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ON LARGE DEVIATIONS AND DENSITY FUNCTIONS

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Abstract

Suppose that $X_1, X_2, \cdots$ is a sequence of absolutely continuous or integer valued random variables with corresponding probability density functions $f_n(x)$. Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of real numbers, then necessary and sufficient conditions are given for $n^{-1}\log f_n(\phi_n)-n^{-1}\log \mathrm{P}(X_n>\phi_n)=o(1)$ as $n\to\infty$.

1. Introduction

Theorems relating the probability density functions of a sequence of random variables to their large deviation were proven in Killeen, Hettmansperger, and Sievers [1]. They showed that if for each $n=1, 2, \cdots, X_n$ is a random variable with p.d.f. $f_n(x)$ and $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of real numbers then, under certain conditions,

$$(1.1)\qquad n^{-1}\log f_n(\phi_n)-n^{-1}\log \mathrm{P}(X_n>\phi_n)=o(1)\qquad \text{as } n\to\infty .$$

The necessity of the conditions of these theorems is not clear, so theorems which present necessary and sufficient conditions for (1.1) are presented here.

2. Main results

In this section we let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of real numbers and assume that $f_n(x)$ is non-increasing in $[\phi_n, \infty)$ with $\mathrm{P}(X_n>\phi_n)>0$ for large n.

THEOREM 1. *Suppose that X_n is an absolutely continuous random variable with density function $f_n(x)$. Then the following condition is necessary and sufficient for* (1.1): *there exists a non-negative sequence $\{\gamma_n\}_{n=1}^{\infty}$ with $n^{-1}\log\gamma_n=o(1)$ as $n\to\infty$ and for all $\varepsilon>0$.*

$$(2.1)\quad n^{-1}\log[(f_n(\phi_n+e^{-\varepsilon n})+\mathrm{P}(X_n\geqq\phi_n+\gamma_n))/f_n(\phi_n)]=o(1)\qquad \textit{as } n\to\infty .$$

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THEOREM 2. *Suppose that X_n is an integer valued random variable with probability function $P_n(k)=P(X_n=k)$ and $f_n(x)=P_n([x])$ where [ ] is the greatest integer function, then* (1.1) *holds if and only if there exists a non-negative sequence $\{\gamma_n\}_{n=1}^{\infty}$ with $n^{-1}\log\gamma_n=o(1)$ as $n\to\infty$ and*

$$(2.2)\quad n^{-1}\log[(f_n(\phi_n+1)+P(X_n\geqq\phi_n+\gamma_n))/f_n(\phi_n)]=o(1)\qquad \textit{as } n\to\infty\,.$$

3. Proof of theorems

PROOF OF THEOREM 1. Suppose that (1.1) is valid, then for any satisfactory sequence $\{\gamma_n\}_{n=1}^{\infty}$ it is easy to show that if $\varepsilon>0$ then $n^{-1}\cdot\log P(X_n\geqq\phi_n+\gamma_n)$ and $n^{-1}\log(f_n(\phi_n+e^{-\varepsilon n})/f_n(\phi_n))$ are bounded above by $o(1)$ as $n\to\infty$. Thus

$$(3.1)\quad n^{-1}\log[(f_n(\phi_n+e^{-\varepsilon n})+P(X_n\geqq\gamma_n+\phi_n))/f_n(\phi_n)]\leqq o(1)\qquad \text{as } n\to\infty\,.$$

However, $f_n(\phi_n)e^{-\varepsilon n}+f_n(\phi_n+e^{-\varepsilon n})\gamma_n+P(X_n\geqq\gamma_n+\phi_n)$ is an upper bound for $P(X_n>\phi_n)$, so (1.1) yields

$$\begin{aligned}&n^{-1}\log[f_n(\phi_n)e^{-\varepsilon n}+f_n(\phi_n+e^{-\varepsilon n})\gamma_n+P(X_n\geqq\gamma_n+\phi_n)]\\&\quad\geqq n^{-1}\log(f_n(\phi_n))+o(1)\end{aligned}$$

as $n\to\infty$. Now notice that $n^{-1}\log(f_n(\phi_n)e^{-\varepsilon n})=n^{-1}\log f_n(\phi_n)-\varepsilon$ and we have

$$\begin{aligned}&n^{-1}\log[f_n(\phi_n+e^{-\varepsilon n})\gamma_n+P(X_n\geqq\gamma_n+\phi_n)]\\&\quad\geqq n^{-1}\log f_n(\phi_n)+o(1)\qquad \text{as } n\to\infty\,.\end{aligned}$$

The fact that $n^{-1}\log\gamma_n=o(1)$ now yields

$$(3.2)\quad\begin{aligned}&n^{-1}\log[f_n(\phi_n+e^{-\varepsilon n})+P(X_n\geqq\gamma_n+\phi_n)]\\&\quad\geqq n^{-1}\log f_n(\phi_n)+o(1)\qquad \text{as } n\to\infty\,.\end{aligned}$$

Combining (3.1) and (3.2) gives (2.1) and necessity.

Suppose that (2.1) is true for all $\varepsilon>0$ and for some non-negative sequence $\{\gamma_n\}_{n=1}^{\infty}$ with $n^{-1}\log\gamma_n=o(1)$ as $n\to\infty$. Using an argument similar to the first part of the proof of Theorem 2.1 in Killeen, Hettmansperger, and Sievers [1], we get

$$(3.3)\quad n^{-1}\log f_n(\phi_n)-n^{-1}\log P(X_n>\phi_n)\geqq o(1)\qquad \text{as } n\to\infty\,.$$

In this argument we replace $P(X_n\geqq\gamma_n)$ by $P(X_n>\gamma_n+\phi_n)$. The fact that $(a+b)\leqq(ae^{-\varepsilon n}+b)e^{\varepsilon n}$ for non-negative numbers a and b and since $f_n(\phi_n+e^{-\varepsilon n})e^{-\varepsilon n}+P(X_n>\phi_n+\gamma_n)$ is a lower bound for $P(X_n>\phi_n)$ we get

$$(3.4)\quad\begin{aligned}&n^{-1}\log f_n(\phi_n)-n^{-1}\log(P(X_n>\phi_n))\\&\quad=n^{-1}\log[(f_n(\phi_n+e^{-\varepsilon n})+P(X_n\geqq\gamma_n+\phi_n))/P(X_n>\phi_n)]+o(1)\end{aligned}$$

$$\leqq n^{-1} \log [(f_n(\phi_n + e^{-\epsilon n})e^{-\epsilon n} + \mathrm{P}(X_n \geqq \gamma_n + \phi_n))/\mathrm{P}(X_n > \phi_n)] \\ + \varepsilon + o(1) \leqq \varepsilon + o(1) \qquad \text{as } n \to \infty .$$

Since ε was arbitrary, combining (3.3) and (3.4) completes the proof of sufficiency.

The proof of Theorem 2 follows in a similar manner.

4. Discussion

The proofs of Theorems 1 and 2 imply that if condition (2.1) or (2.2) is satisfied for some suitable sequence $\{\gamma_n\}_{n=1}^{\infty}$ then it is satisfied for all such sequences. To apply these theorems in practice, it is usually easiest to locate $\{\gamma_n\}$, say $\gamma_n = \exp(n^{1/2}) - \phi_n$, such that $\lim_{n\to\infty} n^{-1} \log (\mathrm{P}(X_n \geqq \gamma_n + \phi_n)/f_n(\phi)) = -\infty$ and then show that $\log(f_n(\phi_n + e^{-\epsilon n})/f_n(\phi_n)) = o(1)$ in the case of Theorem 1, or $\log(f_n(\phi_n + 1)/f_n(\phi_n)) = o(1)$ in the case of Theorem 2.

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REFERENCE

[1] Killeen, T., Hettmansperger, T. and Sievers, G. (1972). An elementary theorem on the probability of large deviations, *Ann. Math. Statist.*, **43**, 181-192.