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SUBSET SELECTION PROCEDURES FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS*

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Abstract

In this paper we are interested in studying multiple decision procedures for k ($k \geq 2$) populations which are themselves unknown but which one assumed to belong to a restricted family. We propose to study a selection procedure for distributions associated with these populations which are convex-ordered with respect to a specified distribution G assuming that there exists a best one. The procedure described here is based on a statistic which is a linear function of the first r order statistics and which reduces to the total life statistics when G is exponential. The infimum of the probability of a correct selection and an asymptotic expression for this probability are obtained using the subset selection approach. Some other properties of this procedure are discussed. Asymptotic relative efficiencies of this rule with respect to some selection procedures proposed by Barlow and Gupta [3] for the star-ordered distributions and by Gupta [8] for the gamma populations with known shape parameters are obtained. A selection procedure for selecting the best population using the indifference zone approach is also studied.

1. Introduction

In many problems, especially those in reliability theory, one is interested in using a model for life length distribution which is not completely specified but belongs, for example, to a family of distributions having increased failure rate (IFR), or increasing failure rate on the average (IFRA). Such distributions form special cases of what are now commonly known as restricted families of probability distributions. The idea of using such families stems from the fact that in many cases the

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experimenter cannot specify the model (distribution) exactly but is able to say whether it comes from a family of distributions such as IFR, IFRA. Families of probability distributions of these types have been studied by several authors, see, for example, Barlow, Marshall and Proschan [4], Barlow and Proschan [5], [6] and Barlow and Doksum [1].

In this paper we are interested in studying multiple decision procedures for k ($k \geqq 2$) populations which are themselves unknown but which are assumed to belong to a restricted family. We now give some definitions of interest to us (see Barlow and Gupta [3]).

(i) F is said to be convex with respect to G (written $F \underset{c}{<} G$) if and only if $G^{-1}F(x)$ is convex on the support of F.

(ii) F is said to be star-shaped with respect to G (written $F \underset{*}{<} G$) if and only if $F(0)=G(0)=0$ and $G^{-1}F(x)/x$ is increasing in $x \geqq 0$ on the support of F.

If $G(x)=1-e^{-x}$, $x \geqq 0$, then $F \underset{c}{<} G$ is equivalent to saying that F has increasing failure rate (IFR). Again if $G(x)=1-e^{-x}$, $x \geqq 0$, $F \underset{*}{<} G$ is equivalent to saying that F has increasing failure rate on average (IFRA).

In the statistical literature, selection problems for restricted families were first investigated by Barlow and Gupta [3]. Some further results in this direction and a review of some important results concerning inequalities for restricted families and problems of inference for such families have been given by Gupta and Panchapakesan [10], [11] and Patel [16].

In Section 2, we propose and study a subset selection rule for distributions which are $\underset{c}{<}$ ordered with respect to a specified distribution G assuming there exists a best one. Some properties of this rule are discussed. The infimum of the probability of a correct selection is obtained and an asymptotic expression is also given. We also study the asymptotic relative efficiencies of this rule with respect to some selection procedures. Section 3 deals with selecting the best population using the indifference zone approach.

2. Selection rules for distributions $\underset{c}{<}$ ordered with respect to a specified distribution G

Before discussing the selection problem, we give some preliminary known results for sake of completeness. Let $\mathscr{F}$ be the class of absolutely continuous distribution functions F on R such that $F(0)=0$ with positive and right-(or left-)continuous density f on the interval where $0<F<1$. For $F\in\mathcal{F}$, we take $F^{-1}(1)$ to be equal to the right-hand endpoint of the support of F and we define $F^{-1}(0)=0$. For F, $G\in\mathcal{F}$, consider the following transformation (see Barlow and Doksum [1])

$$H_F^{-1}(t)=\int_0^{F^{-1}(t)} g[G^{-1}F(u)]du\ , \qquad 0\leqq t\leqq 1\ , \tag{2.1}$$

where g denotes the density of G. We assume that G is always fixed. Since H_F^{-1} (the inverse of H_F) is strictly increasing on $[0,1]$, H_F is a distribution. Barlow and Doksum [1] have shown that $F\underset{c}{<}G$ if and only if H_F is convex on the interval where $0<H_F<1$. Since G is assumed known we can estimate H_F^{-1} by substituting the empirical distribution F_n of F; that is

$$H_n^{-1}(t)=H_{F_n}^{-1}(t)=\int_0^{F_n^{-1}(t)} g[G^{-1}F_n(u)]du \tag{2.2}$$

$$H_n^{-1}\left(\frac{r}{n}\right)=\int_0^{X_{r,n}} g[G^{-1}F_n(u)]du=\sum_{i=1}^{r} g\left[G^{-1}\left(\frac{i-1}{n}\right)\right](X_{i,n}-X_{i-1,n}) \tag{2.3}$$

where $X_{i,n}$ is the ith order statistic in a sample of size n from F and $X_{0,n}\equiv 0$.

If $G(x)=1-e^{-x}$ for $x\geqq 0$, then (2.3) can be written as

$$H_n^{-1}\left(\frac{r}{n}\right)=\frac{1}{n}[X_{1,n}+\cdots+X_{r-1,n}+(n-r+1)X_{r,n}]\ . \tag{2.4}$$

We say that $X_{1,n}+\cdots+X_{r-1,n}+(n-r+1)X_{r,n}$ is the total life statistic until rth failure from F.

(A) *Seletion procedure and its properties*

Let $\pi_1,\cdots,\pi_k$ be k populations. The random variable X_i associated with π_i has distribution function F_i, $i=1,2,\cdots,k$, where $F_i\in\mathcal{F}$. Let $F_{[k]}$ denote the cumulative distribution function (c.d.f.) of the "best" population. We assume that (a) $F_i(x)\geqq F_{[k]}(x)$ for all x, $i=1,\cdots,k$ and (b) there exists a distribution G such that $F_i\lesssim G$, $i=1,\cdots,k$, where $\lesssim$ denotes a partial ordering relation on the space of probability distributions. We are given a sample of size n from each π_i $(i=1,\cdots,k)$. Our goal is to select a subset from the k populations so as to include the population with $F_{[k]}$. Let $\Omega=\{\underline{F}=(F_1,\cdots,F_k)\colon \exists j$ such that $F_i(x)\geqq F_j(x)$ for all x and $i=1,2,\cdots,k\}$. Let

$$T_i=\sum_{j=1}^{r} a_j X_{i;j,n} \qquad \text{for } i=1,\cdots,k \tag{2.5}$$

and

(2.6) $$T=\sum_{j=1}^{r} a_j Y_{j,n}$$

where $X_{i;j,n}$ is the jth order statistic from F_i, $Y_{j,n}$ is the jth order statistic from G, r is a fixed positive integer $(1 \leqq r \leqq n)$, $a_j = gG^{-1}((j-1)/n) - gG^{-1}(j/n)$ for $j=1, \cdots, r-1$ and $a_r = gG^{-1}((r-1)/n)$.

For selecting a subset containing $F_{[k]}$, we propose the selection rule R_1 as follows:

R_1: Select population π_i if and only if

(2.7) $$T_i \geqq c_1 \max_{1 \leqq j \leqq k} T_j$$

where $c_1 = c_1(k, P^*, n, r)$ is the largest number between 0 and 1 which is determined as to satisfy the probability requirement

(2.8) $$\inf_{\Omega} \mathrm{P}[CS \mid R_1] \geqq P^*$$

where CS stands for a correct selection, i.e., the selection of any subset which contains the population with distribution $F_{[k]}$. For a given α $(0<\alpha<1)$, we assume each F_i has a unique α-quantile. Let $F_{[i]}(x) = F_{[i]}$ denote the cumulative distribution function of the population with ith smallest α-quantile. Let $T_{(i)}$ be associated with $F_{[i]}$ and let $W_i(x)$ be the c.d.f. of $T_{(i)}$.

LEMMA 2.1. *Let F_1, F_2 be two distribution functions such that $F_1(x) \geqq F_2(x)$ $\forall x$ and $T_i = \sum_{j \in \Delta} b_j X_{i;j,n}$ $i=1, 2$, where $b_j > 0$ for $j \in \Delta$, $\Delta \subset \{1, 2, \cdots, n\}$ and $X_{i;j,n}$ is the jth order statistic from F_i, $i=1, 2$, then* $\mathrm{P}[T_1 \leqq x] \geqq \mathrm{P}[T_2 \leqq x]$.

PROOF. Let

$$\psi(X_{i1}, \cdots, X_{in}) = \begin{cases} 1 & \text{if } T_i \geqq x \\ 0 & \text{otherwise} \end{cases}$$

where $X_{i1}, \cdots, X_{in}$ are n observations from F_i $(i=1, 2)$. Since $\psi(X_{i1}, \cdots, X_{in})$ is nondecreasing in each of its arguments, it follows by induction (Lehmann [15] p. 112) that $\mathrm{E}\,\psi(X_{11}, \cdots, X_{1n}) \leqq \mathrm{E}\,\psi(X_{21}, \cdots, X_{2n})$. That is $\mathrm{P}[T_1 \geqq x] \leqq \mathrm{P}[T_2 \geqq x]$. This proves the lemma.

We now state and prove the following theorem which is more general than that of Patel [16].

THEOREM 2.1. *If F_i, $G \in \mathcal{F}$, $F_i(x) \geqq F_{[k]}(x)$ $\forall x$ and $i=1, 2, \cdots, k$, $F_{[k]} \underset{c}{<} G$, $a_j \geqq 0$ for $j=1, 2, \cdots, r$, $g(0) \leqq 1$ and $a_r \geqq c_1$, then*

(2.9) $$\inf_{\Omega} \mathrm{P}[CS \mid R_1] = \int_0^{\infty} G_T^{k-1}\left(\frac{x}{c_1}\right) dG_T(x)$$

where $G_T(x)$ is the c.d.f. of T.

PROOF.

$$\begin{aligned}
P[CS|R_1] &= P[T_{(k)} \geq c_1 T_{(i)},\ i=1,\cdots,k-1] \\
&= \int_0^\infty \prod_{i=1}^{k-1} W_i\left(\frac{x}{c_1}\right) dW_k(x) \\
&\geq \int_0^\infty W_k^{k-1}\left(\frac{x}{c_1}\right) dW_k(x) \qquad \text{(By Lemma 2.1)} \\
&= P[Z_k \geq c_1 Z_j,\ j=1,\cdots,k-1]
\end{aligned}$$

where $Z_1,\cdots,Z_k$ are i.i.d. with c.d.f. $W_k(x)$. Let $\phi(x)=G^{-1}F_{[k]}(x)$. Note that $\phi(x)$ is nondecreasing in x. Also we can write

$$Z_i \underset{\text{st}}{=} \sum_{j=1}^{r} a_j X^*_{i;j,n} \qquad i=1,\cdots,k\,, \tag{2.10}$$

where $X^*_{i;j,n}$ is the jth order statistic in a sample of size n from $F_{[k]}$, $i=1,\cdots,k$.

$$P[Z_k \geq c_1 \max_{1\leq j\leq k} Z_j] = P\left[\phi\left(\frac{1}{c_1}Z_k\right) \geq \phi(Z_i),\ i=1,\cdots,k-1\right]\,. \tag{2.11}$$

Since $\sum_{j=1}^{r} a_j = g(0) \leq 1$, $a_j \geq 0\ \forall j=1,\cdots,r$, and $\phi(0)=0$, by Lemma 4.1 of Barlow and Proschan [5] and (2.10), then

$$\phi(Z_i) \underset{\text{st}}{\leq} \sum_{j=1}^{r} a_j \phi(X^*_{i;j,n})\,. \tag{2.12}$$

Since $(1/c_1)a_r \geq 1$, $(1/c_1)\sum_{j=1}^{r} a_j \geq 1$ for $i=1,\cdots,r$, and $\phi(0)=0$, by Lemma 4.3 of Barlow and Proschan [5] and (2.10), we have

$$\phi\left(\frac{1}{c_1}Z_k\right) \underset{\text{st}}{\geq} \frac{1}{c_1}\sum_{j=1}^{r} a_j \phi(X^*_{k;j,n})\,. \tag{2.13}$$

$$\phi(X^*_{i;j,n}) \underset{\text{st}}{=} Y_{i;j,n} \tag{2.14}$$

where $Y_{i;j,n}$ is the jth order statistic from G, $i=1,2,\cdots,k$. Thus from (2.11), (2.12), (2.13), and (2.14),

$$\begin{aligned}
P[Z_k \geq c_1 \max_{1\leq i\leq k} Z_i] &\geq P\left[\sum_{j=1}^{r} a_j Y_{k;j,n} \geq c_1 \sum_{j=1}^{r} a_j Y_{i;j,n},\ i=1,\cdots,k-1\right] \\
&= \int_0^\infty G_T^{k-1}\left(\frac{x}{c_1}\right) dG_T(x)\,.
\end{aligned}$$

This completes the proof.

The constant $c_1=c_1(k, P^*, n, r)$ satisfying (2.8) is the largest number between 0 and 1 determined by

$$\int_0^\infty G_T^{k-1}\left(\frac{x}{c_1}\right)dG_T(x)\geqq P^* \quad \text{and} \quad gG^{-1}\left(\frac{r-1}{n}\right)\geqq c_1 .$$

We now consider two specific distributions $G(x)$. If $G(x)=1-e^{-x}$, $x\geqq 0$, then we have following result which slightly generalizes the result of Patel [16].

COROLLARY 2.1. *If* $F_i(x)\geqq F_{[k]}(x)$ $\forall x$ *and* $i=1,\cdots,k$, $F_{[k]}\underset{c}{<}G$, $G(x)=1-e^{-x}$, $x>0$ *and* $n\geqq\max\{r, (r-1)/(1-c_1)\}$, *then*

$$\inf_{\Omega} \mathrm{P}\,[CS|R_1]=\int_0^\infty H^{k-1}\left(\frac{x}{c_1}\right)dH(x) \tag{2.15}$$

where $H(x)$ *is the c.d.f of a* χ^2 *random variable with* $2r$ *d.f.*

PROOF. If $G(x)=1-e^{-x}$ then $a_j=1/n$ for $j=1,2,\cdots,r-1$ and $a_r=(1/n)(n-r+1)$. Also $(1/c_1)a_r\geqq 1$ iff $n\geqq(r-1)/(1-c_1)$. By Theorem 2.1 and the fact that $2nT$ is distributed as χ^2 with $2r$ d.f., the result follows.

If $G(x)=x$ for $0<x<1$, then we have the following result which is a special case of Theorem 2.1 of Barlow and Gupta [3].

COROLLARY 2.2. *If* $F_i(x)\geqq F_{[k]}(x)$ $\forall x$ *and* $i=1,\cdots,k$, $F_{[k]}\underset{c}{<}G$ *and* $G(x)=x$ *for* $0<x<1$, *then*

$$\inf_{\Omega} \mathrm{P}\,[CS|R_1]=n\binom{n-1}{r-1}\int_0^\infty\left[\sum_{i=r}^{n}\binom{n}{i}\left(\frac{x}{c_1}\right)^i\left(1-\frac{x}{c_1}\right)^{n-i}\right]^{k-1} \cdot x^{r-1}(1-x)^{n-r}dx . \tag{2.16}$$

Actually, the condition $F_{[k]}\underset{c}{<}G$ in Corollary 2.2 can be relaxed to $F_{[k]}\underset{*}{<}G$.

We state and prove the following theorem about the asymptotic evaluations of the probability of a correct selection associated with the rule R_1 in the case where r is so chosen that $r\leqq(n+1)\alpha<r+1$, $0<\alpha<1$. This amounts to selecting populations with large values of the α-quantile for α (and r) as defined above. In this case, $r/n\to\alpha$ as $n\to\infty$. Note that the result holds for all α.

THEOREM 2.2. *If* F_i, $G\in\mathcal{F}$ *for all* $i=1,\cdots,k$ *and*

(i) $F_i(x)\geqq F_{[k]}(x)$ $\forall x$, $i=1,\cdots,k$, $F_{[k]}\underset{c}{<}G$,

(ii) $G(x)$ *has a differentiable density* g *in a neighborhood of its* α-*quan-*

tile η_α, $g(\eta_\alpha)\neq 0$, *and*
(iii) gG^{-1} *is uniformly continuous on* $[0, 1]$, $G^{-1}(x)$ *is convex and there exists a* ξ, $0<\xi<1$, *such that for* $\xi\leqq y<1$, $gG^{-1}(y)/(1-y)$ *is nondecreasing in* y, *then as* $n\to\infty$

$$\text{(2.17)} \qquad \mathrm{P}\,[CS|R_1]\geqq\int_{-\infty}^{\infty}\Phi^{k-1}\left[\frac{x}{c_1}+\frac{1-c_1}{c_1}\eta_\alpha g(\eta_\alpha)\left(\frac{n}{\alpha\bar{\alpha}}\right)^{1/2}\right]d\Phi(x)$$

where $\bar{\alpha}=1-\alpha$ *and* $\Phi(x)$ *is the standard normal c.d.f.*

PROOF. We note that

$$\text{(2.18)} \qquad \mathrm{P}\,[CS|R_1]\geqq\mathrm{P}\,[Z_k\geqq c_1\max_{1\leqq j\leqq k} Z_j]$$

where $Z_1,\cdots, Z_k$ are i.i.d. with c.d.f. $W_k(x)$ and $W_k(x)$ is the c.d.f. of $T_{(k)}$. By Theorem (2.2) of Barlow and Van Zwet [7] and condition (iii), then we have (see Barlow and Doksum [1]), for n large,

$$\text{(2.19)} \qquad Z_i \underset{\text{st}}{\simeq} Y_{i;r,n}$$

where $Y_{i;r,n}$ is the rth order statistic from $H_{F_{[k]}}$ and $H_{F_{[k]}}^{-1}$ (the inverse of $H_{F_{[k]}}$) is defined in (2.1). Now $F_{[k]}\underset{c}{<}G$ if and only if $H_{F_{[k]}}$ is convex. Since $G^{-1}(x)$ is increasing and convex, it follows that $G^{-1}H_{F_{[k]}}(x)$ is convex. Since $H_{F_{[k]}}\underset{c}{<}G$ and $G^{-1}(0)=0$, then $H_{F_{[k]}}\underset{*}{<}G$. In a manner similar to Theorem (2.1) of Barlow and Gupta [3], we have

$$\text{(2.20)} \qquad \mathrm{P}\,[Y_{k;r,n}\geqq c_1\max_{1\leqq i\leqq k} Y_{i;r,n}]\geqq \mathrm{P}\,[Y^*_{k;r,n}\geqq c_1 Y^*_{i;r,n},\ i\neq k]$$

where $Y^*_{i;r,n}$ is the rth order statistic from G, $i=1,\cdots,k$. From (2.18), (2.19), (2.20) and using the fact that

$$\text{(2.21)} \qquad Y^*_{i;r,n}\sim N\left(\eta_\alpha,\ \frac{\alpha\bar{\alpha}}{ng^2(\eta_\alpha)}\right),$$

the theorem follows.

Before we discuss some properties of the selection rule R_1, we introduce some definitions (see Santner [17]).

Define $\mathrm{P}_{\underline{F}}(i)=\mathrm{P}_{\underline{F}}[\pi_{(i)}$ is selected$|R]$ where $\pi_{(i)}$ is associated with $F_{[i]}$.

DEFINITION 2.1. (i) A rule R is strongly monotone in $\pi_{(i)}$ if

$$\mathrm{P}_{\underline{F}}(i) \text{ is } \begin{cases} \uparrow \text{ in } F_{[i]} & \text{when all other components of } \underline{F} \text{ are fixed} \\ \downarrow \text{ in } F_{[j]} & (j\neq i) \text{ when all other components of } \underline{F} \text{ are fixed.} \end{cases}$$

That means, $\mathrm{P}_{\underline{F}_1}(i)\geqq\mathrm{P}_{\underline{F}_1^*}(i)$ when $F_{[i]}\underset{\text{st}}{\geqq}F^*_{[i]}$ and $\mathrm{P}_{\underline{F}_2}(j)\leqq\mathrm{P}_{\underline{F}_2^*}(j)$ when

$F_{[j]} \underset{\text{st}}{\geqq} F^*_{[j]}$ for $j \neq i$, where $\underline{F}_1=(F_{[1]},\cdots,F_{[i]},\cdots,F_{[k]})$, $\underline{F}_1^*=(F_{[1]},\cdots,F^*_{[i]},\cdots,F_{[k]})$, $\underline{F}_2=(F_{[1]},\cdots,F_{[j]},\cdots,F_{[k]})$ and $\underline{F}_2^*=(F_{[1]},\cdots,F^*_{[j]},\cdots,F_{[k]})$.

(ii) A rule R is monotone means $\mathrm{P}_{\underline{F}}(i) \leqq \mathrm{P}_{\underline{F}}(j)$ for all $\underline{F} \in \Omega$ with $F_{[i]}(x) \geqq F_{[j]}(x)$.

(iii) A rule R is unbiased if $\mathrm{P}_{\underline{F}}(i) \leqq \mathrm{P}_{\underline{F}}(k)$ for all $\underline{F} \in \Omega$ with $F_{[i]}(x) \geqq F_{[k]}(x)$.

(iv) A rule R is consistent with respect to Ω' means $\inf_{\Omega'} \mathrm{P}[CS|R] \to 1$ as $n \to \infty$.

Using arguments similar to those in the proof of Lemma 2.1, we can prove the following theorem.

THEOREM 2.3. *If $a_i \geqq 0$ for $i=1,\cdots,r$, then R_1 is strongly monotone in $\pi_{(i)}$.*

Remark 2.1. (1) If a rule R is strongly monotone in $\pi_{(i)}$ for all $i=1,\cdots,k$, then R is monotone and $\inf_{\Omega} \mathrm{P}[CS|R]=\inf_{\Omega_0} \mathrm{P}[CS|R]$ where

$$\Omega_0=\{\underline{F}=(F_1,\cdots,F_k) \in \Omega:\ F_1=\cdots=F_k\}\ .$$

(2) If R is monotone, then it is unbiased.

(3) If $F_i(x)=F(x,\theta_i)$, $i=1,\cdots,k$ and T_i is a consistent estimator of θ_i, then R_1 is consistent with respect to $\Omega=\{\underline{F}=(F_1,\cdots,F_k):\ \exists j$ such that $F_i(x) \geqq F_j(x)$ for all x and $i=1,\cdots,k\}$.

(4) If $F_i,\ G \in \mathcal{F}$, $F_i \underset{\text{c}}{<} G$, $i=1,\cdots,k$ and the condition (iii) of Theorem 2.2 is satisfied, we can show that R_1 is consistent with respect to Ω.

The selection of the population with largest F_i $(i=1,\cdots,k)$ can be handled analogously. We assume $F_{[i]}(x) \leqq F_{[1]}(x)$, $i=1,\cdots,k$, and $F_{[1]} \underset{\text{c}}{<} G$. The rule for selecting the population with $F_{[1]}$ is R_2: Select population π_i if and only if

$$c_2 T_i \leqq \min_{1 \leqq j \leqq k} T_j \tag{2.22}$$

where c_2 $(0<c_2<1)$ is determined so as to satisfy the basic requirement.

In a manner similar to the proof of Theorem 2.1, we have

THEOREM 2.4. *If $F_i,\ G \in \mathcal{F}$, $F_{[i]}(x) \leqq F_{[1]}(x)$ $\forall x$ and $i=1,\cdots,k$, $F_{[1]} \underset{\text{c}}{<} G$, $a_j \geqq 0$ for $j=1,\cdots,r$, $g(0) \leqq 1$ and $a_r \geqq c_2$, then*

$$\inf_{\Omega'} \mathrm{P}[CS|R_2]=\int_0^\infty \bar{G}_T^{k-1}(c_2 x)\,dG_T(x) \tag{2.23}$$

where $\bar{G}_T(x)=1-G_T(x)$ and $\Omega'=\{\underline{F}=(F_1,\cdots,F_k):\ \exists j$ such that $F_i(x) \leqq F_j(x)$ for all x and $i=1,\cdots,k\}$.

(B) *Efficiency of procedure R_1 under slippage configuration*

Under the same notations and conditions of Theorem 2.2 and the comments above Theorem 2.2, we consider slippage configuration $F_{[i]}(x) = F(x/\delta)$, $i=1,2,\cdots,k-1$, and $F_{[k]}(x)=F(x)$, $0<\delta<1$. Let $\mathrm{E}(S|R)$ denote the expected subset size using the rule R. Then $\mathrm{E}(S|R)-\mathrm{P}[CS|R]$ is the expected number of non-best populations included in the selected subset. For a given $\varepsilon>0$, let $n_R(\varepsilon)$ be the asymptotic sample size for which $\mathrm{E}(S|R)-\mathrm{P}[CS|R]=\varepsilon$. We define the asymptotic relative efficiency $\mathrm{ARE}(R, R^*, \delta)$ of R relative to R^* to be the limit as $\varepsilon\to 0$ of the ratio $n_R(\varepsilon)/n_{R^*}(\varepsilon)$ i.e. $\mathrm{ARE}(R, R^*; \delta)=\lim_{\varepsilon\to 0}(n_R(\varepsilon)/n_{R^*}(\varepsilon))$. Under the slippage configuration we have,

$$\mathrm{E}(S|R_1)=\mathrm{P}[CS|R_1]+(k-1)\,\mathrm{P}[T_{(1)}\geqq c_1 \max_{i\neq 1} T_{(i)}]\,. \tag{2.24}$$

If n is large, then from an argument similar to the one in the proof of Theorem 2.2, we have

$$\mathrm{P}[T_{(1)}\geqq c_1 \max_{i\neq 1} T_{(i)}]\approx \mathrm{P}[Y_1\geqq c_1 \max_{i\neq 1} Y_i] \tag{2.25}$$

where $Y_1,\cdots,Y_k$ are independent and Y_i is the rth order statistic from $H_{F_{[i]}}$ for $i=1,\cdots,k$. The right-hand side of (2.25) is asymptotically equal to

$$\int_{-\infty}^{\infty}\Phi\left(\frac{\delta x}{c_1}-a_\alpha h(a_\alpha)\left(1-\frac{\delta}{c_1}\right)\left(\frac{n}{\alpha\bar{\alpha}}\right)^{1/2}\right) \cdot \Phi^{k-2}\left(\frac{x}{c_1}-a_\alpha h(a_\alpha)\left(1-\frac{1}{c_1}\right)\left(\frac{n}{\alpha\bar{\alpha}}\right)^{1/2}\right)d\Phi(x) \tag{2.26}$$

where c_1 is the constant used in defining R_1, a_α is the (unique) α-quantile of $H_{F_{[k]}}(x)$ and $h(x)$ is the density function of $H_{F_{[k]}}(x)$. For $k=2$ and n large,

$$\mathrm{E}(S|R_1)-\mathrm{P}[CS|R_1] \approx\Phi\left(-h(a_\alpha)a_\alpha\left(1-\frac{\delta}{c_1}\right)\left(\frac{n}{\alpha\bar{\alpha}}\right)^{1/2}\left(1+\frac{\delta^2}{c_1^2}\right)^{-1/2}\right)\,. \tag{2.27}$$

Let

$$\int_{-\infty}^{\infty}\Phi^{k-1}\left(\frac{x}{c_1}+(1-c_1)\eta_\alpha g(\eta_\alpha)\frac{1}{c_1}\left(\frac{n}{\alpha\bar{\alpha}}\right)^{1/2}\right)d\Phi(x)=P^*\,.$$

Now setting the right side of (2.27) equal to ε and using $c_1\approx 1-2^{1/2}D/n^{1/2}$, where $D=\Phi^{-1}(P^*)(\alpha\bar{\alpha})^{1/2}/\eta_\alpha g(\eta_\alpha)$, we obtain

$$n_{R_1}(\varepsilon)\approx[-(\alpha\bar{\alpha})^{1/2}\Phi^{-1}(\varepsilon)(1+\delta^2)^{1/2}+\sqrt{2}\,D\delta a_\alpha h(a_\alpha)]^2[a_\alpha^2 h^2(a_\alpha)(1-\delta)^2]^{-1}\,. \tag{2.28}$$

Comparison with Barlow-Gupta procedure

Barlow and Gupta [3] propose a procedure R_3, for the quantile selection problem of star-ordered distributions which is,

R_3: Select population π_i if and only if

$$T_{r,i} \geqq c_3 \max_{1\leq j\leq k} T_{r,j} \tag{2.29}$$

where c_3 $(0<c_3<1)$ is chosen to satisfy $P[CS|R_3]\geqq P^*$ and $T_{r,i}$ is the rth order statistic from F_i where $r\leqq(n+1)\alpha<r+1$. They derive an expression for $n_{R_3}(\varepsilon)$ as follows:

$$n_{R_3}(\varepsilon)\approx[-(\alpha\bar{\alpha})^{1/2}\Phi^{-1}(\varepsilon)(1+\delta^2)^{1/2}+\sqrt{2}\,D\delta\xi_\alpha f(\xi_\alpha)]^2[\xi_\alpha^2 f^2(\xi_\alpha)(1-\delta)^2]^{-1}$$

where f is the density of F with unique α-quantile, ξ_α.

$$\text{ARE}\,(R_1, R_3; \delta)=\lim_{\varepsilon\to 0}\frac{n_{R_1}(\varepsilon)}{n_{R_3}(\varepsilon)}=\frac{\xi_\alpha^2 f^2(\xi_\alpha)}{a_\alpha^2 h^2(a_\alpha)}\,. \tag{2.30}$$

If $G(x)=1-e^{-x}$, $x>0$ and $F_{[1]}(x)=1-e^{-x/\delta}$ and $F_{[2]}(x)=1-e^{-x}$, $x\geqq 0$, $0<\delta<1$, we have,

$$\text{ARE}\,(R_1, R_3; \delta)=\frac{(1-\alpha)^2\log^2(1-\alpha)}{\alpha^2}\leqq 1 \tag{2.31}$$

and the ARE$=0.4803$, $\alpha=1/2$.

Comparison with Gupta procedure

Gupta [8] gave a selection procedure for gamma populations π_i's with densities $(1/\Gamma(a)\theta_i^a)x^{a-1}e^{-x/\theta_i}$ $x>0$, $\theta_i>0$, $i=1, 2, \cdots, k$. The procedure R_4 is

R_4: Select population π_i if and only if

$$\bar{X}_i\geqq c_4 \max_{1\leq j\leq k}\bar{X}_j \tag{2.32}$$

where $\bar{X}_i$ is the sample mean of size n from π_i and c_4 is the largest constant $(0<c_4<1)$ chosen so that $P[CS|R_4]\geqq P^*$. For $k=2$, $\theta_{[1]}=\delta$ and $\theta_{[2]}=1$ (see Barlow and Gupta [3]), we have

$$\text{ARE}\,(R_3, R_4; \delta)=\lim_{\varepsilon\to 0}\frac{n_{R_3}(\varepsilon)}{n_{R_4}(\varepsilon)}=\frac{a(\log\delta)^2\alpha\bar{\alpha}(1+\delta^2)}{2(1-\delta)^2[\xi_\alpha f(\xi_\alpha)]^2}\,. \tag{2.33}$$

Hence

$$\text{ARE}\,(R_1, R_4; \delta)=\text{ARE}\,(R_1, R_3; \delta)\,\text{ARE}\,(R_3, R_4; \delta) = \left\{\frac{\sqrt{a}\,\log\delta\sqrt{\alpha\bar{\alpha}}\sqrt{1+\delta^2}}{\sqrt{2}\,(1-\delta)a_\alpha h(a_\alpha)}\right\}^2. \tag{2.34}$$

If $G(x)=1-e^{-x}$ for $x>0$ and $a=1$,

$$\text{ARE}\,(R_1, R_4; \delta) = \frac{(1-\alpha)(1+\delta^2)\log^2\delta}{2(1-\delta)^2\alpha} \tag{2.35}$$

$$\text{ARE}\,(R_1, R_4; \delta \uparrow 1) = \frac{1-\alpha}{\alpha}\,. \tag{2.36}$$

Comparison of R_1 and R_5 from uniform distribution

Suppose π_1 and π_2 are two independent uniform populations with distribution functions F_i $(i=1, 2)$.

$$F_i(x) = \begin{cases} 0 & x<0 \\ \dfrac{x}{\theta_i} & 0 \leqq x \leqq \theta_i \\ 1 & x>\theta_i \end{cases} \tag{2.37}$$

where $\delta=\theta_{[1]}<\theta_{[2]}=1$. A sample of n independent observations is drawn from each of the two populations. Let T_i^* be the total life statistic until rth failure from π_i $(i=1, 2)$ where $r \leqq (n+1)\alpha < r+1$. The procedure R_5 is given by

R_5: Select population π_i if and only if

$$T_i^* \geqq c_5 \max_{1\leqq j\leqq k} T_j^* \tag{2.38}$$

where c_5 is chosen so that $\text{P}\,[CS|R_5] \geqq P^*$. Let $T_{(i)}^*$ be associated with $\theta_{[i]}$.

$$\text{E}\,(S|R_5) - \text{P}\,[CS|R_5] = \text{P}\,[T_{(1)}^* \geqq c_5 T_{(2)}^*] = \text{P}\left[T_1' \geqq \frac{c_5}{\delta} T_2'\right] \tag{2.39}$$

where T_1', T_2' are two independent total life statistic until rth failure from uniform distribution over $(0, 1)$. By Gupta and Sobel [12],

$$\frac{T_i'-u}{\sigma} \longrightarrow N(0, 1) \qquad \text{as } n\longrightarrow\infty\,, \tag{2.40}$$

where $u=n\alpha(2n-\alpha n+1)/(2n+1) \approx u' = n\alpha(2-\alpha)/2$, $\sigma^2=An$ and $A=\alpha(1-\alpha)(2-\alpha)^2/4+\alpha^3/12$. Hence $u/\sigma \approx u'/\sigma = B\sqrt{n}$ where $B=\alpha(2-\alpha)/2\sqrt{A}$. From (2.39), we have

$$\text{E}\,(S|R_5) - \text{P}\,[CS|R_5] \approx \text{P}\left[Z_1 \geqq \frac{c_5}{\delta} Z_2 + \left(\frac{c_5}{\delta}-1\right) B\sqrt{n}\right]$$

where Z_1, Z_2 are i.i.d. with $N(0, 1)$.

$$\text{E}\,(S|R_5) - \text{P}\,[CS|R_5] = \int_{-\infty}^{\infty} \Phi\left[\frac{\delta}{c_5}x - \left(1-\frac{\delta}{c_5}\right)B\sqrt{n}\right] d\Phi(x)$$

$$=\Phi\left[-\left(1-\frac{\delta}{c_5}\right)B\sqrt{n}\Big/\sqrt{1+\left(\frac{\delta}{c_5}\right)^2}\right].$$

Let $E(S|R_5)-P[CS|R_5]=\varepsilon>0$, we obtain

$$\left(\frac{1}{c_5}-\frac{1}{\delta}\right)\sqrt{n}=\sqrt{\frac{1}{\delta^2}+\frac{1}{c_5^2}}\cdot\frac{\Phi^{-1}(\varepsilon)}{B}. \tag{2.41}$$

Note that $\inf_{\Omega} P[CS|R_5]=P[T_1'\geqq c_5T_2']$,

$$\approx\Phi\left[-\left(1-\frac{1}{c_5}\right)B\sqrt{n}/\sqrt{1+1/c_5^2}\right]$$

where T_1' and T_2' are defined as above. Setting $\inf_{\Omega} P[CS|R_5]=P^*$ and using $c_5\approx1-\sqrt{2}\,\Phi^{-1}(P^*)/\sqrt{n}\,B$ and $1/c_5\approx1+\sqrt{2}\,\Phi^{-1}(P^*)/\sqrt{n}\,B$, from (2.41), we obtain

$$n_{R_5}(\varepsilon)\approx\left\{\frac{\Phi^{-1}(\varepsilon)\sqrt{1+\delta^2}-\sqrt{2}\,\delta\Phi^{-1}(P^*)}{B(1-\delta)}\right\}^2. \tag{2.42}$$

From (2.28) and (2.42),

$$\text{ARE}(R_1, R_5;\delta)=\lim_{\varepsilon\to0}\frac{n_{R_1}(\varepsilon)}{n_{R_5}(\varepsilon)}=\frac{\alpha\bar{\alpha}B^2}{a_\alpha^2h^2(a_\alpha)}. \tag{2.43}$$

If we assume that $G(x)=x$ for $0<x<1$, then

$$\text{ARE}(R_1, R_5;\delta)=\frac{B^2(1-\alpha)}{\alpha}=\frac{3(1-\alpha)(2-\alpha)^2}{3(1-\alpha)(2-\alpha)^2+\alpha^2}<1. \tag{2.44}$$

$\text{ARE}(R_1, R_5;\delta)$ is a decreasing function of α and for $\alpha=1/2$, it is equal to 0.931. Note that in (2.44), R_1 is based on rth ordered statistic and R_5 is based on the total life statistic until rth failure.

(C) *Selection procedure for distribution* $\underset{c}{<}$ *ordered with respect to Weibull distribution*

Assume that the specified distribution $G(x)$ is given by

$$G(x)=\begin{cases}1-e^{-\lambda x^\alpha} & \text{for } x\geqq0\\ 0 & \text{for } x<0\end{cases}$$

where $\lambda>0$ and attention is restricted to $\alpha\geqq1$ which is assumed known. In this case, we use T_i^* as our statistic where

$$T_i^*=\sum_{j=1}^{r-1}X_{i;j,n}^\alpha+(n-r+1)X_{i;r,n}^\alpha,\qquad i=1,\cdots,k.$$

As before, $X_{i;j,n}$ denote the jth order statistic from F_i, $i=1,\cdots,k$.

Since $G(x)$ is convex with respect to the exponential distribution if $\alpha \geqq 1$ and since the convex ordering is transitive, the family of distributions which are convex with respect to Weibull ($\alpha \geqq 1$) will have IFR distribution. Thus our interest here is in a special subclass of IFR distributions. The rule for selecting the population which is associated with $F_{[k]}$ is as follows,

R_6: Select population π_i if and only if

$$T_i^* \geqq c_6 \max_{1 \leqq j \leqq k} T_j^* \tag{2.45}$$

where c_6 $(0 < c_6 < 1)$ is determined so as to satisfy the basic probability requirement. Using the fact that if $F \underset{c}{<} G$ and $F(0) = G(0) = 0$ then $F_\alpha \underset{c}{<} G_\alpha$ or $\alpha \geqq 1$, where F_α is the c.d.f. of X^α, $F(x)$ is the c.d.f. of X, G_α is the c.d.f. of Y^α and $G(y)$ is the c.d.f. of Y. Also, $G_\alpha^{-1} F_\alpha(X_{i,n}^\alpha)$ is stochastically equivalent to the ith order statistic from $G^*(x) = 1 - e^{-x}$, for $x \geqq 0$, where $X_{1,n} \leqq \cdots \leqq X_{n,n}$ are order statistics from F. In a manner similar to the proof of Theorem 2.1, one can prove the following theorem.

THEOREM 2.5. *If $F_i(x) \geqq F_{[k]}(x)$ $\forall x$ and $i = 1, \cdots, k$, $F_{[k]}(0) = 0$, $F_{[k]} \underset{c}{<} G$, $G(x) = 1 - e^{-\lambda x^\alpha}$, $x > 0$, $\lambda > 0$ and α $(\geqq 1)$ is known and $n \geqq \max\{r, (r-1)/(1-c_6)\}$, then*

$$\inf_\Omega \mathrm{P}\,[CS \mid R_6] = \int_0^\infty H^{k-1}\left(\frac{x}{c_6}\right) dH(x) \tag{2.46}$$

where $H(x)$ is the c.d.f. of a χ^2 random variable with $2r$ d.f.

(D) *Selection with respect to the means for Gamma populations*

Let $\pi_1, \cdots, \pi_k$ be k populations with densities

$$f_i(x) = \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\beta x}, \qquad x \geqq 0,\ \beta > 0,\ \alpha_i \geqq 1,\ i = 1, \cdots, k\,.$$

Let $F_i(x)$ be the distribution function of π_i, $i = 1, \cdots, k$. We are given a sample size of n from each π_i. Let T_i^* be total life statistic until rth failure from π_i. Let $\alpha_{[1]} \leqq \cdots \leqq \alpha_{[k]}$ be the ordered values of α_i's. We are interested in selecting the population with the largest value $\alpha_{[k]}$ (unknown). Since the mean of π_i is α_i/β, selection of the population with largest mean is equivalent to selecting the population with largest value, $\alpha_{[k]}$. The subset selection rule based on T_i^* is:

R_7: Select population π_i if and only if

$$T_i^* \geqq c_7 \max_{1 \leqq j \leqq k} T_j^*\,, \tag{2.47}$$

where c_7 $(0<c_7<1)$ is the largest value chosen to satisfy $\mathrm{P}[CS|R_7]\geqq P^*$. Since the rule R_7 is scale invariant, we can assume $\beta=1$.

Case 1. All α_i are unknown and $\geqq 1$. Let $\Omega_1=\{\underline{\alpha}=(\alpha_1,\cdots,\alpha_k)\colon \alpha_i \geqq 1\ \forall i\}$. In this case, by Corollary 2.1 and $F_i \underset{c}{<} G(x)=1-e^{-x}$, $x\geqq 0$, $i=1,\cdots,k$, we have the following result. If $n\geqq \max\{r,(r-1)/(1-c_7)\}$, then $\inf_{\Omega_1} \mathrm{P}[CS|R_7]=\int_0^\infty H^{k-1}(x/c_7)dH(x)$, where $H(x)$ is the c.d.f. of a χ^2 r.v. with $2r$ d.f.

Case 2. α_i are unknown but assume $1\leqq \alpha_i\leqq \Delta$, $i=1,\cdots,k$ and Δ is known. Let $F_\Delta(x)$ be the c.d.f. of X with density function $f_\Delta(x)=(1/\Gamma(\Delta))x^{\Delta-1}e^{-x}$, $x>0$. Let $H(x)$ be the c.d.f. of a χ^2 r.v. with $2r$ d.f. and let $h(x)$ be its density function. The following theorem gives a lower bound for the probability of correct selection without any condition on n.

THEOREM 2.6.

$$\mathrm{P}[CS|R_7]\geqq \int_0^\infty H^{k-1}\left(\frac{2n}{c_7}x\right)\frac{2nh(2ny)}{f_\Delta(y)}e^{-x}dx \tag{2.48}$$

where

$$y=F_\Delta^{-1}(1-e^{-x})\ .$$

PROOF.

$$\mathrm{P}[CS|R_7]=\mathrm{P}[T^*_{(k)}\geqq c_7 \max_{1\leqq j\leqq k-1} T^*_{(j)}]\ ,$$

where $T^*_{(i)}$ is associated with $\alpha_{[i]}$, $i=1,\cdots,k$. Since $F_\Delta(x)\leqq F_i(x)\leqq G(x)=1-e^{-x}$,

$$\mathrm{P}[CS|R_7]\geqq \mathrm{P}[T_k^{**}\geqq c_7 \max_{1\leqq j\leqq k-1} T_j^{**}] \tag{2.49}$$

where T_k^{**} is the total life statistic until rth failure from $G(x)$ and T_j^{**} $(j=1,\cdots,k-1)$ is the total life statistic until rth failure from $F_\Delta(x)$. Since $\Delta\geqq 1$ then $F_\Delta \underset{c}{<} G$. Let $\phi(x)=G^{-1}F_\Delta(x)$

$$\begin{aligned}\mathrm{P}[T_k^{**}\geqq c_7T_j^{**},\ j=1,\cdots,k-1]\\ =\mathrm{P}\left[\phi\left(\frac{1}{n}T_k^{**}\right)\geqq \phi\left(\frac{c_7}{n}T_j^{**}\right)\ j=1,\cdots,k-1\right]\ .\end{aligned} \tag{2.50}$$

By Lemma 4.1 of Barlow and Proschan [5] with $a_1=\cdots=a_{r-1}=c_7/n$, $a_r=(n-r+1)c_7/n$, $a_i=0$ for $i\geqq r+1$ and $\phi(X)\underset{\mathrm{st}}{=}Y$ where $X(Y)$ is a r.v. with distribution function $F_\Delta(G)$ respectively, we have

$$\text{(2.51)} \qquad \mathrm{P}\left[\phi\left(\frac{1}{n}T_k^{**}\right) \geqq \phi\left(\frac{c_7}{n}T_j^{**}\right),\ j=1,\cdots,k-1\right]$$
$$\geqq \mathrm{P}\left[\phi\left(\frac{1}{n}T_k^{**}\right) \geqq \frac{c_7}{2n}Y_j,\ j=1,\cdots,k-1\right]$$

where Y_j $(j=1,\cdots,k-1)$ is a r.v. with χ^2 with $2r$ d.f. From (2.49), (2.50) and (2.51), we have

$$\mathrm{P}\,[CS\,|\,R_7] \geqq \int_0^\infty H^{k-1}\left(\frac{2n}{c_7}x\right)dB(x)\ ,$$

where $B(x)=\mathrm{P}\,[\phi((1/n)T_k^{**})\leqq x]$. Since $B(x)=H[2nF_\Delta^{-1}(1-e^{-x})]$, then

$$\int_0^\infty H^{k-1}\left(\frac{2n}{c_7}x\right)dB(x)=\int_0^\infty H^{k-1}\left(\frac{2n}{c_7}x\right)\frac{2nh(2ny)}{f_\Delta(y)}e^{-x}dx\ .$$

This completes the proof.

Let S denote the size of the selected subset. The expected value of S when R_7 is used is given by

$$\text{(2.52)} \qquad \mathrm{E}\,(S\,|\,R_7)=\sum_{i=1}^{k}\mathrm{P}\,[T_i^*\geqq c_7 \max_{1\leqq j\leqq k} T_j^*]\ .$$

Let $\Omega'=\{\underline{\alpha}=(\alpha_1,\cdots,\alpha_k):\ 1\leqq\alpha_i\leqq\Delta,\ i=1,\cdots,k\}$. For $\underline{\alpha}\in\Omega'$ since $F_\Delta(x)\leqq F_i(x)\leqq G(x)=1-e^{-x}$, then

$$\mathrm{E}\,(S\,|\,R_7)\leqq k\,\mathrm{P}\,[T_1^{**}\geqq c_7 \max_{2\leqq j\leqq k} T_j^{**}]$$

where T_1^{**} is the total life statistic until rth failure from $F_\Delta(x)$ and T_j^{**} $(j=2,\cdots,k)$ is the total life statistic until rth failure from $G(x)$. Thus

$$\text{(2.53)} \qquad \sup_{\Omega'}\mathrm{E}\,(S\,|\,R_7)=k\int_0^\infty H^{k-1}\left(\frac{2x}{c_7}\right)dS(x)$$

where $H(x)$ is the c.d.f. of a χ^2 r.v. with $2r$ d.f. and $S(x)$ is the c.d.f. of the total life statistic until rth failure from $F_\Delta(X)$.

Remark 2.2. (i) We can show that the lower bound for Case 2 in Theorem 2.6 is less than or equal to the lower bound for Case 1.

(ii) Now we are dealing with the problem in Case 2. Let $\int_0^\infty H^{k-1}(x/c_7)dH(x)=P^*$, then c_7 can be determined. If $n\geqq\max\{r,(r-1)/(1-c_7)\}$, then we should use the lower bound for Case 1. If $r\leqq n<(r-1)/(1-c_7)$, then the lower bound for Case 1 cannot be applied. In this case, we can use the lower bound for Case 2.

(iii) Sometimes, the distribution function $S(x)$ which is defined above Remark 2.2 is hard to compute. Using $\mathrm{E}(S|R_7) \leqq k\,\mathrm{P}[T_1^{**} \geqq c_7 T_j^{**}, j=2,\cdots,k]$ where T_1^{**} is the total life statistic until rth failure from F_Δ and T_j^{**} $(j=2,\cdots,k)$ is the total life statistic until rth failure from $G(x)$ and employing similar arguments as in the proof of Theorem 2.6, we can get

$$\mathrm{E}(S|R_7) \leqq k \int_0^\infty H^{k-1}\left[\frac{2n}{c_7} F_\Delta^{-1}(1-e^{-x/2n})\right] dH(x)$$

where $H(x)$ is the c.d.f. of a χ^2 r.v. with $2r$ d.f. In this case, the upper bound of $\mathrm{E}(S|R_7)$ can be computed.

3. Selecting a best population—using indifference zone approach

Let $\pi_1,\cdots,\pi_k$ be k populations. The random variable X_i associated with π_i has an absolutely continuous distribution F_i. We assume there exists an $F_{[k]}(x)$ such that $F_{[i]}(x) \geqq F_{[k]}(x/\delta)$ for all x, $i=1,\cdots,k-1$ and δ $(0<\delta<1)$ is specified. Let

$$(3.1) \quad \Omega(\delta) = \{\underline{F}=(F_1,\cdots,F_k):\ \exists j \text{ such that } F_i(x) \geqq F_j(x/\delta)\ \forall i \neq j\}\ .$$

The correct selection is the choice of any population which is associated with $F_{[k]}$. We propose the selection rule R_8: Select population π_i if and only if

$$(3.2) \qquad T_i = \max_{1\leqq j\leqq k} T_j \qquad \text{where } T_i \text{ is defined as in (2.5)}\ .$$

We want the $\mathrm{P}[CS|R_8] \geqq P^*$, for all $\underline{F} \in \Omega(\delta)$, where P^* is specified.

THEOREM 3.1. *If F_i, $G \in \mathcal{F}$, $i=1,\cdots,k$, $F_{[k]} \underset{c}{<} G$, $a_j \geqq 0$, $j=1,\cdots,r$, $g(0) \leqq 1$ and $a_r \geqq \delta$, then*

$$(3.3) \qquad \inf_{\Omega(\delta)} \mathrm{P}[CS|R_8] = \int_0^\infty G_T^{k-1}\left(\frac{x}{\delta}\right) dG_T(x)$$

where $G_T(x)$ is the c.d.f. of T defined in (2.6).

PROOF.

$$\mathrm{P}[CS|R_8] = \mathrm{P}[T_{(k)} \geqq \max_{1\leqq j\leqq k} T_{(j)}]\ .$$

Since $F_{[i]}(\delta x) \geqq F_{[k]}(x)$, $i=1,\cdots,k-1$ and by Lemma 2.1, then

$$\mathrm{P}[CS|R_8] = \mathrm{P}\left[T_{(k)} \geqq \delta \frac{T_{(j)}}{\delta}\ \ \forall j \neq k\right] \geqq \mathrm{P}[T_{(k)} \geqq \delta T_j^*\ \ \forall j \neq k]$$

where $T_1^*,\cdots,T_{k-1}^*$, $T_{(k)}$ are i.i.d. with c.d.f. $W_k(x)$. Using the same argument as in Theorem 2.1, we have our theorem.

For given k, δ, P^* and $G(x)$, we can possibly find the values of the pair (n, r), $(n \geqq r)$ which satisfy

$$a_r \geqq \delta \quad \text{and} \quad \int_0^\infty G_T^{k-1}\left(\frac{x}{\delta}\right) dG_T(x) \geqq P^* . \tag{3.4}$$

If $G(x)=x$ for $0<x<1$, we can always find the values of the pair (n, r), $(n \geqq r)$ which satisfy

$$n\binom{n-1}{r-1}\int_0^\infty \left[\sum_{i=r}^n \binom{n}{i}\left(\frac{x}{\delta}\right)^i\left(1-\frac{x}{\delta}\right)^{n-i}\right]^{k-1} x^{r-1}(1-x)^{n-r}dx \geqq P^* .$$

If $G(x)=1-e^{-x}$ for $x \geqq 0$, we can find the smallest integer r, say r_0, which satisfies $\int_0^\infty H^{k-1}(x/\delta)dH(x) \geqq P^*$ where $H(x)$ is the c.d.f. of a χ^2 random variable with $2r$ d.f. Since $(1/\delta)a_r \geqq 1$ iff $n \geqq (r-1)/(1-\delta)$, we can find the minimum n satisfying $n \geqq \max\{r_0, (r_0-1)/(1-\delta)\}$.

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