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ON BAHADUR'S REPRESENTATION OF SAMPLE QUANTILES

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Abstract

We extend the well known transformation technique for order statistics to get less restrictive conditions for the Bahadur representation of sample quantiles.

1. Introduction and summary

Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with common distribution function F. Fix $p \in (0, 1)$ and suppose $F(\zeta_p)=p$. Let $F_n(x)$ be the empirical distribution function based on $(X_1, \cdots, X_n)$; i.e. $nF_n(x)$=number of X_i less than or equal to x $(1 \leqq i \leqq n)$. Let $Y_{p;n}$ be the sample p-quantile based on $(X_1, \cdots, X_n)$; take e.g. $Y_{p;n}$ the $[np]$th order statistic among $(X_1, \cdots, X_n)$. Bahadur [2] proved that if F is twice differentiable in a neighbourhood of ζ_p, F'' is bounded there and $F'(\zeta_p)>0$, then

$$Y_{p;n}=\zeta_p+(G_n(\zeta_p)-q)/F'(\zeta_p)+R_n \tag{1.1}$$

where $R_n=O(n^{-3/4}\log n)$ a.s. as $n \to \infty$, with $G_n(x)=1-F_n(x)$ and $q=1-p$. An exact order of R_n has been given by Kiefer [8]. Sen [9] extended Bahadur's result to a sequence of dependent variables.

Instead of the exact p-quantile $Y_{p;n}$ one can take an approximate p-quantile $Y_{p_n;n}$ with p_n close to p. We will assume throughout that np_n is an integer. J. K. Ghosh [4] obtained the representation

$$Y_{p_n;n}=\zeta_{p_n}+(G_n(\zeta_p)-q)/F'(\zeta_p)+R_n$$

with the weaker result that $n^{1/2}R_n \to 0$ in probability as $n \to \infty$ for $n^{1/2}(p_n-p)$ bounded as $n \to \infty$ under the weaker assumption that $F'(\zeta_p)$ exists and is strictly positive. M. Ghosh and S. Sukathme [5] proved, under the assumption

$$\lim_{h \to 0} |F(\zeta_p+h)-F(\zeta_p)||h|^{-\rho}=M$$

for some $\rho>0$, $M>0$ and $n^{1/2}(p_n-p)$ bounded as $n\to\infty$ that

$$M|Y_{p_n;n}-\zeta_p|^\rho \operatorname{sgn}(Y_{p_n;n}-\zeta_p)=G_n(\zeta_p)-q+R_n \tag{1.2}$$

where $n^{1/2}R_n\to 0$ in probability as $n\to\infty$. Under the assumption

$$|F(\zeta_p+h)-F(\zeta_p)|=M|h|^\rho(1+O(|h|)) \qquad \text{as } h\to 0$$

and $n^{3/4}(\log n)^{-1}(p_n-p)$ bounded as $n\to\infty$ they also obtained an extension of (1.1) namely:

$$M|Y_{p_n;n}-\zeta_p|^\rho \operatorname{sgn}(Y_{p_n;n}-\zeta_p)=G_n(\zeta_p)-q+R_n \tag{1.3}$$

where $R_n=O(n^{-1/2-\gamma}\log n)$ a.s. as $n\to\infty$ and $\gamma=\min(1/4, 1/2\rho)$.

In Section 2 we will prove the following result.

THEOREM 1 (weak convergence result). *Suppose* $\lim_{n\to\infty} n^{1/2}(p_n-p)=0$ *and* $\rho>0$. *There exists a sequence of positive constants* $\{a_n\}$ *such that*

$$n^{-1/2}a_n^{-\rho}|Y_{p_n;n}-\zeta_p|^\rho \operatorname{sgn}(Y_{p_n;n}-\zeta_p)=G_n(\zeta_p)-q+R_n \tag{1.4}$$

with $n^{1/2}R_n\to 0$ *in probability as* $n\to\infty$ *if and only if*

$$\lim_{t\downarrow 0}\frac{F(\zeta_p+xt)-F(\zeta_p)}{F(\zeta_p+t)-F(\zeta_p)}=|x|^\rho \operatorname{sgn} x \qquad \textit{for all } x\neq 0\,. \tag{1.5}$$

Here $a_n\sim F^{-1}(p+n^{-1/2})-F^{-1}(p):=\inf\{x\,|\,F(\zeta_p+x)-F(\zeta_p)\geqq n^{-1/2}\}$ *as* $n\to\infty$.

COROLLARY. *This shows in particular that for J. K. Ghosh's representation the condition* $F'(\zeta_p)>0$ *is necessary as well.*

Note that Theorem 1 gives a necessary and sufficient condition for a more general version of (1.2).

In Section 3 we will prove

THEOREM 2. *Suppose* $(n/\log\log n)^{3/4}(p_n-p)$ *bounded as* $n\to\infty$. *If*

$$|F(\zeta_p+h)-F(\zeta_p)|=M|h|^\rho(1+O|h|^{\rho/2}) \qquad \textit{with } M>0,\ \rho>0 \tag{1.6}$$

then

$$M|Y_{p_n;n}-\zeta_p|^\rho \operatorname{sgn}(Y_{p_n;n}-\zeta_p)=G_n(\zeta_p)-q+R_n$$

with $(n/\log\log n)^{3/4}R_n$ *bounded with probability* 1 *as* $n\to\infty$.

Comparing this result with that of Ghosh and Sukathme, one sees that if $\rho\leqq 2$ our statement is slightly stronger, but our conditions are weaker. If $\rho>2$ both our result and our condition on F are stronger.

COROLLARY. *Suppose* F' *exists in a neighbourhood of* ζ_p *and*

$|y|^{-1/2}\{F'(\zeta_p+y)-F'(\zeta_p)\}$ is bounded, then the conditions of Theorem 2 are fulfilled with $\rho=1$. In particular this is the case if $|F''(\zeta_p)|<\infty$ (cf. Bahadur's condition requiring boundedness of F'' in a neighbourhood of ζ_p).

2. Proof of Theorem 1

We use the following lemma

LEMMA 1. *Let $\{V_n\}$ and $\{W_n\}$ be two sequences of random variables satisfying the following conditions:*

a) *For all $\delta>0$ there exists $\lambda=\lambda(\delta)$ such that $\mathrm{P}\{|W_n|>\lambda\}<\delta$ for all n.*

b) *For all k and $\varepsilon>0$*

i) $\lim_{n\to\infty} \mathrm{P}\{V_n\leqq k;\ W_n\geqq k+\varepsilon\}=0$

ii) $\lim_{n\to\infty} \mathrm{P}\{V_n\geqq k+\varepsilon;\ W_n\leqq k\}=0$.

Then $V_n-W_n\to 0$ in probability as $n\to\infty$.

For a proof of this lemma we refer to J. K. Ghosh [4].

Let $\{np_n\}$ be a sequence of positive integers such that $p_n=p+o(n^{-1/2})$ and let $Y_{p_n;n}$ be a sample p_n-quantile. For the proof of (1.4) we use the lemma and set

$$V_n=\left|\frac{Y_{p_n;n}-\zeta_p}{a_n}\right|^\rho \operatorname{sgn}\left(\frac{Y_{p_n;n}-\zeta_p}{a_n}\right)$$

$$t_n=\sqrt{n}\,\{F(\zeta_p+a_n|t|^{1/\rho}\operatorname{sgn} t)-F(\zeta_p)\}$$

$$Z_{t;n}=\sqrt{n}\,\{G_n(\zeta_p+a_n|t|^{1/\rho}\operatorname{sgn} t)-G(\zeta_p+a_n|t|^{1/\rho}\operatorname{sgn} t)\}$$

with $G(x)=1-F(x)$, $G_n(x)=1-F_n(x)$ and $W_n=\sqrt{n}\,\{G_n(\zeta_p)-G(\zeta_p)\}$. We have $n^{1/2}R_n=V_n-W_n$ and it is sufficient to show that V_n and W_n satisfy the conditions of Lemma 1.

As in Ghosh [4] we have

$$V_n\leqq t \Leftrightarrow Z_{t;n}\leqq t_n+\sqrt{n}\,(p_n-p) \tag{2.1}$$

and

$$t_n=\sqrt{n}\,\{F(\zeta_p+a_n|t|^{1/\rho}\operatorname{sgn} t)-F(\zeta_p)\}\sim\frac{F(\zeta_p+a_n|t|^{1/\rho}\operatorname{sgn} t)-F(\zeta_p)}{F(\zeta_p+a_n)-F(\zeta_p)}$$

as $n\to\infty$, since the function $F(\zeta_p+x)-F(\zeta_p)$ is regularly varying as $x\downarrow 0$. Hence

$$\lim_{n\to\infty}\{t_n+\sqrt{n}\,(p_n-p)\}=t\,. \tag{2.2}$$

Furthermore, since $n(Z_{t;n}-W_n)$ has a binomial distribution we have

$$\mathrm{E}\,(Z_{t;n}-W_n)^2=\rho_{t;n}(1-\rho_{t;n})$$

where $\rho_{t;n}=|F(\zeta_p+a_n|t|^{1/\rho}\operatorname{sgn} t)-F(\zeta_p)|\to 0$ as $n\to\infty$. So

$$Z_{t;n}-W_n\to 0 \quad \text{in probability as } n\to\infty\,. \tag{2.3}$$

We verify that the conditions of Lemma 1 are fulfilled. Condition a) is true by the asymptotic normality of W_n. For condition b,i) note that with $t=k$ for sufficiently large n (using (2.1) and (2.2)) $V_n\leqq t\Rightarrow Z_{t;n}\leqq t_n+\sqrt{n}(p_n-p)\Rightarrow Z_{t;n}\leqq t+\varepsilon/2$. Hence by (2.3) $\lim_{n\to\infty}\mathrm{P}\{V_n\leqq t;\ W_n\geqq t+\varepsilon\}=0$. Similarly one verifies b,ii). Hence (1.4) is true.

It is clear that for this part of the proof the boundedness of $\sqrt{n}(p_n-p)$ is sufficient.

Next we prove the converse i.e.: If for some $\rho>0$, some sequence $\{a_n\}$ and $\sqrt{n}(p_n-p)\to 0$ we have $n^{1/2}R_n\to 0$ in probability, then (1.5) holds. By the asymptotic normality of W_n and $n^{1/2}R_n=V_n-W_p\to 0$, $V_n=|(Y_{p_n;n}-\zeta_p)/a_n|^\rho\operatorname{sgn}((Y_{p_n;n}-\zeta_p)/a_n)$ asymptotically has a normal distribution. Hence the sequence $(Y_{p_n;n}-\zeta_p)/a_n$ converges weakly to the distribution of $U^{1/\rho}\operatorname{sgn} U$, where U has a $N(0,p(p-1))$ distribution. From Smirnov's [10] necessary and sufficient conditions for the convergence of the normalized $Y_{p_n;n}$ we then have (1.5).

3. Proof of Theorem 2

Theorem 2 has been formulated in this way because of the relative simplicity of its result and conditions. Before proving Theorem 2 we wish to prove a more general representation with less appealing conditions. Theorem 2 will turn out to be a special case.

THEOREM 2′. *Suppose $(n/\log\log n)^{3/4}(p_n-p)$ bounded as $n\to\infty$. If*

$$\left(\frac{n}{\log\log n}\right)^{3/4}\frac{1}{\sqrt{n}}\left[\frac{U(x(\sqrt{(\log\log n)/n}))}{U(1/\sqrt{n})}-x\sqrt{\log\log n}\right] \tag{3.1}$$

is bounded uniformly for all $|x|\leqq p(1-p)$ with

$$U(x):=|F^{-1}(p+x)-F^{-1}(p)|^\rho\operatorname{sgn} x\,,\qquad \rho>0\,;$$

then

$$\operatorname{sgn}(Y_{p_n;n}-F^{-1}(p))\cdot\frac{1}{\sqrt{n}}\left|\frac{Y_{p_n;n}-F^{-1}(p)}{a_n}\right|^\rho=p-F_n(\zeta_p)+R_n^0$$

with $a_n=F^{-1}(p+1/\sqrt{n})-F^{-1}(p)$ and $(n/\log\log n)^{3/4}R_n^0$ bounded with probability 1 as $n\to\infty$.

Let $Z_1, Z_2, \cdots$ be i.i.d. random variables with a uniform distribution on $[0, 1]$. Let $H_n(x)$ be the empirical distribution function based on $(Z_1, \cdots, Z_n)$ and let $V_{p;n}$ be the samlpe p-quantile corresponding to $(Z_1, \cdots, Z_n)$ and $0<p<1$. We define

$$r_n(p)=(V_{p;n}-p)+(H_n(p)-p) .$$

We know (Kiefer [8]) that

$$\overline{\lim_{n\to\infty}}\, r_n(p)[2^{5/4}\cdot 3^{-3/4}\cdot(p(1-p))^{1/4}\cdot n^{-3/4}(\log\log n)^{3/4}]^{-1}=1$$

with probability 1.

We now use the well-known transformation technique for order statistics (cf. e.g. de Haan [6]). Let F be a distribution function with $F(\zeta_p)=p$. Let $X_i:=F^{-1}(Z_i)$ for $i=1, 2, \cdots$ then $X_1, X_2, \cdots$ are i.i.d. random variables with distribution function F. Let $Y_{p;n}$ be the sample p-quantile from $(X_1, X_2, \cdots, X_n)$ then $Y_{p;n}=F^{-1}(V_{p;n})$. Furthermore

$$\begin{aligned} nF_n(\zeta_p) &= \{\text{number of } X_i \text{ less than or equal } \zeta_p \ (1\leqq i\leqq n)\} \\ &= \{\text{number of } Z_i \text{ less than or equal } p \ (1\leqq i\leqq n)\} \\ &= nH_n(p) . \end{aligned}$$

PROOF OF THEOREM 2′. Define $R_n^0:=\operatorname{sgn}(Y_{p_n;n}-F^{-1}(p))(1/\sqrt{n})|(Y_{p_n;n}-F^{-1}(p))/a_n|^{\rho}+F_n(\zeta_p)-p$. Then

$$R_n^0-r_n=\operatorname{sgn}(Y_{p_n;n}-F^{-1}(p))\cdot\frac{1}{\sqrt{n}}\left|\frac{Y_{p_n;n}-F^{-1}(p)}{a_n}\right|^{\rho}-(V_{p;n}-p)$$

with $(n/\log\log n)^{3/4}r_n$ bounded a.s. as $n\to\infty$.

For the proof of Theorem 2′ it is sufficient to prove that $(n/\log\log n)^{3/4}(R_n^0-r_n)$ is bounded a.s. as $n\to\infty$.

LEMMA 2. *If $p_n=p+O(n^{-3/4}(\log\log n)^{3/4})$ and $V_{p;n}$ a sample p-quantile based on $(Z_1, \cdots, Z_n)$ with $\{Z_i\}$ a sequence of independent uniformly distributed random variables, then*

$$\left[\frac{n}{\log\log n}\right]^{3/4}(V_{p_n;n}-V_{p;n}) \qquad \text{bounded a.s. as } n\to\infty .$$

The proof of this lemma can be found in the appendix, due to W. R. van Zwet.

From the result of this lemma we conclude that our representation holds if the following expression can be shown bounded with probability 1.

$$\left[\frac{n}{\log\log n}\right]^{3/4}\left[\operatorname{sgn}(Y_{p_n;n}-F^{-1}(p))\left|\frac{Y_{p_n;n}-F^{-1}(p)}{a_n}\right|^{\rho}\frac{1}{\sqrt{n}}\right. \tag{3.2}$$

$$-(V_{p_n;n}-p)\Big] .$$

Substituting $U(x)=|F^{-1}(p+x)-F^{-1}(p)|^\rho \operatorname{sgn} x$, we can write (3.2) as

$$\left[\frac{n}{\log\log n}\right]^{3/4}\left[\operatorname{sgn}(V_{p_n;n}-p)\cdot\frac{1}{\sqrt{n}}\left|\frac{F^{-1}(p+(V_{p_n;n}-p))-F^{-1}(p)}{F^{-1}(p+1/\sqrt{n})-F^{-1}(p)}\right|^\rho -(V_{p_n;n}-p)\right]$$

$$=\left[\frac{n}{\log\log n}\right]^{3/4}\cdot\frac{1}{\sqrt{n}}\left[\frac{U(V_{p_n;n}-p)}{U(1/\sqrt{n})}-\sqrt{n}\,(V_{p_n;n}-p)\right] . \tag{3.3}$$

According to the law of iterated logarithm (Bahadur [2]) and Lemma 2

$$\lim_{n\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\sqrt{n}\,(V_{p_n;n}-p)}{\sqrt{\log\log n}}=\begin{matrix}+p(1-p)\\ -p(1-p)\end{matrix}\qquad\text{with probability 1}.$$

Hence (3.3) is bounded with probability one if (3.1) is bounded uniformly for all $|x|\leqq p(1-p)$ as $n\to\infty$.

Remark. We point out a necessary condition for (3.1). With $x=v/\sqrt{\log\log n}$ and $v>0$ constant (3.1) implies $[n/\log\log n]^{3/4}\cdot v/\sqrt{n}\cdot [(U(v/\sqrt{n})/vU(1/\sqrt{n}))-1]$ bounded. From this it follows that $U(v/\sqrt{n})/vU(1/\sqrt{n})\to 1$ as $n\to\infty$ and hence $((U(v/\sqrt{n})/vU(1/\sqrt{n}))-1)\sim\log(U(v/\sqrt{n})/vU(1/\sqrt{n}))$. So with $\Psi(t)=\log(U(t)/t)$ (3.1) implies for all $v>0$ $n^{1/4}/(\log\log n)^{3/4}\{\Psi(v/\sqrt{n})-\Psi(1/\sqrt{n})\}$ bounded $(n\to\infty)$. According to Ash, Erdös and Rubel [1] this implies $\lim_{t\downarrow 0}(U(t)/t)$ exists and is finite and positive, i.e.

$$\lim_{t\downarrow 0}\frac{(F^{-1}(p+t)-F^{-1}(p))^\rho}{t}=\frac{1}{M} .$$

Hence a necessary condition for (3.1) is

$$\lim_{h\to 0}\frac{|F(\zeta_p+h)-F(\zeta_p)|}{|h|^\rho}=M . \tag{3.4}$$

Now we turn to the proof of the representation of Theorem 2.

PROOF OF THEOREM 2. We adapt the previous proof. From (1.6) it follows clearly

$$|t|^{-1/2}\left[\frac{|F^{-1}(p+t)-F^{-1}(p)|}{|t|}-\frac{1}{M}\right]\qquad\text{is bounded for } t\to 0 . \tag{3.5}$$

Defining

$$R_n:=M|Y_{p_n;n}-\zeta_p|^\rho\operatorname{sgn}(Y_{p_n;n}-\zeta_p)+F_n(\zeta_p)-p$$

we have

$$R_n - r_n = M|Y_{p_n;n} - \zeta_p|^\rho \operatorname{sgn}(Y_{p_n;n} - \zeta_p) - (V_{p;n} - p) .$$

It is sufficient to prove that $[n/\log\log n]^{3/4}(R_n - r_n)$ is bounded a.s. as $n \to \infty$.

Using Lemma 2 once again it is clear as in the proof of Theorem 2′ that it is sufficient to prove

$$\left[\frac{n}{\log\log n}\right]^{3/4} [MU(V_{p_n;n} - p) - (V_{p_n;n} - p)]$$

bounded with probability 1 as $n \to \infty$. In view of the law of iterated logarithm for uniform order statistics it is sufficient to prove

$$\left[\frac{n}{\log\log n}\right]^{3/4} \left[MU\left(x\sqrt{\frac{\log\log n}{n}}\right) - x\sqrt{\frac{\log\log n}{n}}\right]$$

bounded for all $|x| \leqq p(1-p)$. This follows from (3.5) by substituting $t = x\sqrt{(\log\log n)/n}$.

Remark. Under the condition of Theorem 2 also the condition of Theorem 2′ is valid. This can be seen if one substracts (3.5) with $t = 1/\sqrt{n}$ from (3.5) with $t = x\sqrt{(\log\log n)/n}$.

4. Appendix*

Let $U_{1:N} < U_{2:N} < \cdots < U_{N:N}$ be uniform order statistics.

LEMMA 1. *For* $N = 1, 2, \cdots$ *and* $k = 1, 2, \cdots, N$, $\mathrm{P}(U_{k:N} \geqq 2k/N) \leqq e^{-(k+2)/4}$.

PROOF. For $k \geqq N/2$ the lemma is trivial. For $k = 1$, $\mathrm{P}(U_{1:N} \geqq 2/N) = (1 - 2/N)^N \leqq e^{-2}$. Consider therefore the case $2 \leqq k < N/2$. Let S be the number of values in $[2k/N, 1]$ among $U_{1:N}, \cdots, U_{N:N}$. Clearly S has a binomial distribution with parameters N and $p = 1 - 2k/N$. By Bernstein's inequality (see Hoeffding [7])

$$\mathrm{P}\left(U_{k:N} \geqq \frac{2k}{N}\right) = \mathrm{P}(S \geqq N - k + 1) = \mathrm{P}\left(\frac{S}{N} - p \geqq \frac{k+1}{N}\right) \leqq e^{-\tau h(\lambda)} ,$$

where

$$\lambda = \frac{(1-p)((k+1)/N)}{p(1-p)} = \frac{k+1}{N-2k} , \qquad \tau = \frac{N((k+1)/N)}{1-p} = \frac{(k+1)N}{2k} ,$$

$$h(\lambda) = \frac{3\lambda}{2(3+\lambda)} ,$$

* Communicated by W. R. van Zwet.

so that

$$\tau h(\lambda)=\frac{3}{4}N\frac{(k+1)^2}{k(3N-5k+1)}\geqq\frac{1}{4}(k+2)\ .$$

Note that for $k=1$, Hoeffding's condition $t<b$ is violated.

For $N=1, 2,\cdots$, define $U_{0:N}=0$ a.s. and $U_{N+1:N}=1$ a.s.

LEMMA 2.
For $N=1, 2,\cdots$ *and* $1\leqq k<m\leqq N+1$,

$$\mathrm{P}\left(\frac{U_{m:N}-U_{k:N}}{U_{m:N}}\geqq\frac{2(m-k)}{m-1}\right)\leqq e^{-(m-k+2)/4}\ ;$$

for $N=1, 2,\cdots$ *and* $0\leqq k<m\leqq N$,

$$\mathrm{P}\left(\frac{U_{m:N}-U_{k:N}}{1-U_{k:N}}\geqq\frac{2(m-k)}{N-k}\right)\leqq e^{-(m-k+2)/4}\ .$$

PROOF. Immediate from Lemma 1 because the random variables involved are distributed as $U_{m-k:m-1}$ and $U_{m-k:N-k}$ respectively.

THEOREM. *Let* $\{k_N\}$ *and* $\{m_N\}$ *be sequences of integers with* $0\leqq k_N<m_N\leqq N+1$ *for* $N=1, 2,\cdots$. *Then, with probability* 1,

$$\limsup_N\frac{N}{(m_N-k_N)\vee 4\log N}(U_{m_N:N}-U_{k_N:N})\leqq 2\ .$$

Here $x\vee y$ *denotes the larger of* x *and* y; $x\wedge y$ *will denote the smaller of* x *and* y *and* $[x]$ *the integer part of* x.

PROOF. We split up the natural numbers into three disjoint subsets $A=\{N: m_N-k_N\geqq N/2\}$, $B=\{N: m_N-k_N<N/2, m_N\geqq N/2\}$ and $C=\{N: m_N-k_N<N/2, m_N<N/2\}$ and consider the lim sup over N in these subsets separately. Because $U_{m_N:N}-U_{k_N:N}\leqq 1$ a.s., the $\limsup_{N\in A}$ is certainly $\leqq 2$. Next we consider the set B. Choose $\varepsilon>0$ and define, for $N\in B$, $k'_N=k_N\wedge[m_N-(4+\varepsilon)\log N]$ so that $m_N-k'_N\geqq(4+\varepsilon)\log N$ but

$$1\leqq\limsup_{N\in B}\frac{m_N-k'_N}{(m_N-k_N)\vee 4\log N}\leqq 1+\frac{\varepsilon}{4}\ .$$

For sufficiently large $N\in B$, we have $1\leqq k'_N\leqq k_N$ because $m_N\leqq N/2$ and $k_N>m_N-N/2\geqq 0$. Hence

$$\limsup_{N\in B}\frac{N}{(m_N-k_N)\vee 4\log N}(U_{m_N:N}-U_{k_N:N})\leqq\left(1+\frac{\varepsilon}{4}\right)\tag{4.1}$$

$$\cdot \limsup_{N \in B} \frac{N}{m_N - k'_N} (U_{m_N:N} - - U_{k'_N:N}) .$$

Because $m_N - k'_N \geqq (4+\varepsilon) \log N$, the first part of Lemma 2 and the Borel-Cantelli lemma ensure that

$$\limsup_{N \in B} \frac{N}{m_N - k'_N} (U_{m_N:N} - U_{k'_N:N}) \cdot \frac{(m_N - 1)/N}{U_{m_N:N}} \leqq 2 \qquad \text{a.s.} \tag{4.2}$$

Since $m_N/N \geqq 1/2$, $(m_N - 1)(NU_{m_N:N})^{-1} \to 1$ a.s. by the Glivenko-Cantelli theorem and as $\varepsilon > 0$ is arbitrary, (4.1) and (4.2) imply that

$$\limsup_{N \in B} \frac{N}{(m_N - k_N) \vee 4 \log N} (U_{m_N:N} - U_{k_N:N}) \leqq 2 \qquad \text{a.s.}$$

Finally $\limsup_{N \in C}$ may be handled in a similar way with the aid of the second part of Lemma 2, or even easier by a simple symmetry argument replacing $(U_{m_N:N} - U_{k_N:N})$ by $(U_{N-k_N:N} - U_{N-m_N:N})$. The proof is complete.

Note that we have been overly careful in the proof of the theorem not to increase the constant 4 in the denominator which originates from the constant 1/4 in Lemma 1. Further improvements would have to come from sharpening Lemma 1. We note that the proof will still work for triangular arrays $U_{k:N}^{(N)}$ because we use the joint behavior for different N only when applying Glivenko-Cantelli and that remains valid for arrays.

An alternative way of proving the theorem is by using results of M. Csörgö and P. Révész [3] on the approximation of the quantile process by Brownian bridges.

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