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MULTI-FOLDING THE NORMAL DISTRIBUTION AND MUTUAL TRANSFORMATION BETWEEN UNIFORM AND NORMAL RANDOM VARIABLES

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1. Introduction and summary

Various techniques are known for transforming uniform random variables into normal random variables. Atkinson and Pearce [1] give a good summary of methods in general use. However, when we must attach much importance to reliability of the samples, namely, goodness of fit and clearness of identity, one may certainly use the simple rejection technique. Unfortunately, the resulting algorithm is not fast and requires considerable numbers of uniform samples. The method presented in Section 5 rose out of an improvement of simple rejection technique with the idea to reclaim uniform samples to be rejected in the former methods in so far as the excellence of the simple rejection technique preserves.

We begin in Section 2 with a description of multiply folded normal density functions formed by folding the standard normal density function at fixed intervals. There we shall define a function sequence

$$\begin{aligned} S&=\{s_i(x);\ i=0, 1, 2, \cdots\} \\ &=\{\phi(x), \phi(x-2a), \phi(x+2a), \phi(x-4a), \phi(x+4a), \cdots\} \end{aligned}$$

to consider the function series

$$f_n(x)=2\sum_{i=0}^{n} s_i(x)$$

and their limit

$$f_\infty(x)=\lim_{n\to\infty} f_n(x)=2\sum_{i=0}^{\infty} s_i(x) ,$$

where $\phi(x)$ is the standard normal density function and $a>0$.

In Sections 3 and 4, several inequalities concerning $f_n(x)$ including $f_\infty(x)$ and their derived functions will be shown, which provide the theoretical endorsement of efficacy of the mutual transformation between uniform and normal random variables discussed in Section 5.

The *Normal to Uniform* transformation, described in the last section, will serve ratification of Poincaré's consideration about "Chance" [3]. One will find that a normal random variable is generated, by the *Uniform to Normal* transformation, very rapidly from always two uniform variables and the normal samples are surely very reliable as well as by simple rejection methods.

2. Multiply folded normal density functions

We shall write $\phi(x)$ to denote the standard normal density function and let

$$f_n(x) \qquad [n=0, 1, 2, \cdots]$$

be the function series given as follows;

$$f_{2m}(x)=2\sum_{t=-m}^{m}\phi(x+2ta)\ , \qquad f_{2m+1}(x)=2\sum_{t=-m-1}^{m}\phi(x+2ta)$$
$$[a>0,\ m=0, 1, 2, \cdots]\ .$$

We also apply the notation like $f_n(x|a)$ instead of $f_n(x)$ as occasion demands. The following lemma is easily obtained;

LEMMA 1.

$$\begin{aligned} f_{2m}(x)&=2\Big[\phi(x)+\sum_{t=1}^{m}(\phi(2ta-x)+\phi(2ta+x))\Big] \\ &=f_{2m-1}(x)+2\phi(2ma+x)\ , \end{aligned}$$

$$\begin{aligned} f_{2m+1}(x)&=2\sum_{t=0}^{m}[\phi(2ta+x)+\phi(2(t+1)a-x)] \\ &=f_{2m}(x)+2\phi((2m+1)a+(a-x)) \end{aligned}$$

provided that we let $f_{-1}(x)\equiv 0$ *for convenience' sake.*

COROLLARY 1.

$$f_{2m}(x)=f_{2m}(-x)\ , \qquad f_{2m+1}(a+x)=f_{2m+1}(a-x)$$

so that the derived functions

$$f_n^{(2r-1)}(x)=\frac{d^{2r-1}}{dx^{2r-1}}f_n(x) \qquad [r=1, 2, 3, \cdots]$$

satisfy

$$f_{2m}^{(2r-1)}(0)=f_{2m+1}^{(2r-1)}(a)=0\ .$$

COROLLARY 2. *Let $\tilde{f}_n(x)$ denote the probability density functions (p.d.f.'s) defined as follows;*

1) *when $x \neq 0$ and $x \neq a$,*

$$\tilde{f}_0(x) = \frac{d}{dx} \Pr [|X| \leqq x] ,$$

$$\tilde{f}_1(x) = \frac{d}{dx} \Pr [a - |a - |X|| \leqq x] ,$$

$$\tilde{f}_2(x) = \frac{d}{dx} \Pr [|a - |a - |X||| \leqq x] ,$$

$$\tilde{f}_3(x) = \frac{d}{dx} \Pr [a - |a - |a - |a - |X|||| \leqq x] ,$$

$$\cdots$$

2) $\tilde{f}_n(0) = \lim_{x \to +0} \tilde{f}_n(x)$, $\tilde{f}_n(a) = \lim_{x \to a-0} \tilde{f}_n(x)$,

where X is a random variable distributed with $N(0, 1)$. Then $f_n(x)$ and $\tilde{f}_n(x)$ are identical for $x \in [0, a]$.

These $\tilde{f}_n(x)$ are regarded as being formed by folding the standard normal p.d.f. about the vertical axis and about the axis of $x = a$ alternately, and then adding up the piled parts. $\tilde{f}_n(x)$ are to be called multiply folded normal density function and the suffix n signifies (number of times being folded)-1. We investigate the properties of $f_n(x)$ as substitute of $\tilde{f}_n(x)$ and then restore the results to $\tilde{f}_n(x)$.

LEMMA 2.

$$f_n(x \mid a) + f_n(a - x \mid a) = f_{2n+1}(x \mid a/2) .$$

The above easily verified lemma displays that the two following operations
1) folding $\phi(x)$ $2(n+1)$-times between the points $x = 0$ and $x = a/2$,
2) folding $\phi(x)$ $(n+1)$-times between $x = 0$ and $x = a$ then folding the piled part once more about the axis of $x = a/2$,

provide the same p.d.f. $\tilde{f}_{2n+1}(x \mid a/2)$.

The existence of $f_\infty(x) = \lim_{n \to \infty} f_n(x)$ follows from the fact that the series

$$\sum_{t=0}^{\infty} \phi(2ta + x) = \lim_{m \to \infty} \sum_{t=0}^{m} \phi(2ta + x)$$

converge. The following theorem gives an explicit form of $f_\infty(x)$.

THEOREM 1.

$$f_\infty(x)=\frac{1}{a}\left[1+2\sum_{k=1}^{\infty}\exp\left(-\pi^2k^2/(2a^2)\right)\cos\left(\pi kx/a\right)\right] .$$

PROOF. We can write

$$\begin{aligned} f_\infty(x)&=2\sum_{t=-\infty}^{\infty}\phi(2ta+x)\\ &=\sqrt{\frac{2}{\pi}}\exp\left(-x^2/2\right)\cdot\sum_{t=-\infty}^{\infty}\exp\left(-2t^2a^2\right)\cdot\exp\left(-2tax\right)\\ &=\sqrt{\frac{2}{\pi}}\exp\left(-x^2/2\right)\left[1+2\sum_{t=1}^{\infty}\exp\left(-2t^2a^2\right)\cosh\left(2tax\right)\right] \end{aligned}$$

which is rewritten with a theta function, one of elliptic functions of the third kind, (q.v. [4])

$$\theta_3(\xi\,|\,\tau)=1+2\sum_{t=1}^{\infty}\exp\left(i\pi\tau t^2\right)\cos\left(2\pi t\xi\right)$$

as follows;

$$\begin{aligned} f_\infty(x)&=\sqrt{\frac{2}{\pi}}\exp\left(-x^2/2\right)\theta_3\left(\frac{a}{\pi}ix\,\middle|\,\frac{2a^2}{\pi}i\right)\\ &=\sqrt{\frac{2}{\pi}}\exp\left(-x^2/2\right)\cdot\sqrt{\frac{\pi}{2}}\frac{1}{a}\exp\left(x^2/2\right)\theta_3\left(\frac{x}{2a}\,\middle|\,\frac{\pi}{2a^2}i\right)\\ &\qquad\qquad\text{(Jacobi's imaginary transformation)}\\ &=\frac{1}{a}\left[1+2\sum_{k=1}^{\infty}\exp\left(-\pi^2k^2/(2a^2)\right)\cos\left(\pi kx/a\right)\right] . \end{aligned}$$

COROLLARY 3. *$f_\infty(x)$ is a periodic function of fundamental period $2a$ and Theorem 1 gives the Fourier series of $f_\infty(x)$.*

COROLLARY 4. *The following equations are easily obtained from Corollary 1 and Lemma 2 adding a reaffirmation through Theorem 1.*

1) $f_\infty(x)=f_\infty(-x)$,
2) $f_\infty(a+x)=f_\infty(a-x)$,
3) $f_\infty^{(2r-1)}(0)=f_\infty^{(2r-1)}(a)=0\ [r=1,2,3,\cdots]$,
4) $f_\infty(x\,|\,a)+f_\infty(a-x\,|\,a)=f_\infty(x\,|\,a/2)$.

COROLLARY 5. *$\tilde{f}_\infty(x)=\lim_{n\to\infty}\tilde{f}_n(x)$ is in existence being a p.d.f. over $[0,a]$, since*

$$\int_0^a f_\infty(x)dx=1 .$$

3. Some properties of $f_\infty(x)$

In this section, we shall be interested in several inequalities concerning $f_\infty(x)$ and its derived functions.

First, we shall prove the lemma which provides signum of $f_\infty^{(2r-1)}(x)$, the $(2r-1)$th derived function of $f_\infty(x)$.

LEMMA 3.

$$(-1)^r f_\infty^{(2r-1)}(x)>0 \qquad [r=1, 2, 3, \cdots],$$

if $0<x<a\leqq\pi(3/(2(2r+1)\ln 2))^{1/2}$.

PROOF. (1) When $a/2\leqq x<a$; concerning $h(k)=k^{2r}\exp(-c^2k^2)$ $[c^2=\pi^2/(2a^2)]$, we can obtain from the given condition that

$$[h(2)/h(1)]=2^{2r}\exp(-3c^2)\leqq 1/2$$

and

$$\exp(-2c^2)\leqq 2^{-2(2r+1)/3}\leqq 1/4 .$$

Hence for $k\geqq 1$ we have

$$[h(k+2)/h(k+1)]=[k(k+2)/(k+1)^2]\exp(-2c^2)[h(k+1)/h(k)]$$
$$<\frac{1}{4}[h(k+1)/h(k)]\leqq\frac{1}{8} .$$

Now we let $y=a-x$ then we have $0<y\leqq a/2$ and

$$\sin\frac{\pi}{a}x\geqq\frac{2}{a}y , \qquad \sin\frac{k\pi}{a}x\geqq-\frac{k\pi}{a}y .$$

Therefore

$$(-1)^r f_\infty^{(2r-1)}(x)=\frac{2\pi^{2r-1}}{a^{2r}}\sum_{k=1}^{\infty}k^{2r-1}\exp(-c^2k^2)\sin\left(\frac{\pi k}{a}x\right)$$
$$>\frac{2\pi^{2r}}{a^{2r}}y\left[\frac{2}{\pi}h(1)-\sum_{k=2}^{\infty}h(k)\right]$$
$$>\frac{2\pi^{2r}}{a^{2r}}y\left[\frac{2}{\pi}h(1)-h(2)\sum_{k=0}^{\infty}\left(\frac{1}{8}\right)^k\right]>0 .$$

(2) When $0<x<a/2$; we can write x with integer $m\geqq 1$ and $0<y\leqq 2^{-m-1}a$ as

$$x=2^{-m}a-y .$$

Now, supposing that

$$(-1)^r f_\infty^{(2r-1)}(x) \leqq 0$$

then we have, since

$$(-1)^r f_\infty^{(2r-1)}(a-x) < 0$$

from (1), that

$$(-1)^r f_\infty^{(2r-1)}(x \mid 2^{-1}a) = (-1)^r [f_\infty^{(2r-1)}(x \mid a) + f_\infty^{(2r-1)}(a-x \mid a)]$$
$$< 0 \qquad \text{[Corollary 4]} .$$

Similarly from

$$(-1)^r f_\infty^{(2r-1)}(2^{-s}a - x \mid 2^{-s}a) < 0 \qquad [s=0, 1, \cdots, m-1]$$

we obtain

$$(-1)^r f_\infty^{(2r-1)}(x \mid 2^{-m}a) < 0$$

in consequence, which is inconsistent with (1). Therefore we have the lemma also in this case.

COROLLARY 6. *For any* $a > 0$ *and* $0 < x < a$, *it hold that*

$$f_\infty'(x) < 0$$

and

$$f_\infty(0) > 1/a > f_\infty(a) .$$

PROOF. If $a \leqq 2$, the first inequality of the corollary is identical with Lemma 3, hence we assume $a > 2$. Then, the second derived function of $\phi(x)$ satisfies

$$\phi''(x) > 0 \qquad \text{if } x \geqq a/2 > 1 ,$$

so that, if $1 \leqq x < a$, it follows that $f_\infty''(x) > 0$ which implies $f_\infty'(x) < 0$ since $f_\infty'(a) = 0$ [Corollary 4].

When $0 < x < 1 < a/2$, the proof is done similarly as in (2) of the proof of Lemma 3.

The second inequalities are simple consequences of the first.

COROLLARY 7. *For* $0 < x < a \leqq \sqrt{3+\sqrt{6}}$

$$f_\infty'''(x) > 0$$

and

$$f_\infty''(x) < 0 < f_\infty''(a) .$$

PROOF. Notice that

$$\phi^{(iv)}(x)<0 , \qquad \text{if } \sqrt{3-\sqrt{6}}<x<\sqrt{3+\sqrt{6}}$$

with $2\sqrt{3-\sqrt{6}}<\pi\sqrt{3/(10\ln 2)}$, moreover for the second inequality that

$$\int_0^a f''_\infty(x)dx=0 .$$

LEMMA 4. *The least upperbound of deviation of $a\cdot f_\infty(x)$ from its mean* $(=1)$,

$$\varepsilon(a)=\min|a\cdot f_\infty(x)-1| ,$$

satisfies

$$\varepsilon(a)=2\sum_{k=1}^{\infty}\exp(-\pi^2k^2/(2a^2))<\frac{2\exp(-\pi^2k^2/(2a^2))}{1-\exp(-3\pi^2/(2a^2))} .$$

PROOF. It can be easily obtained that

$$a\cdot f_\infty(0)-1>1-a\cdot f_\infty(a)>0 .$$

Hence, from the fact that

$$a\cdot f_\infty(0)\geqq a\cdot f_\infty(x)\geqq a\cdot f_\infty(a)$$

we have

$$\varepsilon(a)=a\cdot f_\infty(0)-1=2\sum_{k=1}^{\infty}\exp(-\pi^2k^2/(2a^2))$$

which is bounded from above by the geometrical series

$$\varepsilon(a)<2\sum_{k=1}^{\infty}\exp(-\pi^2(3k-2)/(2a^2))=\frac{2\exp(-\pi^2/(2a^2))}{1-\exp(-3\pi^2/(2a^2))} .$$

$\varepsilon(a)$ and $\varepsilon(a/2)$ are so small [see Table 1] that the figure of $\tilde{f}_\infty(x)$

Table 1 Bound of deviation $|a\cdot f_\infty(x)-1|$

a	$\varepsilon(a)$	$\varepsilon(a/2)$
1	$<1.5\times10^{-2}$	$<5.4\times10^{-9}$
1/2	$<5.4\times10^{-9}$	$<1.1\times10^{-34}$
1/3	$<1.1\times10^{-19}$	$<1.5\times10^{-77}$
1/4	$<1.1\times10^{-34}$	$<1.4\times10^{-137}$
1/5	$<5.3\times10^{-54}$	$<9.7\times10^{-215}$
1/8	$<1.4\times10^{-137}$	$<4.5\times10^{-549}$
1/16	$<4.5\times10^{-549}$	$<5.2\times10^{-2195}$

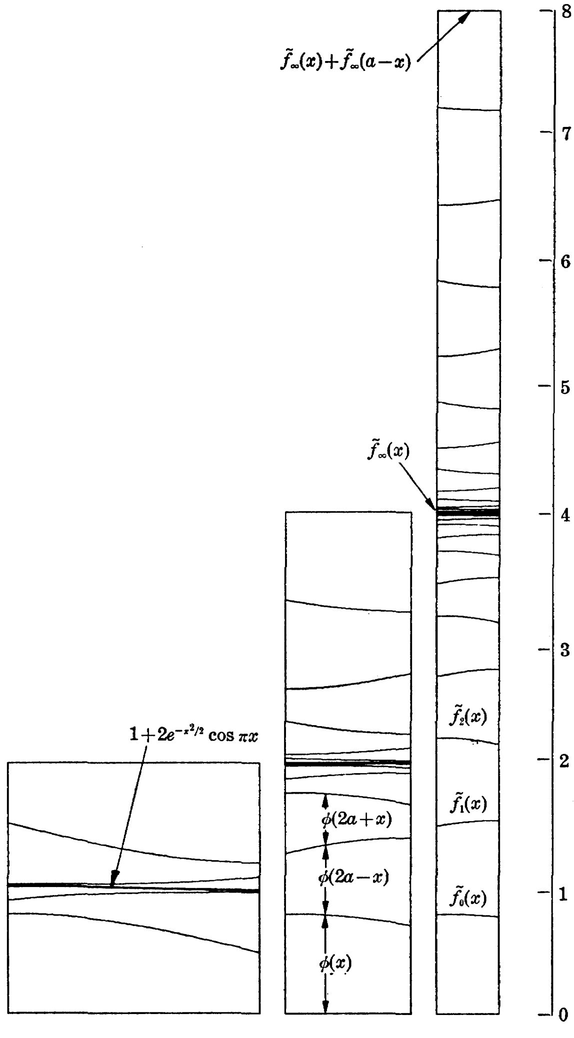


Fig. 1 Folded normal density functions.

and $\tilde{f}_\infty(x)+\tilde{f}_\infty(a-x)$ appear to be just rectangles [see Fig. 1].

In addition to Lemma 4, we can prove other inequalities such as

$$|a\cdot f_\infty(x)-[1+2\exp(-\pi^2/(2a^2))\cos(\pi x/a)]|<\frac{2\exp(-2\pi^2/a^2)}{1-\exp(-5\pi^2/(2a^2))}$$

and so forth.

4. Inequalities concerning $f_n(x)$ and their derived functions

We have mentioned, in Section 2, some equalities concerning $f_n(x)$

and their derived functions. In this section we shall discuss inequalities regarding them.

LEMMA 5. *Let $f_n''(x)$ denote the second derived function of $f_n(x)$. Then the following inequalities hold for $0<x<a$;*

$$f_0''(x)>f_1''(x)>\cdots>f_{M-2}''(x)>f_{M-1}''(x)$$

and

$$f_M''(x)<f_{M+1}''(x)<\cdots<f_\infty''(x)$$

where M is the largest integer smaller than $1/a$.

PROOF. This lemma is easy to prove. Only to notice the sign of $[f_n''(x)-f_{n-1}''(x)]$ may suffice for it.

The preceding theorem and lemmas leads to an important property concerning the first and second derived functions of $f_n(x)$, which is summarized by the following theorem.

THEOREM 2. *Let $p=\sqrt{\pi^2-1/e}$ $[\simeq 3.08]$, $\rho(a)=p/a^2-3$ and let K be the largest odd integer not exceeding $\rho(a)$. Then, if $0<a\leqq\sqrt{p/6}$ $[\simeq 0.71]$, there exists an odd integer $N\geqq K$ such that,*

for $n=0, 1, 2,\cdots, N$,

1) $f_n''(x)<0 \qquad [0\leqq x\leqq a]$,

2) $f_n'(x)<0 \qquad [0<x\leqq a]$ *if n is even* ,
$f_n'(x)>0 \qquad [0\leqq x<a]$ *if n is odd* .

To prove Theorem 2, it suffices to show that the second derived function of $f_n(x)$ is negative in the range $0\leqq x\leqq a$. Concerning 2), we can obtain the inequalities easily through Corollary 1 $[f_{2m}'(0)=f_{2m+1}'(a)=0;\ m=0, 1, 2,\cdots]$ when we have proved 1).

Now prior to it, we shall demonstrate a lemma to assist the proof of the theorem.

LEMMA 6. *For any integer r which satisfies $2ra\geqq\sqrt{3+\sqrt{6}}$ and $0\leqq x\leqq a$, the following inequality holds;*

$$f_\infty''(x)-f_{2r-1}''(x)>(32/\sqrt{3\pi})(r+1)^2a^2\exp[-2(r+1)^2a^2] .$$

PROOF. On the given condition the fourth derived function of $\phi(x)$,

$$\phi^{(iv)}(x)=(x^4-6x^2+3)\phi(x) ,$$

satisfies

$$\phi^{(iv)}(2ra+y)>0 \qquad \text{if } y>0$$

which is followed by

$$\phi''(2ra+x)+\phi''((2r+1)a+(a-x))>2\phi''((2r+1)a)$$

and

$$\phi''((2r+1)a)+\phi''((2r+3)a)>2\phi''(2(r+1)a) .$$

Therefore the left-hand side of the lemma is bounded from below as

$$\begin{aligned} f''_\infty(x)-f''_{2r-1}(x) &> 4\sum_{s=r}^{\infty}\phi''((2s+1)a) \\ &> 8\phi''(2(r+1)a) \\ &= 4\sqrt{2/\pi}\,[4(r+1)^2a^2-1]\exp[-2(r+1)^2a^2] . \end{aligned}$$

But since $4(r+1)^2a^2>3+\sqrt{6}$, the bound is evaluated as follows;

$$\begin{aligned} &> 4\sqrt{2/\pi}\,[1-1/(3+\sqrt{6})]\cdot 4(r+1)^2a^2\exp(-2(r+1)^2a^2) \\ &= (32/\sqrt{3\pi})(r+1)^2a^2\exp(-2(r+1)^2a^2) . \end{aligned}$$

PROOF OF THEOREM 2. From Corollary 7 we have for $0\leqq x\leqq a\leqq \sqrt{p/6}<\sqrt{3+\sqrt{6}}$

$$\begin{aligned} f''_\infty(x)\leqq f''_\infty(a) &= (2\pi^2/a^3)\sum_{k=1}^{\infty}(-1)^{k-1}k^2\exp(-\pi^2k^2/(2a^2)) \\ &< (2\pi^2/a^3)\exp(-\pi^2/(2a^2)) . \end{aligned}$$

Now if $\sqrt{p/8}<a\leqq\sqrt{p/6}$, K is equal to 3 and we easily obtain

$$(K+1)a=4a>4\sqrt{p/8}>\sqrt{3+\sqrt{6}} .$$

On the other hand, if $0<a\leqq\sqrt{p/8}$, then we have

$$\begin{aligned} (K+1)a>(\rho(a)-1) &= p/a-4a \\ &\geqq p/\sqrt{p/8}-4\sqrt{p/8} \\ &= 4\sqrt{p/8}>\sqrt{3+\sqrt{6}} . \end{aligned}$$

Hence for $0<a\leqq\sqrt{p/6}$, we can apply Lemma 6 with $r=(K+1)/2$ to obtain

$$\begin{aligned} f''_K(x) &= f''_\infty(x)-[f''_\infty(x)-f''_K(x)] \\ &< (2\pi^2/a^3)\exp(-\pi^2/(2a^2))-(8/\sqrt{3\pi})(K+3)^2a^2\exp[-(K+3)^2a^2/2] . \end{aligned}$$

Since $(K+3)^2a^2\leqq p^2$ and the second term decreases as $u=(K+3)a$ $[>p/a-2a\geqq 4\sqrt{p/6})]$ increases, the left-hand side of the above inequality is bounded by

$$f_K''(x) < a^{-3}[2\pi^2 - (8/\sqrt{3\pi})p^2 a \exp(-1/(2ea^2))] \exp(-\pi^2/(2a^2))$$
$$< a^{-3}[2\pi^2 - (8/\sqrt{3\pi})p^2] \exp(-\pi^2/(2a^2)) < 0 .$$

Thus we have proved that $f_K(x)$ is strictly concave for $x \in [0, a]$. It follows that there exists $N \geqq K$ and $f_n(x)$ $[n=0, 1, 2, \cdots, N;\ 0 \leqq x \leqq a]$ are also concave through Lemma 5 and Corollary 7 $[f_\infty''(a) > 0]$.

COROLLARY 8. *For* $0 \leqq m < n \leqq N$, $0 \leqq x \leqq a$ *and* $0 \leqq y \leqq a$,

$$0 < f_m(x) < f_n(y) < 1/a .$$

PROOF. It is easy to show $f_m(x) < f_n(y)$ from 2) of Theorem 2, adding

$$f_n(y) \leqq f_N(a) < f_\infty(a) < 1/a .$$

To supplement Theorem 2, we shall prove the following lemma which is to be used in the next section.

LEMMA 7. *For any even integer* $2r = 0, 2, 4, \cdots$,

$$f_{2r}(0) > f_{2r}(a) .$$

PROOF. (1) If a is not exceeding $\sqrt{p/6}$, the lemma holds for $2r < N$ from Theorem 2. Then we only consider the case that $2r > N$.
(2) Otherwise, we consider $2r \geqq 2$ since it is obvious that $f_0(0) > f_0(a)$.

In both cases we get $2ra > 1$, so that $\phi''(na + x) > 0$ for $n \geqq 2r$ and $x \geqq 0$. From Lemma 1 and Corollary 6 we have

$$f_{2r}(0) - f_{2r}(a) = [f_\infty(0) - f_\infty(a)] + 2 \sum_{s=r+1}^{\infty} [\phi(2sa - a) + \phi(2sa + a) - 2\phi(2sa)]$$
$$> 0 .$$

5. Mutual transformation between uniform and normal variables

Normal to Uniform

Let X's be independent normal variables with mean μ and variance σ^2. We define random variable U as follows:

(1) $U = \mathrm{mod}\,1(X, a) = X - a \cdot \mathrm{floor}\,(X/a)$.

(2) $U = \mathrm{mod}\,2(X, a) + a \cdot \mathrm{neg}\,(X)$
$\quad = X - a \cdot \mathrm{trunc}\,(X/a) + a \cdot [\text{if } X < 0 \text{ then } 1, \text{ else } 0]$
$\quad [= \mathrm{mod}\,1(X, a) \text{ for } U \in (0, a)]$

where floor (x) represents the largest integer not exceeding x and trunc $(x) = \mathrm{sign}\,(x) \cdot \mathrm{floor}\,(|x|)$. The function mod $1(x, y)$ corresponds to the builtin function MOD(x, y) in PL/I and ALGOL, and also to the residue operation $y|x$ in APL. On the other hand, mod 2(x, y) corresponds to AMOD(x, y) in FORTRAN.

For both definition, we obtain that $0 \leqq U \leqq a$ and from Theorem 1 that

$$\begin{aligned}
\Pr[U \leqq u] &= \sum_{n=-\infty}^{\infty} \Pr[0 \leqq x \leqq u \mid X = na + x] \\
&= \frac{1}{\sigma} \sum_{n=-\infty}^{\infty} \int_0^u \phi\left(\frac{na + x - \mu}{\sigma}\right) dx \\
&= \int_0^u \frac{1}{\sigma} \sum_{n=-\infty}^{\infty} \phi\left(\frac{a}{\sigma} n + \frac{x-\mu}{\sigma}\right) dx \\
&= \int_0^u \frac{1}{2\sigma} f_\infty\left(\frac{x-\mu}{\sigma} \,\middle|\, \frac{a}{2\sigma}\right) dx \,.
\end{aligned}$$

Therefore U has the probability density $(1/2\sigma) f_\infty((x-\mu)/\sigma \mid a/2\sigma))$ provided $0 \leqq x \leqq a$. If we may neglect the maximum approximation error $(1/a) \cdot \varepsilon(a/2\sigma)$, we can say that U has the uniform distribution over $[0, a]$. The affinity between the two distribution is bounded from below by $1 - 1/2 \cdot \varepsilon(a/2\sqrt{2}\,\sigma)$, since

$$\begin{aligned}
\frac{1}{a} \int_{-\mu}^{a-\mu} (1 + 2\Sigma)^{1/2} dx > \frac{1}{a} \int_{-\mu}^{a-\mu} (1 + \Sigma - 2\Sigma^2) dx \\
= 1 - \sum_{k=1}^{\infty} e^{-4\pi^2\sigma^2k^2a^{-2}} \\
= 1 - \frac{1}{2} \varepsilon\left(\frac{a}{2\sqrt{2}\,\sigma}\right) ,
\end{aligned}$$

where

$$\Sigma = \sum_{k=1}^{\infty} e^{-2\pi^2\sigma^2k^2a^{-2}} \cos 2\pi k a^{-1} x \,.$$

See [2] as to the notion of "affinity".

This result is one which ratifies Poincaré's consideration about "chance" [3] citing the minor planets on the Zodiac and the game of roulette. That is to say, the unit interval a may be regarded as the circumference of the Zodiac or the roulette wheel and imagine the Creator has thrown the minor planets, or a operator spins balls, imparting to them different initial speeds according to a normal distribution.

Uniform to Normal

The random variable X determined by the following theorem can be regarded to be normally distributed with mean zero and unit variance.

THEOREM 3. *Let U, V be independent uniform random variables* *over* $(0, 1)$ *and let* $u=|2U-1|a$ *and if* $U\geqq 1/2$ *then set* $s=1$ *else set* $s=-1$. *Moreover, for* $n=-1, 0, 1, \cdots, N$, *let* $g_n(x)=af_n(x)$ *or* $af_n(a-x)$ *according as* n *is even or odd. Now put* X *for each* $n\geqq 0$ *as*

(1) $X=s(na+u)$ *if* $g_{n-1}(0)\leqq V<g_n(a)$,

(2) *if* $g_n(a)\leqq V<g_n(0)$

(2a) $X=s(na+u)$ *if* $V<g_n(u)$,

(2b) $x=s[(n+2)a-u]$ *otherwise*,

(3) $X=X^*(u)$ *if* $g_N(x)\leqq V$;

then the distribution function of the random variable X *is identical with the standard normal distribution function,* $\Phi(x)$, *in the range* $|x|<R=(N+1)a$, *where* $X^*(x)$ *is an arbitrary function which satisfies* $X^*(x)>R$ *for* $0<x<a$.

PROOF. It is easily obtained that $g_n(x)$ is monotone decreasing (Theorem 2) and greater than $g_m(y)$ in the case that $n>m$ and $x, y\in[0, a]$ but always smaller than one (Corollary 8). Therefore the conditional expressions (1)–(3) include all the possible cases and are exclusive each other.

We can gather (1) and (2a) as

$$X=s(na+u) \qquad \text{if } g_{n-1}(0)\leqq V<g_n(u) .$$

Consider (3) and we have;

(i) For $2ra\leqq x\leqq(2r+1)a$ $[r=0, 1, \cdots, (N-1)/2]$:

$$\begin{aligned}\Pr[X\leqq x]=&\Pr[X\leqq 2ra]+\Pr[2ra+u\leqq x,\ g_{2r-1}(0)\leqq V<g_{2r}(u)]\\ &+\Pr[(2r+1)a-u\leqq x,\ g_{2r-1}(u)\leqq V<g_{2r-1}(0)] .\end{aligned}$$

Since both u and $a-u$ are uniform over $(0, a)$, to exchange them each other exerts no influence on the value of the probability. Hence we can rewrite the above expression as follows;

$$\begin{aligned}\Pr[X\leqq x]&=\Pr[X\leqq 2ra]+\Pr[2ra+u\leqq x,\ g_{2r-1}(a-u)\leqq V<g_{2r}(u)]\\ &=\Pr[X\leqq 2ra]+\int_0^{x-2ra}[af_{2r}(u)-af_{2r-1}(u)]\frac{1}{2a}du\\ &=\Pr[X\leqq 2ra]+\int_{2ra}^{x}\phi(u)du\\ &=\Pr[X\leqq 2ra]+\Phi(x)-\Phi(2ra) .\end{aligned}$$

(ii) For $(2r-1)a\leqq x\leqq 2ra$ $[r=1, 2, \cdots, (N+1)/2]$: similarly to (i) we have

$$\begin{aligned}\Pr[X\leqq x]&=\Pr[X\leqq(2r-1)a]+\Pr[2ra-u\leqq x,\ g_{2r-2}(u)\leqq V<g_{2r-1}(a-u)]\\ &=\Pr[X\leqq(2r-1)a]-\int_x^{(2r-1)a}\phi(u)du\\ &=\Pr[X\leqq(2r-1)a]-\Phi[(2r-1)a]+\Phi(x) .\end{aligned}$$

Since the probability density of X is obviously symmetrical about zero, it holds that

$$\Pr[X \leqq 0] = 1/2 = \Phi(0) ,$$

so that we have the result

$$\Pr[X \leqq x] = \Phi(x) \qquad \text{for } -R \leqq x \leqq R = (N+1)a .$$

Generation methods based on this theorem must hold some convenient properties to put to practical use for faster generation of normal variables.

(a) If a is enough small [say $a \leqq 1/2$] the random variable X can be regarded as exactly normal except the probability as small as impossible. Table 2 shows the normal range and the probability to overflow.

Table 2 Range where normality holds

a	K ($\leqq N$)	$(K+1)a$ [$\leqq R$]	$2\Pr[X>R]$
1/2	9	5.0	$<10^{-6}$
1/3	23	8.0	$<10^{-14}$
1/4	47	12.0	$<10^{-32}$
1/5	73	14.8	$<10^{-48}$
1/8	193	24.25	$<10^{-129}$
1/16	785	49.125	$<10^{-526}$

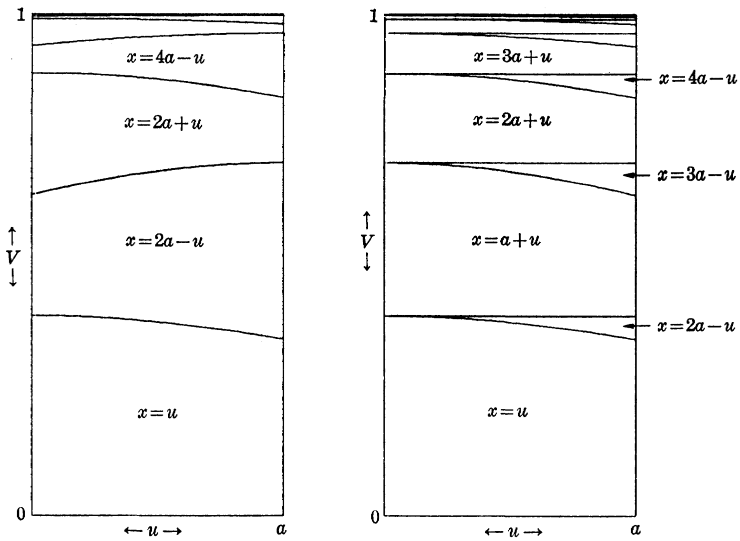
(b) Only two uniform random variables are required to generate a normal variable on all occasions. If a few bits loss of accuracy is allowed, it is possible to frame generating procedures which require less uniform samples towards 1. This property is favorable particularly to systems with rather "expensive" supply of uniform samples.

(c) When X is determined by (1) of Theorem 3, the generation speed is very fast only with simple comparisons with constants. And the probability that we must compute $g_n(u)$, namely, that X is determined by (2) of Theorem 3, is very small as discussed in the following.

(d) Since $g_n(x)$ are symmetrical about $x=0$, the polynomial expansions of them contain no term of odd degree. Therefore we can obtain the sufficient approximation of $g_n(x)$ with just a few terms.

(e) The principle of generation is illustrated in Fig. 2. As one may gasp the general idea easily, this method can be called *rejection-free rejection method* or *selection method*. The least significant digits of X's are uniform and equal to u's which is important for sampling experiments taking the nature of "tail" as one of main concerns.

The concrete procedure of this method will be published in near



(1) Basic idea (2) Revised form

Fig. 2 Principle of the generation method.

future. In conclusion we will discuss the remaining problem of (c).

THEOREM 4. *The probability that X is determined by* (2) [(2a) *or* (2b)] *of Theorem* 3 *is less than* $a/\sqrt{2\pi}$.

PROOF. Let P denote the probability concerned. We can write it as

$$\begin{aligned} P &= \sum_{n=0}^{N} \Pr\,[g_n(a) \leqq V < g_n(0)] \\ &= \sum_{n=0}^{N} [g_n(0) - g_n(a)] \\ &= g_0(0) + \sum_{n=1}^{N} [g_n(0) - g_{n-1}(a)] - g_N(a) \\ &= 2a \sum_{n=0}^{N} \phi(na) - g_N(a) \\ &= a\phi(0) + [g_{N-1}(0) + g_N(0)]/2 - g_N(a)\ . \end{aligned}$$

From Lemma 7 we have, since N is an odd integer,

$$g_{N+1}(a) - g_{N+1}(0) < 0\ ,$$

which implies that

$$\begin{aligned} g_N(0) - g_N(a) &= [g_{N+1}(a) - 2a\phi((N+2)a)] - [g_{N+1}(0) - 2a\phi((N+1)a)] \\ &< 2a\phi((N+1)a) = g_N(a) - g_{N-1}(0)\ . \end{aligned}$$

Thus we obtain that

$$P = a\phi(0) + [g_N(0) - g_N(a)]/2 - [g_N(a) - g_{N-1}(0)]/2$$
$$< a\phi(0) = a/\sqrt{2\pi}\ ,$$

which completes the proof.

Note. Since $g_n(x)$ are concave on $(0, a)$, the diagonal lines $(0, g_n(0))$–$(a, g_n(a))$ do not intersect $g_n(x)$ except on their both ends. Therefore the probability that we are obliged to compute $g_n(x)$ is reduced to $a/(2\sqrt{2\pi})$. Moreover, using quadratic approximation of them may make the probability negligible small.

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CORRECTIONS TO "MULTI-FOLDING THE NORMAL DISTRIBUTION AND MUTUAL TRANSFORMATION BETWEEN UNIFORM AND NORMAL RANDOM VARIABLES"

NAOTO NIKI

(This Annals 31 (1979), 125–140)

Page	Line	Error	Correct
129	15	$[k(k+2)/(k+1)^2]$	$[k(k+2)/(k+1)^2]^{2r}$
131	6	min	max
	8	$\dfrac{2\exp(-\pi^2k^2/(2a^2))}{1-\exp(-3\pi^2/(2a^2))}$	$\dfrac{2\exp(-\pi^2/(2a^2))}{1-\exp(-3\pi^2/(2a^2))}$
132	Fig. 1	$\phi(2a+x)$	$2\phi(2a+x)$
		$\phi(2a-x)$	$2\phi(2a-x)$
		$\phi(x)$	$2\phi(x)$
134	3 ↑	$(K+3)^2a^2\leqq p^2$	$(K+3)^2a^4\leqq p^2$
137	7	(2b) $x=$	(2b) $X=$
	8	$g_N(x)\leqq V$	$g_N(0)\leqq V$

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