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ON A LACK OF MEMORY PROPERTY OF THE EXPONENTIAL DISTRIBUTION

R. SHIMIZU

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Summary

Let X be a positive random variable with the distribution F and let G_0 be a monotone non-decreasing function such that $\mathrm{E}\{G_0(X)\}$ exists and is positive. Then under some additional conditions on F and G_0, $\mathrm{E}\{G_0(X-x)|X>x\}=\mathrm{E}\{G_0(X)\}$, $x\geqq 0$ implies that F is exponential.

1. Introduction

We consider a positive random variable X with the distribution F. Suppose y is a positive number such that $F(y)<1$. We say that the distribution F lacks memory (in the strong sense) at y if the condition

$$(1) \qquad \Pr\{X>x+y|X>y\}=\Pr\{X>x\}, \qquad x\geqq 0,$$

is satisfied. In terms of the distribution function $F(x)$, the distribution F lacks memory at y if and only if

$$(2) \qquad 1-F(x+y)=(1-F(x))(1-F(y)), \qquad x\geqq 0.$$

It is known that F satisfies (2) for two values of y, y_1 and y_2, say, such that y_1/y_2 is an irrational number if and only if F is the exponential distribution (Marsaglia-Tubilla [7]). Introducing the condition

$$(3) \qquad \Pr\{X>Y+x|X>Y\}=\Pr\{X>x\}, \qquad x\geqq 0.$$

where Y is a random variable independent of X, Ramachandran [8] generalized this characterization. He showed that if the distribution G_0 of Y is non-lattice and if $G_0(0)=0$, then (3) implies the exponentiality of X. His result, originally proved under the assumption of existence of the moment generating function of X, was also obtained by Huang [4], Shimizu [11], and Ramachandran [9] without the assumption. Huang pointed out that the condition $G_0(0)=0$ can be relaxed but only to the extent that $G_0(0)<\Pr\{X>Y\}$. Ramachandran's theorem is much more reaching than it appears. For one thing it contains some other characterization theorems such as "$|X_1-X_2|$ *has the same distribution as* X_1 *if and only if* X_1 *is exponentially distributed, where* X_1 *and* X_2 *are i.i.d. variables.*" And for another it is easy to generalize it further. In fact (3) is equivalent to (if $G_0(0)=0$)

$$(4)\qquad \int_0^\infty (1-F(x+y))dG_0(y)=(1-F(x))\int_0^\infty (1-F(y))dG_0(y)\,, \qquad x\geqq 0$$

which in turn implies

$$(5)\qquad \mathrm{E}\{G_0(X-x)\,|\,X>x\}=\mathrm{E}\{G_0(X)\}\,,$$

for all $x\geqq 0$, and it is clear that G_0 need not be a distribution function. We shall say that X lacks memory (in the weak sense) at x with respect to G_0 if $\mathrm{E}\{G_0(X)\}$ exists and the condition (5) is satisfied. Now, we can ask "to what extent can we relax the condition on G_0 in order that the fulfilment of (5) for all $x\geqq 0$ implies the exponentiality of X?" In the next section we shall give a fairly general sufficient condition. The result contains the ones obtained by Laurent ([6], $G_0(x)=x^2$), Azlarov-Dzamirzaev-Sultanova ([1], $G_0(x)=x^2$), Dallas ([2], $G_0(x)=x^2$), and Sahobov-Geshev ([10], $G_0(x)=x^k$, $k\geqq 2$).

A similar result was obtained by Klebanov [5] under somewhat different conditions using a different method.

Our result depends on the following theorem which can easily be obtained from Theorems 1 and 2 of Shimizu [11].

THEOREM 1. *If* $G(x)$ *is a distribution function on* $(0, \infty)$ *such that*

$$(6)\qquad \int_0^\infty e^{\delta x}dG(x)<\infty \qquad \text{for some } \delta>0$$

and if $H(x)$ *is a non-negative right-continuous function defined for* $x\geqq 0$ *and satisfies the condition*

$$(7)\qquad H(x+y)\leqq e^{\lambda y}H(x) \qquad x, y\geqq 0$$

where λ *is a positive constant, then the equation*

$$(8)\qquad H(x)=\int_0^\infty H(x+y)dG(y)$$

is satisfied if and only if $H(x)$ *is a periodic function with period* ρ *for every point* ρ *of increase of* G.

2. A characterization theorem

Let X be a positive random variable with distribution F, and let

$G_0(x)$ be a right continuous monotone non-decreasing function which satisfies the following conditions

(i) $G_0(-0)=G_0(+0)=0$

(ii) $\mu\equiv\mathrm{E}\{G_0(X)\}$ exists and is positive, and

(iii) there exists a positive number ξ such that

$$\mu<\int_0^\infty e^{-\xi x}dG_0(x)<\infty\ .$$

We shall prove

THEOREM 2. *Suppose the conditions stated above are satisfied. Let Ω be the set of all points of increase of the non-decreasing function G_0. If the equality*

$$\mathrm{E}\{G_0(X-x)\,|\,X>x\}=\mathrm{E}\{G_0(x)\}\ ,\qquad x\geqq 0 \tag{9}$$

holds, then F can be put in the form

$$F(x)=1-H(x)e^{-\lambda x}\ ,\qquad x\geqq 0 \tag{10}$$

where λ is a positive constant and $H(x)$ is a periodic function with period u for all $u\in\Omega$. In particular (9) implies the exponentiality of X if G_0 is not concentrated on the lattice points $0, \rho, 2\rho, 3\rho, \cdots$ for any $\rho>0$.

PROOF. We first show that

$$\mu(x)\equiv\int_x^\infty G_0(y-x)dF(y)=\int_0^\infty(1-F(x+y))dG_0(y)\leqq\mu(0)=\mu\ , \quad \text{for all } x\geqq 0\ . \tag{11}$$

In fact, if $A>x$, integration by parts gives

$$\begin{aligned}\int_x^A G_0(y-x)dF(y)&=\int_0^{A-x}G_0(y)d_yF(y+x)\\&=-(1-F(A))G_0(A-x)+\int_0^{A-x}(1-F(x+y))dG_0(y)\ .\end{aligned}$$

But as $\mu=\mu(0)=\int_0^\infty G_0(y)dF(y)$ exists we have

$$0\leqq(1-F(A))G_0(A-x)\leqq(1-F(A))G_0(A)\leqq\int_A^\infty G_0(y)dF(y)\to 0\ ,\quad \text{as } A\to\infty\ .$$

It follows that

$$\mu(x)=\lim_{A\to\infty}\int_x^A G_0(y-x)dF(y)=\int_0^\infty(1-F(x+y))dG_0(y)$$

as claimed. The inequality in (11) follows from the monotonicity of F.

As the conditional distribution function $F_x(y)$ of X given $X>x$ is

$$F_x(y)=\frac{F(y)-F(x)}{1-F(x)}\,, \qquad y>x\,,$$

we have in view of (11)

$$\begin{aligned}\mathrm{E}\,\{G_0(X-x)\,|\,X>x\}&=\int_x^\infty G_0(y-x)d_yF_x(y)\\&=(1-F(x))^{-1}\int_x^\infty G_0(y-x)dF(y)\\&=(1-F(x))^{-1}\int_0^\infty (1-F(x+y))dG_0(y)\,.\end{aligned}$$

It follows from the condition (9) of Theorem 2 that the equation

$$\int_0^\infty (1-F(x+y))dG_0(y)=\mu(1-F(x))\,, \qquad x\geqq 0 \tag{12}$$

holds.

On the other hand by the assumption (iii) we can find the unique positive number λ such that $\lambda>\xi$ and

$$\int_0^\infty e^{-\lambda x}dG_0(x)=\mu\,.$$

Let G be the distribution function defined by $dG(x)=\mu^{-1}e^{-\lambda x}dG_0(x)$. Introducing $H(x)\equiv(1-F(x))e^{\lambda x}$, (12) becomes

$$H(x)=\int_0^\infty H(x+y)dG(y)\,, \qquad x\geqq 0\,.$$

Note that the condition (6) is satisfied with $\delta=\lambda-\xi>0$ and that G and G_0 have a common set Ω of points of increase. The desired result follows from Theorem 1. Q.E.D.

Remark. The condition (iii) is satisfied if

$$\int_0^\infty e^{-\varepsilon x}dG_0(x)<\infty \qquad \text{for every } \varepsilon>0\,. \tag{13}$$

In fact, if we take A sufficiently large, we can make

$$\mu_1\equiv\int_0^A dG_0(y)>\mu=\int_0^\infty (1-F(x))dG_0(x)\,.$$

Let $\xi>0$ be so small that $\mu_1e^{-\xi A}>\mu$.

Then

$$\mu<\mu_1e^{-\xi A}\leqq e^{-\xi A}\int_0^A dG_0(x)\leqq\int_0^A e^{-\xi x}dG_0(x)\leqq\int_0^\infty e^{-\xi x}dG_0(x)<\infty\,.$$

In particular if $\mathrm{E}|X|^{\gamma}$ *exists for some positive* γ, *then*

$$\mathrm{E}\{|X-x|^{\gamma}|X>x\}=\mathrm{E}\{|X|^{\gamma}\}, \qquad x\geqq 0$$

implies the exponentiality of X.

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* These papers were not available to the author at the time of writing this paper (c.f. Galambos-Kotz [3]).