow the convention

$$a_{k,n}^{(0)}=1 ,$$

and denote $-\log(1-Q)$ by Q_1, $-\log(1-P)$ by P_1, $(1-Q)/(P-Q)$ by Q_2, and $(1-P)/(P-Q)$ by P_2. Now the probability density function of $X_{k,n}$ $(1\leqq k\leqq n)$ is given (David [1], p. 8) by

$$f_{k,n}(x)=\{B(k,\, n-k+1)\}^{-1}\{F(x)\}^{k-1}\{1-F(x)\}^{n-k}f(x) ,$$

where $B(\cdot,\cdot)$ stands for the complete beta function. Substituting for $f(x)$ and $F(x)$, this can be written as

$$f_{k,n}(x)=\{B(k,\, n-k+1)\}^{-1}(P-Q)^{-n}(1-Q-e^{-x})^{k-1} \cdot (P-1+e^{-x})^{n-k}e^{-x} , \qquad Q_1\leqq x\leqq P_1 .$$

Consequently

$$a_{k,n}^{(i)}=\int_{Q_1}^{P_1} x^i f_{k,n}(x)dx . \tag{2}$$

RELATION I. For $i\geqq 1$,

$$a_{1,1}^{(i)}=Q_1^iQ_2-P_1^iP_2+ia_{1,1}^{(i-1)} . \tag{3}$$

PROOF. Setting $k=n=1$ in equation (2), we have

$$a_{1,1}^{(i)}=(P-Q)^{-1}\int_{Q_1}^{P_1} x^i e^{-x}dx .$$

Integrating by parts by treating x^i for differentiation and e^{-x} for integration, the result follows.

RELATION II. For $n\geqq 2$, and $i\geqq 1$,

$$a_{n,n}^{(i)}=a_{n-1,n-1}^{(i)}Q_2-P_1^iP_2+(i/n)a_{n,n}^{(i-1)} , \tag{4}$$

$$a_{1,n}^{(i)}=Q_1^iQ_2-a_{1,n-1}^{(i)}P_2+(i/n)a_{1,n}^{(i-1)} . \tag{5}$$

PROOF. To prove (4), note that

$$a_{n,n}^{(i-1)}=n(P-Q)^{-n}\int_{Q_1}^{P_1} x^{i-1}(1-Q-e^{-x})^{n-1}e^{-x}dx .$$

Integrating by parts, but now treating x^{i-1} for integration and noting that

$$\frac{d}{dx}\{(1-Q-e^{-x})^{n-1}e^{-x}\}$$
$$=-n(1-Q-e^{-x})^{n-1}e^{-x}+(n-1)(1-Q)(1-Q-e^{-x})^{n-2}e^{-x} ,$$

we get

$$a_{n,n}^{(i-1)}=(n/i)(P-Q)^{-n}\Big\{x^i(1-Q-e^{-x})^{n-1}e^{-x}\Big|_{Q_1}^{P_1}$$
$$+n\int_{Q_1}^{P_1} x^i(1-Q-e^{-x})^{n-1}e^{-x}dx$$
$$-(n-1)(1-Q)\int_{Q_1}^{P_1} x^i(1-Q-e^{-x})^{n-2}e^{-x}dx\Big\}$$
$$=(n/i)[P_1^iP_2+a_{n,n}^{(i)}-a_{n-1,n-1}^{(i)}Q_2] .$$

Rewriting in terms of $a_{n,n}^{(i)}$, the result (4) follows. The proof of (5) is similar.

RELATION III. For $n\geqq 3$, $2\leqq k\leqq n-1$, and $i\geqq 1$,

$$a_{k,n}^{(i)}=a_{k-1,n-1}^{(i)}Q_2-a_{k,n-1}^{(i)}P_2+(i/n)a_{k,n}^{(i-1)} . \tag{6}$$

PROOF. Now

$$a_{k,n}^{(i-1)}=\{B(k,\ n-k+1)\}^{-1}(P-Q)^{-n}\int_{Q_1}^{P_1} x^{i-1}h(x)dx ,$$

where

$$h(x)=(1-Q-e^{-x})^{k-1}(P-1+e^{-x})^{n-k}e^{-x} .$$

The result (6) is obtained by integration by parts and using that

$$\frac{d}{dx}h(x)=-n(1-Q-e^{-x})^{k-1}(P-1+e^{-x})^{n-k}e^{-x}$$
$$+(k-1)(1-Q)(1-Q-e^{-x})^{k-2}(P-1+e^{-x})^{n-k}e^{-x}$$
$$-(n-k)(1-P)(1-Q-e^{-x})^{k-1}(P-1+e^{-x})^{n-k-1}e^{-x} .$$

In order to calculate $a_{k,n}^{(i)}$, the Relations I, II, and III are applied successively. For example, for $i=1$, $a_{1,1}^{(1)}$ is first obtained from equation (3), the moments of largest and smallest order statistics are then evaluated from equations (4) and (5) respectively, and finally equation (6) is used to obtain other moments. Next the moments for $i=2$ are evaluated in the same order, and so on.

As pointed out in Joshi [2] for the right truncation case, here also the moments can be obtained by means of a simple computer program without introducing serious rounding errors, at least for small values of n. Further, any particular pattern of truncation can be handled.

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