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MATHEMATICAL EXPRESSION OF AN INEQUALITY FOR A BLOCK DESIGN

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1. Introduction

An n-ary block design is an arrangement of v treatments in b blocks of the jth block size k_j $(j=1, 2, \cdots, b)$ such that the ith treatment occurs r_i times $(i=1, 2, \cdots, v)$ and the ith treatment occurs in the jth block n_{ij} times, where n_{ij} can take any of the values, $0, 1, \cdots$, or $n-1$. When $n=2$ $(r_1=r_2=\cdots=r_v)$, the design is said to be binary (equireplicated). Otherwise, it is called nonbinary (unequal-replicated).

There are some discussions about the expression of lower bounds on the number of blocks for a block design. However, almost all of them are given for equireplicated block designs (cf. Kageyama and Tsuji [4], [5]). Some of them are presented for unequal-replicated block designs with special structure like a variance-balanced design or a partially variance-balanced design (cf. Kageyama [2]). Little attention has been given to discussions in the form of including unequal-replicated block designs. In this paper, we aim at a systematic management of bounds on the number of blocks for an n-ary block design by use of one idea. A mathematical expression of a bound on the number of blocks is successfully presented for an n-ary block design which is of the most general type. The approach is mainly based on properties of the C-matrix which plays an important role in a block design. The bound derived here will lead to a number of inequalities known for various block designs of binary and nonbinary cases.

Since a design uniquely determines its incidence matrix and vice versa, both a design and its incidence matrix are denoted by the same symbol throughout this paper. The designs considered here are assumed to be connected (cf. Bose [1]). For convenience, the following notations are used: I_s is the identity matrix of order s. $E_{s\times t}$ $(O_{s\times t})$ is an $s\times t$ matrix all of whose elements are unity (zero). As a special case, $E_{s\times s}$ is denoted by G_s. A' is the transpose of the matrix A. $D_r=\operatorname{diag}\{r_1, r_2, \cdots, r_v\}$ which is a $v\times v$ diagonal matrix with diagonal elements $r_1, r_2, \cdots, r_v$. $D_k=\operatorname{diag}\{k_1, k_2, \cdots, k_b\}$. $D_{\sqrt{k}}=\operatorname{diag}\{\sqrt{k_1}, \sqrt{k_2}, \cdots, \sqrt{k_b}\}$.

$D_{\sqrt{r}}=\text{diag}\{\sqrt{r_1}, \sqrt{r_2}, \cdots, \sqrt{r_v}\}$.

2. Bound on the number of blocks

We consider an n-ary connected block design N with parameters v, b, r_i and k_j ($i=1,2,\cdots,v$; $j=1,2,\cdots,b$) whose C-matrix is given by

$$C=D_r-ND_k^{-1}N',$$

in which case rank$(C)=v-1$, since the design is connected. Furthermore, it is known (cf. Yamamoto and Fujikoshi [7]) that the minimum eigenvalue, 0, of $D_{\sqrt{r}}^{-1}CD_{\sqrt{r}}^{-1}$ is simple and other eigenvalues, θ_l, say, satisfy $0<\theta_l\leqq 1$. Thus, as a spectral expansion of $D_{\sqrt{r}}^{-1}CD_{\sqrt{r}}^{-1}$ we can put

$$(2.1)\qquad D_{\sqrt{r}}^{-1}CD_{\sqrt{r}}^{-1}=I_v-D_{\sqrt{r}}^{-1}ND^{-1}N'D_{\sqrt{r}}^{-1}=\sum_{l=1}^{q}\theta_l P_l+0P_0,$$

where P_l and P_0 are the projections corresponding to non-zero distinct eigenvalues θ_l ($0<\theta_l\leqq 1$) and 0 (zero), respectively, and $q\leqq v-1$ and $\sum_{l=1}^{q}\text{rank}(P_l)=v-1$.

Remark. (i) When the design N is equireplicated, we can choose P_0 as $(1/v)G_v$, since $D_{\sqrt{r}}^{-1}CD_{\sqrt{r}}^{-1}$ and G_v are orthogonal to each other. (ii) The non-zero eigenvalues of $D_{\sqrt{r}}^{-1}CD_{\sqrt{r}}^{-1}$ and of CD_r^{-1} are identical with the same multiplicities.

Now let β be the multiplicity of the maximum eigenvalue 1 of CD_r^{-1}; $\beta=0$ if it does not exist (i.e., $\theta_l<1$ for all l). From (2.1) we have

$$(2.2)\qquad D_{\sqrt{r}}^{-1}ND_k^{-1}N'D_{\sqrt{r}}^{-1}=P_0+\sum_{l=1}^{q}(1-\theta_l)P_l$$

which implies that

$$v-\beta=\text{rank}\,(D_{\sqrt{r}}^{-1}ND_k^{-1}N'D_{\sqrt{r}}^{-1})=\text{rank}\,(ND_k^{-1}N')=\text{rank}\,(N)\leqq b,$$

i.e., an inequality $b\geqq v-\beta$ holds. Furthermore, since $D_{\sqrt{r}}^{-1}ND_k^{-1}N'D_{\sqrt{r}}^{-1}=(D_{\sqrt{r}}^{-1}ND_{\sqrt{k}}^{-1})(D_{\sqrt{r}}^{-1}ND_{\sqrt{k}}^{-1})'$, we get, from (2.2), a spectral expansion of $D_{\sqrt{k}}^{-1}\cdot N'D_r^{-1}ND_{\sqrt{k}}^{-1}$ as

$$D_{\sqrt{k}}^{-1}N'D_r^{-1}ND_{\sqrt{k}}^{-1}=(D_{\sqrt{k}}^{-1}N'D_{\sqrt{r}}^{-1})P_0(D_{\sqrt{r}}^{-1}ND_{\sqrt{k}}^{-1})$$
$$+\sum_m(1-\theta_m)\left\{\frac{1}{1-\theta_m}(D_{\sqrt{k}}^{-1}N'D_{\sqrt{r}}^{-1})P_m\right.$$
$$\left.\cdot(D_{\sqrt{r}}^{-1}ND_{\sqrt{k}}^{-1})\right\}+0Q_0,$$

where Q_0 is the projection corresponding to 0 (zero), the summation $\sum_m$ extends over all the integers m satisfying $1-\theta_m>0$ for $m=1, 2, \cdots, q$ ($\leqq v-1$), and rank $(Q_0)=b-(v-\beta)$. Furthermore, it follows that $b=v-\beta$ if and only if $Q_0=O_{b\times b}$, in which case,

$$D_{\sqrt{k}}^{-1}N'D_{\sqrt{r}}^{-1}P_0D_{\sqrt{r}}^{-1}ND_{\sqrt{k}}^{-1}+\sum_m \frac{1}{1-\theta_m}(D_{\sqrt{k}}^{-1}N'D_{\sqrt{r}}^{-1})P_m(D_{\sqrt{r}}^{-1}ND_{\sqrt{k}}^{-1})=I_b$$

or

$$N'D_{\sqrt{r}}^{-1}P_0D_{\sqrt{r}}^{-1}N+\sum_m \frac{1}{1-\theta_m}N'D_{\sqrt{r}}^{-1}P_mD_{\sqrt{r}}^{-1}N=D_k .$$

Thus, we can establish the following.

THEOREM A. *For an n-ary block design N with parameters v, b,* r_i, k_j *(*$i=1, 2, \cdots, v$; $j=1, 2, \cdots, b$*) in which* $C=D_r-ND_k^{-1}N'$, *the following inequality holds*:

$$b\geqq v-\beta ,$$

where β *is the multiplicity of the maximum eigenvalue,* 1, *of the matrix* $D_{\sqrt{r}}^{-1}CD_{\sqrt{r}}^{-1}$. *Especially, the equality sign holds if and only if the projection corresponding to zero eigenvalue of* $D_{\sqrt{r}}^{-1}CD_{\sqrt{r}}^{-1}$ *is a zero matrix. In this case,*

$$D_k=N'D_{\sqrt{r}}^{-1}P_0D_{\sqrt{r}}^{-1}N+\sum_m \frac{1}{1-\theta_m}N'D_{\sqrt{r}}^{-1}P_mD_{\sqrt{r}}^{-1}N ,$$

where the summation extends over all the integers m satisfying $1-\theta_m>0$ *for* $m=1, 2, \cdots, q$ ($\leqq v-1$).

Furthermore, letting $\beta=0$ in Theorem A, we obtain the following.

COROLLARY A. *For an n-ary block design with parameters v, b,* r_i *and* k_j ($i=1, 2, \cdots, v$; $j=1, 2, \cdots, b$) *having* θ_l ($l=1, 2, \cdots, q$) *as nonzero eigenvalues of* $D_{\sqrt{r}}^{-1}CD_{\sqrt{r}}^{-1}$, *if* $\theta_l<1$ *for all* $l=1, 2, \cdots, q$ ($\leqq v-1$), *then an inequality* $b\geqq v$ *holds.*

The theory developed here is the most general regarding Fisher's inequality, $b\geqq v$, first known for a balanced incomplete block (BIB) design with parameters v, b, r, k and λ, in the sense of including bounds on the number of blocks for unequal-replicated block designs. Of course, Theorem A and Corollary A include a number of the results known for various block designs. Thus, the mathematical expression of an inequality derived here appears to be the best for a wide class of block designs.

3. Derivations of known main results

As far as the author knows except for block designs with special structure, all the known results regarding bounds on the number of blocks have been given only for equireplicated block designs. We shall consider various special cases of the results in the preceding section. These cases are important especially when $n=2$ (i.e., a design is binary). Two cases of incomplete block designs are considered here.

Case I. Equireplicated n-ary block designs (i.e., $r_1=r_2=\cdots=r_v=r$, say).

As mentioned in the remark, we can get $P_0=(1/v)G_v$. Furthermore, from (2.1) the eigenvalue θ_l can be replaced by ρ_l/r for $l=1, 2, \cdots, q$ $(\leqq v-1)$, where ρ_l's are non-zero distinct eigenvalues of the matrix C $(=rI_v-ND_k^{-1}N')$. Hence, β is equal to the multiplicity ($=\alpha$, say) of the maximum eigenvalue, r, of the matrix C. In this case, Theorem A yields the following main results of Kageyama and Tsuji [5].

COROLLARY 3.1. *For an equireplicated n-ary block design N with parameters v, b, r and k_j $(j=1, 2, \cdots, b)$ in which $C=rI_v-ND_k^{-1}N'=\sum_{i=1}^{q}\rho_i P_i$, the following inequality holds:*

$$b \geqq v-\alpha ,$$

where P_i's are projections corresponding to the eigenvalues ρ_i's of C and α is the multiplicity of the maximum eigenvalue r of C. In particular, the equality sign holds if and only if the projection corresponding to zero eigenvalue of $D_{\sqrt{k}}^{-1}N'ND_{\sqrt{k}}^{-1}$ is a zero matrix. In this case,

$$D_k=\frac{1}{vr}N'G_vN+\sum_m \frac{1}{r-\rho_m}N'P_mN ,$$

where the summation extends over all the integers m satisfying $r-\rho_m>0$ for $m=1, 2, \cdots, q$ $(\leqq v-1)$.

On the other hand, Corollary A yields the following.

COROLLARY 3.2. *For an equireplicated n-ary block design with parameters v, b, r and k_j $(j=1, 2, \cdots, b)$ having ρ_l $(l=1, 2, \cdots, q)$ as non-zero eigenvalues of C, if $\rho_l<r$ for all $l=1, 2, \cdots, q$ $(\leqq v-1)$, then an inequality $b \geqq v$ holds.*

For further discussions relating to the bounds on the number of blocks of equireplicated n-ary block designs including μ-resolvable designs, we refer to Kageyama and Tsuji [5].

Case II. Equireplicated and equiblock-sized n-ary block designs (i.e., $r_1=r_2=\cdots=r_v=r$, say, and $k_1=k_2=\cdots=k_b=k$, say).

We consider an n-ary block design with parameters v, b, r and k in which $C=rI_v-(1/k)NN'$. Since it is clear that the multiplicity of the eigenvalue r of C coincides with the multiplicity of zero eigenvalue of NN', Theorem A yields, from Case I, a generalization of Theorem 1.1 of Kageyama and Tsuji [4] as follows.

COROLLARY 3.3. *For an n-ary block design N with parameters v, b, r and k, an inequality $b\geqq v-\alpha$ holds, where α is the multiplicity of zero eigenvalue of NN'. In particular, the equality sign holds if and only if the projection corresponding to zero eigenvalue of $N'N$ is a zero matrix. In this case,*

$$I_b=\frac{1}{b}G_b+\sum_m \frac{1}{k(r-\rho_m)}N'P_mN\,,$$

where the summation extends over all the integers m satisfying $r-\rho_m>0$ for $m=1,2,\cdots,q$ $(\leqq v-1)$.

Note that if all the eigenvalues of NN' for a block design N are non-zero, then Fisher's inequality $b\geqq v$ holds. Sufficient conditions for the validity of Fisher's inequality for various block designs are recently treated in Kageyama [3] and Kageyama and Tsuji [6].

Note that Case II includes well-known BIB designs and partially balanced incomplete block designs. Further discussions can be referred to Kageyama and Tsuji [4] in detail.

The approach adopted here is based on the spectral expansion of $D^{-1}_{\sqrt{r}}CD^{-1}_{\sqrt{r}}$. This consideration succeeds the general derivation of an inequality and a condition of attaining the bound, *even for* unequal-replicated block designs. The author believes that the expression of Theorem A includes all the known results of bounds on the number of blocks for a block design.

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