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THE DISTRIBUTION AND THE EXACT PERCENTAGE POINTS FOR WILKS' L_{mvc} CRITERION

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Summary

Wilks' L_{mvc} is the likelihood ratio criterion for testing the hypothesis that the mean values are equal, the variances are equal and the covariances are equal, in a p-variate normal population. In this article the exact null distribution as well as the exact percentage points are given for the first time. The distribution is obtained for the most general cases and the inverse tables, namely, the values of u for given values of $F(u)$ are computed for the values of $F(u)=0.01, 0.02, 0.05$ and for the various values of n and p where $F(u)$ is the exact distribution function of the test statistic, $n=N-1$ and N is the sample size. The exact tables are given for $p=2, 3, 4, 5, 6, 7, 8, 9$.

1. Introduction

Consider a p-variate normal population $N_p(\mu, \Sigma)$ where $\Sigma=(\sigma_{ij})$ is positive definite. Consider the hypothesis $\mu_1=\mu_2=\cdots=\mu_p=\mu$, $\mu'=(\mu_1, \cdots, \mu_p)$, $\sigma_{ii}=\sigma$, $i=1,\cdots,p$ and $\sigma_{ij}=\sigma^*$, $i\neq j=1,\cdots,p$, where μ, σ, σ^* are some unknown scalars and μ' denotes the transpose of the mean vector μ. Wilks [5] derived the likelihood ratio test statistic λ for this hypothesis which is,

$$(1.1)\qquad U=\lambda^{2/N}=|S|\Big/\left\{[s+(p-1)s_1]\left[s-s_1+(N/(p-1))\sum_{j=1}^{p}(\bar{x}_j-\bar{x})^2\right]^{p-1}\right\}$$

where

$$S=(s_{ij})\,,\qquad s_{ij}=\sum_{k=1}^{N}(x_{ik}-\bar{x}_i)(x_{jk}-\bar{x}_j)\,,\qquad \bar{x}_i=\sum_{k=1}^{N}x_{ik}/N\,,$$

$$\bar{x}=\sum_{i=1}^{p}\bar{x}_i/p\,,\qquad s=\sum_{i=1}^{p}s_{ii}/p\,,\qquad s_1=\sum_{i\neq j=1}^{p}s_{ij}/[p(p-1)]\,,$$

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N being the sample size and (x_{ij}) denotes the observation matrix. This test statistic λ is known as Wilks' L_{mvc} criterion. Wilks [5] also obtained the h-th null moment of U, that is, the h-th moment of U when the null hypothesis is assumed to be true. Mathai [2] has obtained the exact non-null moments, and thereby the exact null moments by using an alternate simpler approach. The non-null moments are represented in terms of a hypergeometric function of several variables in the category of Lauricella's functions.

In this article we discuss the exact null distribution in the most general case and then compute the exact percentage points by using this distribution. The h-th null moment is the following.

$$\text{(1.2)} \qquad \mathrm{E}(U^h)=\prod_{j=0}^{p-2}\{\Gamma((n-1)/2+h-j/2)\Gamma((n+1)/2+j/(p-1))/ [\Gamma((n-1)/2-j/2)\Gamma((n+1)/2+h+j/(p-1))]\} .$$

2. The exact null distribution

The exact null distribution, in the general case, is not known to have been worked out so far. It can be worked out by inverting the moment expression in (1.2) with the help of inverse Mellin transform and then using the techniques discussed in Mathai and Saxena [3]. In this article we will outline the main steps of the derivations and give the final results, deleting all the details of the derivations. The computations are carried out by using the general expressions given in this article. Denoting the gammas containing h by $\Phi(h)$ and the remaining part by C the moments given in (1.2) becomes,

$$\text{(2.1)} \qquad \mathrm{E}(U^h)=C\Phi(h)$$

and the density of U, denoted by $f(u)$, can be written as

$$\text{(2.2)} \qquad f(u)=u^{-1}(2\pi i)^{-1}\int_L \Phi(h)u^{-h}dh$$

where $i=(-1)^{1/2}$ and L is a suitable contour. It can be seen from Mathai and Saxena [3] that $f(u)$ can be represented as a series with the help of calculus of residues. Since the expressions are different, the cases when p is odd and when p is even are considered separately. Since the technique is the same as the one used in Mathai and Saxena [3] only the final results are given here.

Case I. p-odd

$$\text{(2.3)} \qquad f(u)=u^{n/2-(p-1)/2-1}\left[\sum_{j=1}^{\infty}R_j+\sum_{j=1}^{\infty}R'_j\right], \qquad 0<u\leqq 1$$

where

$$(2.4)\qquad R_j=\frac{u^{j-1/2}}{(a_j-1)!}\sum_{r=0}^{a_j-1}\binom{a_j-1}{r}(-\log_e u)^{a_j-1-r}\left[\sum_{r_1=0}^{r-1}\binom{r-1}{r_1}A_j^{(r-1-r_1)}\cdots\right]B_j$$

$$(2.5)\qquad a_j=\begin{cases} j\,, & j=1,\cdots,(p-1)/2\\ (p-3)/2\,, & j\geqq(p+1)/2\,;\end{cases}\qquad b_j=\begin{cases} j\,, & j=1,\cdots,(p-3)/2\\ (p-1)/2\,, & j=(p-1)/2,\,(p+1)/2\\ (p-3)/2\,, & j\geqq(p+3)/2\end{cases}$$

and

$$(2.6)\qquad B_j=\Bigg\{\prod_{k=2}^{(p-3)/2-j}\Gamma(k)[\Gamma(1)]^{j+1}\prod_{k=1}^{(p-3)/2}\Gamma(k-j-1/2)\Big/\left[\prod_{k=1}^{j-1}(-j+k)^k(-j+1/2+(p-1)/2)(-j+(p-1)/2)\right.$$
$$\left.\cdot(-j-1/2+(p-1)/2)\right]\Bigg\}\gamma_1\qquad\text{for } j\leqq(p-5)/2\,,$$
$$=\Bigg\{[\Gamma(1)]^{(p-3)/2}\prod_{k=1}^{(p-3)/2}\Gamma(k-j-1/2)\Big/\Big[(-j+1/2+(p-1)/2)$$
$$\cdot(-j+(p-1)/2)(-j-1/2+(p-1)/2)$$
$$\cdot\prod_{k=1}^{j+1-(p-1)/2}(-k)^{(p-1)/2-1}\prod_{k=1}^{(p-1)/2-2}(-j+k)^k\Big]\Bigg\}\gamma_1$$
$$\text{for } j\geqq(p-1)/2-1$$

where

$$\gamma_1=\Gamma(1/2+p/2)\Gamma(1+p/2)\prod_{\substack{k=1\\k\neq(p-1)/2}}^{p-2}\prod_{m=1}^{j}(1/2+p/2+k/(p-1)-m)/$$
$$[(2\pi)^{(p-2)/2}(p-1)^{-(p-1)(1/2+p/2)+1/2}\Gamma((p-1)(1/2+p/2))]\,.$$

The following conditions are used in writing the above expressions as well as the expressions to follow. If $p-2$ is less than unity then $\gamma_1=1$ and $\prod_{k=a}^{b}(\ )=1$ if $b<a$, that is, an empty product is interpreted as unity and correspondingly an empty sum is interpreted as zero. The A_j's appearing in (2.4) are the following.

$$(2.7)\qquad A_j^{(0)}=A_j=\sum_{k=2}^{(p-3)/2-j}\Psi(k)+(j+1)\Psi(1)+\sum_{k=1}^{(p-3)/2}\Psi(k-j-1/2)$$
$$+\sum_{k=1}^{j-1}(k/(-k+j))+1/(j-(p-1)/2-1/2)$$
$$+1/(j-(p-1)/2)+1/(j+1/2-(p-1)/2)$$

$$+\Big[\Psi(1/2+p/2)+\Psi(1+p/2)+\sum_{\substack{k=1\\k\neq(p-1)/2}}^{p-2}\sum_{m=1}^{j}(1/(1/2+p/2+k/(p-1)-m))+(p-1)\log_e(p-1)-(p-1)\cdot\Psi((p-1)(1/2+p/2))\Big]\quad\text{for } j\leqq(p-1)/2-2$$

$$=((p-1)/2-1)\Psi(1)+\sum_{k=1}^{(p-3)/2}\Psi(k-j-1/2)+\sum_{k=1}^{j+1-(p-1)/2}\{((p-1)/2-1)/k\}+\sum_{k=1}^{(p-1)/2-2}(k/(j-k))+1/(j-1/2-(p-1)/2)+1/(j-(p-1)/2)+1/(j+1/2-(p-1)/2)+\Big[\Psi(1+p/2)+\sum_{\substack{k=1\\k\neq(p-1)/2}}^{p-2}\sum_{m=1}^{j}(1/(1/2+p/2+k/(p-1)-m))+(p-1)\log_e(p-1)-\Psi((p-1)(1/2+p/2))+\Psi(1/2+p/2)\Big],\quad\text{for } j\geqq(p-1)/2-1\,.$$

The quantity in [ ] is to be taken as zero if $p-2<1$.

(2.8) $$(A_j^{(t)},\ t\geqq 1)=(-1)^{t+1}t!\Big\{\sum_{k=2}^{(p-3)/2-j}\zeta(t+1,k)+(j+1)\zeta(t+1,1)+\sum_{k=1}^{(p-3)/2}\zeta(t+1,k-j-1/2)+\sum_{k=1}^{j-1}(k/(-j+k)^{t+1})+1/(1/2-j+(p-1)/2)^{t+1}+1/(-j+(p-1)/2)^{t+1}+1/(-j+(p-1)/2-1/2)^{t+1}+\beta_1\Big\}\quad\text{for } j\leqq(p-1)/2-2$$

$$=(-1)^{t+1}t!\Big\{((p-1)/2-1)\zeta(t+1,1)+\sum_{k=1}^{(p-3)/2}\zeta(t+1,k-j-1/2)+\sum_{k=1}^{j+1-(p-1)/2}((p-1)/2-1)/(-k)^{t+1}+\sum_{k=1}^{(p-1)/2-2}(k/(-j+k)^{t+1})+1/(-j+1/2+(p-1)/2)^{t+1}+1/(-j+(p-1)/2)^{t+1}+1/(-j+(p-1)/2-1/2)^{t+1}+\beta_1\Big\}\quad\text{for } j\geqq(p-1)/2-1$$

where

$$\beta_1=\zeta(t+1,1/2+p/2)+\zeta(t+1,1+p/2)-\sum_{\substack{k=1\\k\neq(p-1)/2}}^{p-2}\sum_{m=1}^{j}\{1/(1/2+p/2+k/(p-1)-m)^{t+1}\}$$

$$-(p-1)^{t+1}\zeta((p-1)(1/2+p/2))$$

with β_1 being zero whenever $p-2<1$ and $\Psi(z)$ and $\zeta(r, z)$ are the psi function and the generalized Riemann zeta function respectively which are the successive logarithmic derivatives of the gamma function where

$$(2.9) \quad \Psi(z)=-\gamma+\sum_{n=0}^{\infty}(z-1)/((z+n)(n+1)), \qquad z\neq 0, -1, -2, \cdots$$
$$=0.5772156649015329 \quad \text{(Euler's constant)}$$

$$(2.10) \quad \zeta(s, v)=\sum_{n=0}^{\infty}(v+n)^{-s}, \qquad R(s)>1,\ v\neq 0, -1, -2, \cdots.$$

It should be remarked that one can obtain A_j and $A_j^{(t)}$ from B_j by using the following procedure. Introduce a dummy variable y in every factor of B_j. Take the logarithm to the base e and then take the first derivative with respect to y and evaluate it at $y=0$ to obtain A_j. Take successive derivatives with respect to y and evaluate at $y=0$ to obtain $A_j^{(t)}$, $t\geqq 1$. Also R'_j is available from R_j by replacing $u^{j-1/2}$ by u^{j-1}, a_j by b_j, A_j by A'_j and $A_j^{(\cdot)}$ by $A_j'^{(\cdot)}$. We can obtain A'_j and $A_j'^{(\cdot)}$ from B'_j by using the procedure discussed above. Hence we give only B'_j here.

$$(2.11) \quad B'_j=\left\{\prod_{k=2}^{(p-3)/2-j}\Gamma(k)[\Gamma(1)]^{j+1}\prod_{k=1}^{(p-3)/2}\Gamma(k-j+1/2)\Big/\Big[(1+j+(p-1)/2)\right.$$
$$\left.\cdot(-j+p/2)(-j+(p-1)/2)\prod_{k=1}^{j-1}(-j+k)^k\Big]\right\}\delta_1$$
$$\text{for } j\leqq(p-1)/2-2$$
$$=\left\{[\Gamma(1)]^{(p-1)/2-1}\prod_{k=1}^{(p-3)/2}\Gamma(k-j+1/2)\Big/\Big[(1-j+(p-1)/2)\right.$$
$$\cdot(-j+p/2)(-j+(p-1)/2)\sum_{k=1}^{j+1-(p-1)/2}(-k)^{(p-1)/2-1}$$
$$\left.\cdot\prod_{k=1}^{(p-1)/2-2}(-j+k)^k\Big]\right\}\delta_1 \qquad \text{for } j\geqq(p-1)/2-1$$

where

$$\delta_1=\Gamma(1+p/2)\Gamma(3/2+p/2)\prod_{\substack{k=1\\ k\neq(p-1)/2}}^{p-2}\prod_{m=1}^{j}(1+p/2+k/(p-1)-m)/$$
$$[(2\pi)^{(p-2)/2}(p-1)^{-(p-1)(1+p/2)+1/2}\Gamma((p-1)(1+p/2))]$$

and δ_1 is to be interpreted as unity when $p-2<1$.

Case II. p-even

The expression for the density $f(u)$ remains the same as in (2.3) with R_j defined as in (2.4) and R'_j obtained in a similar way as in the case of p-odd and

(2.12)
$$a_j=\begin{cases} j\,, & j=1,\cdots,p/2-2 \\ p/2-1\,, & j=p/2-1,\,p/2,\cdots; \end{cases}$$
$$b_j=\begin{cases} j\,, & j=1,\cdots,p/2 \\ p/2-1\,, & j=p/2+1,\,p/2+2,\cdots. \end{cases}$$

Here we will give the expressions for B_j and B'_j since A_j, $A_j^{(t)}$, A'_j, $A_j'^{(t)}$ are all available from B_j and B'_j by using the techniques discussed in the case p-odd. After some simplifications it can be seen that B_j and B'_j in the case p-even are given by the following expressions.

(2.13)
$$\begin{aligned} B_j=&\Big\{[\Gamma(1)]^{j+1}\prod_{k=2}^{-j+p/2-2}\Gamma(k)\prod_{k=1}^{p/2-1}\Gamma(-j-1/2+k)\Big/ \\ &\Big[(-j-1/2+p/2)\prod_{k=1}^{j-1}(-j+k)^k\Big]\Big\}\eta_1\,, \qquad \text{for } j\leqq p/2-2 \\ =&\Big\{[\Gamma(1)]^{p/2-1}\prod_{k=1}^{p/2-1}\Gamma(-j-1/2+k)\Big/\Big[(-j-1/2+p/2) \\ &\cdot\prod_{k=1}^{j-p/2+1}(-k)^{p/2-1}\prod_{k=1}^{p/2-2}(-j+k)^k\Big]\Big\}\eta_1\,, \qquad \text{for } j\geqq p/2-1 \\ B'_j=&\Big\{[\Gamma(1)]^{j+1}\prod_{k=2}^{-j+p/2-2}\Gamma(k)\prod_{k=1}^{p/2-1}\Gamma(-j+1/2+k)\Big/ \\ &\Big[(-j+p/2)\prod_{k=1}^{j-1}(-j+k)^k\prod_{k=1}^{p-2}\Gamma(1-j+p/2+k/(p-1))\Big]\Big\}\eta_2\,, \\ &\qquad \text{for } j\leqq p/2-2 \\ =&\Big\{[\Gamma(1)]^{p/2-1}\prod_{k=1}^{p/2-1}\Gamma(-j+k+1/2)\Big/\Big[(-j+p/2) \\ &\cdot\prod_{k=1}^{j-p/2+1}(-k)^{p/2-1}\prod_{k=1}^{p/2-2}(-j+k)^k\Big]\Big\}\eta_2\,, \qquad \text{for } j\geqq p/2-1\,, \end{aligned}$$

where

$$\begin{aligned}\eta_1=&\Gamma(1/2+p/2)\prod_{k=1}^{p-2}\prod_{m=1}^{j}(1/2+p/2+k/(p-1)-m)/[(2\pi)^{(p-2)/2} \\ &\cdot(p-1)^{-(p-1)(1/2+p/2)+1/2}\Gamma((p-1)(1/2+p/2))]\end{aligned}$$

and

$$\begin{aligned}\eta_2=&\Gamma(1+p/2)\prod_{k=1}^{p-2}\prod_{m=1}^{j}(1+p/2+k/(p-1)-m)/[(2\pi)^{(p-2)/2} \\ &\cdot(p-1)^{-(p-1)(1+p/2)+1/2}\Gamma((p-1)(1+p/2))]\,.\end{aligned}$$

As before, η_1 and η_2 are to be interpreted as unities when $p-2<1$. Both in the p-odd and p-even cases whenever a denominator factor in B_j or B'_j becomes zero it is to be taken as unity. Such factors get cancelled in the evaluations but in the simplified notations given above such factors appear to be present.

3. Computations

The computations of the exact percentage points are carried out by using $F(u)=\int_0^u f(t)dt$ where $f(t)$ is given in (2.3). In (2.3) the only factors containing u are of the form $u^{(\cdot)}(-\log u)^{(\cdot)}$. Hence $F(u)$ is available by term by term integration.

The computation is carried out by using the series representation available from (2.3). First $F(u)$ is computed for various values of u. It is checked for monotonicity and for the conditions $F(u)\to 0$ as $u\to 0$ and $F(u)\to 1$ as $u\to 1$. Also particular cases are numerically checked. Then u is computed for various values of p, n and $F(u)$. These are given in the following tables. A seven place accuracy is kept through-

Table 1. (Values of u for $F(u)=0.01$)

n	p			
	2	3	4	5
2	0.0001000			
3	0.0100000	$0.0^{4}1118572$		
4	0.0464159	0.001763885	$0.0^{5}2202921$	
5	0.1000000	0.01115978	$0.0^{3}4415756$	$0.0^{6}5358574$
6	0.1584893	0.02995321	0.003385223	$0.0^{3}1266084$
7	0.2154435	0.05584883	0.01055743	0.001120148
8	0.2682696	0.08598411	0.02215824	0.003933051
9	0.3162278	0.1181197	0.03751544	0.009099945
10	0.3593814	0.1507436	0.05570045	0.01670364
11	0.3981072	0.1829078	0.07582779	0.02654009
12	0.4328761	0.2140516	0.09715967	0.03827068
13	0.4641589	0.2438674	0.1191214	0.05152025
14	0.4923883	0.2722082	0.1412833	0.06592995
15	0.5179475	0.2990271	0.1633328	0.08118129

n	p			
	6	7	8	9
6	$0.0^{6}1452619$			
7	$0.0^{4}3895182$	$0.0^{7}4202690$		
8	$0.0^{3}3871240$	$0.0^{4}1250161$	$0.0^{7}1269454$	
9	0.001502049	$0.0^{3}1371644$	$0.0^{5}4125409$	$0.0^{8}3953076$
10	0.003781937	$0.0^{3}5807160$	$0.0^{4}4935747$	$0.0^{5}1388016$
11	0.007458511	0.001576760	$0.0^{3}2259046$	$0.0^{4}1794217$
12	0.01260072	0.003319102	$0.0^{3}6569022$	$0.0^{4}8813789$
13	0.01915808	0.005934021	0.001468426	$0.0^{3}2729742$
14	0.02700670	0.009479613	0.002767693	$0.0^{3}6452932$
15	0.03598557	0.01395776	0.004632544	0.001278165

Table 2. (Values of u for $F(u)=0.02$)

n	p			
	2	3	4	5
2	0.00040000			
3	0.02000000	$0.0^{4}4504573$		
4	0.07368063	0.003616813	$0.0^{5}8903738$	
5	0.1414214	0.01829720	$0.0^{3}9195076$	$0.0^{5}2171395$
6	0.2091279	0.04382327	0.005669217	$0.0^{3}2666515$
7	0.2714418	0.07619017	0.01580695	0.001906283
8	0.3270243	0.1118416	0.03094026	0.005995122
9	0.3760603	0.1484000	0.04991765	0.01294247
10	0.4192288	0.1844399	0.07152412	0.02263478
11	0.4573051	0.2191682	0.09473376	0.03469096
12	0.4910168	0.2521834	0.1187587	0.04864097
13	0.5210007	0.2833185	0.1430242	0.06402267
14	0.5477974	0.3125432	0.1671258	0.08042526
15	0.5718604	0.3399049	0.1907873	0.09750324

n	p			
	6	7	8	9
6	$0.0^{6}5898295$			
7	$0.0^{4}8279095$	$0.0^{6}1709419$		
8	$0.0^{3}6675150$	$0.0^{4}2677844$	$0.0^{7}5171193$	
9	0.002323793	$0.0^{3}2391721$	$0.0^{5}8896586$	$0.0^{7}1612478$
10	0.005462578	$0.0^{3}9099562$	$0.0^{4}8691138$	$0.0^{5}3011398$
11	0.01026455	0.002308175	$0.0^{3}3579858$	$0.0^{4}3187090$
12	0.01672347	0.004630121	$0.0^{3}9731165$	$0.0^{3}1410842$
13	0.02471361	0.007982048	0.002073363	$0.0^{3}4087261$
14	0.03404687	0.01239079	0.003768095	$0.0^{3}9211545$
15	0.04451207	0.01782389	0.006127669	0.001759358

out. For higher values of p it is seen that the accuracy is being lost. Hence the tables are given only for values of p from 2 to 9. If any experimenter desires to get the tables for higher values of p and n then a Box's type approximation can be used for the computations. This is available in any standard text book on multivariate analysis, see for example Anderson [1]. In order to avoid overflow and underflow in the computer the psi, zeta and gamma functions appearing in B_j, B'_j, A_j etc. are all simplified by using the properties of these functions and then they are programmed for computations.

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Table 3. (Values of u for $F(u)=0.05$)

n	p = 2	3	4	5
2	0.0025000			
3	0.0500000	$0.0^{3}2873848$		
4	0.1357209	0.009527998	$0.0^{4}5742281$	
5	0.2236068	0.03581760	0.002498649	$0.0^{4}1410921$
6	0.3017088	0.07362451	0.01153078	$0.0^{3}7409645$
7	0.3684031	0.1164594	0.02763251	0.003989206
8	0.4248906	0.1602486	0.04917261	0.01080303
9	0.4728708	0.2028226	0.07424717	0.02120394
10	0.5139043	0.2431463	0.1012903	0.03468052
11	0.5492803	0.2808080	0.1291631	0.05055390
12	0.5800282	0.3157255	0.1570866	0.06816257
13	0.6069622	0.3479837	0.1845508	0.08693557
14	0.6307272	0.3777458	0.2112354	0.1064101
15	0.6518363	0.4052064	0.2369509	0.1262248

n	p = 6	7	8	9
6	$0.0^{5}3855067$			
7	$0.0^{3}2341692$	$0.0^{5}1122720$		
8	0.001429330	$0.0^{4}7687131$	$0.0^{6}3410719$	
9	0.004292873	$0.0^{3}5221565$	$0.0^{4}2586781$	$0.0^{6}1067517$
10	0.009178630	0.001717085	$0.0^{3}1929666$	$0.0^{5}8855519$
11	0.01612501	0.003963855	$0.0^{3}6881935$	$0.0^{4}7182631$
12	0.02497112	0.007433530	0.001703605	$0.0^{3}2757524$
13	0.03545759	0.01217595	0.003393672	$0.0^{3}7279932$
14	0.04729342	0.01815145	0.005858787	0.001534242
15	0.06019423	0.02526353	0.009146180	0.002783310

[4] on Wilks' L_{vc} criterion. The statistics L_{vc} and L_{mvc} are structurally different and also it is easy to notice that the method of Nagarsenker [4] is more involved. But from the moment structures it is evident that his method is also readily applicable here. Since his computational procedure involves Box's type approximations if his procedure is used to compute the percentage points for Wilks' L_{mvc} one can expect the points to agree with those in our table for the first 4 or 5 decimal places except for small values of p and n. For small values of p and n his points are likely to be slightly away from the exact points given in this table.

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