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EXACT ROBUSTNESS STUDIES OF THE TEST OF INDEPENDENCE BASED ON FOUR MULTIVARIATE CRITERIA AND THEIR DISTRIBUTION PROBLEMS UNDER VIOLATIONS

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Summary

Exact robustness studies against non-normality have been carried out for test of independence based on the four multivariate criteria: Hotelling's trace, $U^{(p)}$, Pillai's trace, $V^{(p)}$, Wilks' criterion, $W^{(p)}$, and Roy's largest root, $L_{(p)}$. The density functions of $U^{(p)}$, $W^{(p)}$ and $L_{(p)}$ have been obtained in the canonical correlation case and further the moments of $U^{(p)}$ and m.g.f. of $V^{(p)}$ have been derived. All of the study is based on Pillai's distribution of the characteristic roots under violations. Numerical results for the power function have been tabulated for the two-roots case. Slight non-normality does not affect the independence test seriously. $V^{(2)}$ is found to be most robust against non-normality.

1. Introduction

Let $\boldsymbol{S}_1$ $(p\times p)$ have a non-central Wishart distribution $W(p, n_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Omega})$ and $\boldsymbol{S}_2$ $(p\times p)$, an independent central Wishart distribution $W(p, n_2, \boldsymbol{\Sigma}_2, \boldsymbol{0})$. Pillai [7] has derived the joint density of the latent roots $r_1, r_2, \cdots, r_p$, of $\boldsymbol{S}_1\boldsymbol{S}_2^{-1}$ under violations, having an assumption on $\boldsymbol{\Sigma}_1^{1/2}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_1^{1/2}$, in the form:

$$(1.1) \quad C(p, m, n) \exp(-\operatorname{tr} \boldsymbol{\Omega})|\boldsymbol{\Lambda}|^{-n_1/2}|\boldsymbol{R}|^m|\boldsymbol{I}+\lambda\boldsymbol{R}|^{-(n_1+n_2)/2}\prod_{i>j}(r_i-r_j) \cdot \sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}C_\kappa\{\lambda\boldsymbol{R}(\boldsymbol{I}+\lambda\boldsymbol{R})^{-1}\}F_p\,, \qquad 0<r_1<\cdots<r_p<\infty\,,$$

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where F_p denotes the function of matrix arguments of order p defined by

$$(1.2)\qquad F_p=\sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta(-\lambda^{-1}\boldsymbol{\Lambda}^{-1})L_\delta^m(\boldsymbol{\Omega})}{(n_1/2)_\delta C_\delta(\boldsymbol{I})C_\delta(\boldsymbol{I})}\ ,$$

$$(1.3)\qquad C(p, m, n)=\frac{\pi^{p/2}\prod_{i=1}^{p}\Gamma((2m+2n+p+i+2)/2)}{\prod_{i=1}^{p}\{\Gamma((2m+i+1)/2)\Gamma((2n+i+1)/2)\Gamma(i/2)\}}\ ,$$

$m=(n_1-p-1)/2$, $n=(n_2-p-1)/2$, $\lambda>0$, $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_p)$, $\lambda_1,\cdots,\lambda_p$ being the latent roots of $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$ and $\boldsymbol{R}=\mathrm{diag}(r_1,\cdots,r_p)$, $C_\kappa(S)$ is the zonal polynomial [1] of degree k of the symmetric matrix S corresponding to the partition $\kappa=(k_1,\cdots,k_p)$ of k such that $k_1\geqq k_2\geqq\cdots\geqq k_p\geqq 0$ and $k=k_1+\cdots+k_p$. The generalized Laguerre polynomial $L_\kappa^\gamma(S)$ is defined in Equation (14) of [2] and $a_{\kappa,\delta}$ are constants defined in Equation (20) of [2].

The density (1.1) has been derived under the assumption that $\boldsymbol{\Sigma}_1^{1/2}\cdot\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_1^{1/2}$ is partial random, denoted "random". Here "random" implies diagonalization by an orthogonal transformation $\boldsymbol{H}$ and integration over $\boldsymbol{H}$; in other words putting a Haar prior on $\boldsymbol{H}$ leaving the latent roots non-random. The density (1.1) has been useful to study 1) the robustness against non-normality in testing $\boldsymbol{\Sigma}_1=\boldsymbol{\Sigma}_2$ in two p-variate normal populations and 2) against the violation of the assumption of a common covariance matrix in MANOVA. Pillai and Sudjana [8] have carried out a numerical study of 1) and 2) using (1.1) for $p=2$ based on the following four criteria:

(i) Hotelling's trace $U^{(p)}=\sum_{i=1}^{p}r_i$,

(ii) Pillai's trace $V^{(p)}=\sum_{i=1}^{p}\{r_i/(1+r_i)\}$,

(iii) Wilks' criterion $W^{(p)}=\prod_{i=1}^{p}(1+r_i)^{-1}$, and

(iv) Roy's largest root r_p.

In this paper, an attempt is made to study 3) the robustness against non-normality of the test of independence between a p-set and a q-set in a $(p+q)$-variate normal population based on each of the above criteria. The motivations of these exact studies of robustness are discussed in Pillai [7].

For example, for 3), denoting $L(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2, \boldsymbol{\Omega})=W(n_1, p, \boldsymbol{\Sigma}_1, \boldsymbol{\Omega})W(n_2, p, \boldsymbol{\Sigma}_2)$,

$$(1.4)\qquad L(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2, \boldsymbol{\Omega})=L(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Omega})|\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}|^{n_2/2}{}_0F_0\left(\frac{1}{2}(\boldsymbol{\Sigma}_1^{-1}-\boldsymbol{\Sigma}_2^{-2})\boldsymbol{S}_2\right)\ ,$$

where ${}_0F_0(\boldsymbol{T})$ denotes the hypergeometric function of the matrix variate $\boldsymbol{T}$. Now $L(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Omega})$ leads to the distribution of the sample canonical correlation coefficients in the normal non-central case when $\boldsymbol{\Omega}$ is made random, and expression in (1.4) equals $L(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Omega})$ when $\boldsymbol{\Sigma}_1=\boldsymbol{\Sigma}_2$. If $\boldsymbol{\Sigma}_1\neq\boldsymbol{\Sigma}_2$, the series ${}_0F_0$ leads to a non-normal situation.

It may be observed that non-normality occurs in the study due to a) heterogeneity of covariance matrices and b) kurtosis, [6]. The heterogeneity is involved in view of unequal covariance matrices in (1.4). Kurtosis is caused by the introduction of the "random" approach on $\boldsymbol{\Sigma}_1^{1/2}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_1^{1/2}$. For example, using the "random" approach on $\boldsymbol{\Sigma}$ in the Wishart distribution $W(p, n, \boldsymbol{\Sigma}, \boldsymbol{S})$ we get

$$(1.5)\quad \mathrm{E}\,(\exp(-t\,\mathrm{tr}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{S})) =\exp(-\mathrm{tr}\,\boldsymbol{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}\int_{0(p)}\frac{C_\kappa[(\boldsymbol{I}+2t\boldsymbol{H}\boldsymbol{D}_r\boldsymbol{H}'\boldsymbol{\Sigma}^{-1})^{-1}\boldsymbol{\Omega}]}{|\boldsymbol{\Sigma}\boldsymbol{H}\boldsymbol{D}_{1/r}\boldsymbol{H}'+2t\boldsymbol{I}|^{n/2}k!}\,d\boldsymbol{H}\,,$$

where $0(p)$ is the group of all orthogonal $p\times p$ matrices $\boldsymbol{H}$, $d\boldsymbol{H}$ is the Haar measure normalized so that the measure of the whole group is unity and $\boldsymbol{D}_r$ is the diagonal matrix of the latent roots of $\boldsymbol{\Sigma}$. It may be seen that without the "random" approach we get

$$(1.6)\quad \mathrm{E}\,(\exp(-t\,\mathrm{tr}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{S}))=\exp(-\mathrm{tr}\,\boldsymbol{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}C_\kappa(\boldsymbol{\Omega})/[k!(1+2t)^{np/2+k}]\,,$$

which is different from the expression in (1.5). Further

$$(1.7)\quad \mathrm{E}\,(\mathrm{tr}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{S})^2/n=\beta_{2,p}+\left(\frac{2}{n}\right)\sum_{i<j=2}\mathrm{E}\,\{[(\boldsymbol{X}_i-\bar{\boldsymbol{X}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}_i-\bar{\boldsymbol{X}})] \cdot[(\boldsymbol{X}_j-\bar{\boldsymbol{X}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}_j-\bar{\boldsymbol{X}})]\}\,,$$

where $\beta_{2,p}$ is Mardia's measure of multivariate kurtosis replacing $\boldsymbol{\mu}$ by $\bar{\boldsymbol{X}}$ [5], $X_1,\cdots,X_n$ is a random sample of size n from $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and $\bar{\boldsymbol{X}}$ is the mean vector. The above discussion indicates that kurtosis is also involved in this robustness study. However, skewness is not involved since the distributional investigation here starts with Wishart distributions.

In the following sections, the density functions of $U^{(p)}$, $W^{(p)}$ and r_p, mgf of $V^{(p)}$ and moments of $U^{(p)}$ are obtained starting from Pillai's density of the canonical correlations under violations (2.2). Further, cdf's of the four criteria for the two-roots case are derived explicitly in terms of incomplete beta functions and their powers are computed to study the robustness of the test of independence against non-normality. Some inferences are given based on the tabulations.

2. Canonical correlation distribution under violations

Again, Pillai [7] has obtained the density of $r_1, \cdots, r_p$ for canonical correlation under violations by making $\boldsymbol{\Omega}$ completely random in (1.1) in the following manner:

Consider $\boldsymbol{\Omega}$ as a random matrix $(1/2)\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{MYY'M'\Sigma}_1^{-1/2}$ where $\boldsymbol{YY'}$ has a central Wishart distribution $W(q, n_1+n_2, \boldsymbol{\Sigma}_3)$ given by

$$(2.1)\qquad \left\{\Gamma_q\left(\frac{1}{2}(n_1+n_2)\right)|2\boldsymbol{\Sigma}_3|^{(n_1+n_2)/2}\right\}^{-1}|\boldsymbol{YY'}|^{(n_1+n_2-q-1)/2} \cdot \exp\left(\operatorname{tr}\left(-\frac{1}{2}\boldsymbol{\Sigma}_3^{-1}\boldsymbol{YY'}\right)\right).$$

Now, expand the generalized Laguerre polynomial in (1.1) in terms of zonal polynomials and multiply (1.1) by (2.1). Then, integrating YY' using Theorem 1 of [1], we have the joint density of $r_1, \cdots, r_p$ in the form

$$(2.2)\qquad C(p, m, n)|\boldsymbol{\Lambda}|^{-n_1/2}|\boldsymbol{I}+\boldsymbol{\Omega}_1|^{-(n_1+n_2)/2}|\boldsymbol{R}|^{(n_1-p-1)/2}|\boldsymbol{I}+\lambda\boldsymbol{R}|^{-(n_1+n_2)/2} \cdot \prod_{i>j}(r_i-r_j)\sum_{k=0}^{\infty}\sum_{\kappa}\left(\frac{1}{2}(n_1+n_2)\right)_{\kappa}\frac{C_\kappa(\lambda\boldsymbol{R}(\boldsymbol{I}+\lambda\boldsymbol{R})^{-1})}{k!} \cdot \sum_{d=0}^{k}\sum_{\delta}a_{\kappa,\delta}\frac{C_\delta(-\lambda^{-1}\boldsymbol{\Lambda}^{-1})}{C_\delta(\boldsymbol{I})} \cdot \sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu[(\boldsymbol{I}+\boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Omega}_1]}{(n_1/2)_\nu C_\nu(\boldsymbol{I})}$$

where $\boldsymbol{\Omega}_1=\boldsymbol{\Sigma}_3^{1/2}\boldsymbol{M'\Sigma}_1^{-1}\boldsymbol{M\Sigma}_3^{1/2}$. The distribution of the canonical correlation is a special case of (2.2).

3. Density functions of $U^{(p)}$, $W^{(p)}$ and r_p for canonical correlation under violations

In this section, first we consider the density function of Hotelling's trace.

Density of $U^{(p)}=\lambda \operatorname{tr} \boldsymbol{S}_1\boldsymbol{S}_2^{-1}$. Use the density of $U^{(p)}$ for MANOVA under violations given in Equation (2.1) of [8] and consider $\boldsymbol{\Omega}$ as a random matrix $(1/2)\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{MYY'M'\Sigma}_1^{-1/2}$ where $\boldsymbol{YY'}$ has a central Wishart distribution $W(q, n_1+n_2, \boldsymbol{\Sigma}_3, \boldsymbol{0})$ given in (2.1). Then multiplying the density of $U^{(p)}$ from [8] by (2.1) and integrating out $\boldsymbol{YY'}$ we get the density of $U^{(p)}$ as:

$$(3.1)\qquad C|\lambda\boldsymbol{\Lambda}|^{-n_1/2}|\boldsymbol{I}-\boldsymbol{P}^2|^{(n_1+n_2)/2}(U^{(p)})^{pn_1/2-1} \cdot \sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa(-U^{(p)})^k C_\kappa(\lambda^{-1}\boldsymbol{\Lambda}^{-1})(n_1/2)_\kappa}{k!\,\Gamma(pn_1/2+k)}$$

$$\cdot \sum_{n=0}^{k} \sum_{\nu} \frac{(-1)^n a_{\kappa,\nu}((n_1+n_2)/2)_\nu C_\nu(\boldsymbol{P}^2)}{(n_1/2)_\nu C_\nu(\boldsymbol{I})},$$

where $C=\Gamma_p((n_1+n_2)/2)/\Gamma_p(n_2/2)$, $\boldsymbol{P}^2=(\boldsymbol{I}+\boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_1=\boldsymbol{\Sigma}_3^{1/2}\boldsymbol{M}'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{M}\boldsymbol{\Sigma}_3^{1/2}$. The series (3.1) converges only for $0<|U^{(p)}|<1$.

Density of $W^{(p)}$. Using the density of $W^{(p)}=|\boldsymbol{I}+\lambda\boldsymbol{R}|^{-1}$ for MANOVA under violations in Equation (3.5) of [8] and considering $\boldsymbol{\Omega}$ random and proceeding as before we get the density of $W^{(p)}$ as

$$\begin{aligned}(3.2)\qquad & C|\lambda\boldsymbol{\Lambda}|^{-n_1/2}(W^{(p)})^{(n_2-p-1)/2} \\ & \cdot \sum_{k=0}^{\infty}\sum_{\kappa}\left\{\left(\frac{1}{2}(n_1+n_2)\right)_\kappa\left(\frac{1}{2}n_1\right)_\kappa C_\kappa(\boldsymbol{I})/k!\right\}|\boldsymbol{I}-\boldsymbol{P}^2|^{(n_1+n_2)/2} \\ & \cdot G_{p,p}^{p,0}\left(W^{(p)}\left|\begin{matrix}a_1\cdots a_p\\ b_1\cdots b_p\end{matrix}\right.\right)\sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta(-\lambda^{-1}\Lambda^{-1})}{C_\delta(\boldsymbol{I})} \\ & \cdot \sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu(\boldsymbol{P}^2)}{(n_1/2)_\nu C_\nu(\boldsymbol{I})},\end{aligned}$$

where $G(\ )$ denotes Meijer's G-function $a_i=n_1/2+k_{p-i+1}+b_i$ and $b_i=(i-1)/2$.

Density of r_p. Again using Equation (4.6) of [8] we have the density of r_p in the form:

$$\begin{aligned}(3.3)\qquad & C_1|\boldsymbol{\Lambda}|^{-n_1/2}r_p^{pn_1/2-1}|\boldsymbol{I}-\boldsymbol{P}^2|^{(n_1+n_2)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(p/2+1)_\kappa((p-1)/2)_\kappa}{k!((n_1+p+1)/2)_\kappa(p/2)_\kappa} \\ & \cdot \sum_{t=0}^{\infty}\sum_{\tau}\frac{(-1)^t}{t!}\sum_{\delta} g_{\kappa,\tau}^{\delta}\left(\frac{1}{2}(n_1+n_2)\right)_\delta C_\delta(r_p\boldsymbol{\Lambda}^{-1}) \\ & \cdot \sum_{i=0}^{d}\sum_{\nu}\frac{(-1)^i a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu(\boldsymbol{P}^2)}{(n_1/2)_\nu C_\nu(\boldsymbol{I})}\end{aligned}$$

where $C_1=\Gamma(1/2)\Gamma_p((n_1+n_2)/2)\Gamma_{p-1}(p/2+1)/\{\Gamma(p/2)\Gamma(n_1/2)\Gamma_p(n_2/2)\Gamma_{p-1}((n_1+p+1)/2)\}$, where $g_{\kappa,\tau}^{\delta}$ are constants defined as $C_\kappa(\boldsymbol{A})C_\tau(\boldsymbol{A})=\sum_{\delta} g_{\kappa,\tau}^{\delta}C_\delta(\boldsymbol{A})$ where δ is a partition of $k+t=d$ and ν that of i.

4. Mgf of $V^{(p)}$ and moments of $U^{(p)}$

We derive two forms for the m.g.f. of $V^{(p)}=\mathrm{tr}\,[\boldsymbol{R}(\boldsymbol{I}+\boldsymbol{R})^{-1}]$,

Mgf of $V^{(p)}$. The following theorem gives the two forms of the mgf of $V^{(p)}$:

THEOREM 4.1. *The m.g.f. of* $V^{(p)}$ *for canonical correlation under violations can be obtained in the following two forms:*

$$(4.1)\quad \text{(i)}\quad \mathrm{E}(\exp(tV^{(p)}))=|\boldsymbol{\Lambda}|^{-n_1/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n_1/2)_\kappa C_\kappa(\boldsymbol{I})}{k!((n_1+n_2)/2)_\kappa}$$
$$\cdot\sum_{d=0}^{k}\sum_{\delta}a_{\kappa,\delta}\left(\frac{1}{2}(n_1+n_2)\right)_\delta t^{k-d}$$
$$\cdot\sum_{n=0}^{d}\sum_{\nu}\frac{a_{\kappa,\nu}C_\nu(-\boldsymbol{\Lambda}^{-1})}{C_\nu(\boldsymbol{I})}$$
$$\cdot\sum_{i=0}^{n}\sum_{\eta}\frac{(-1)^i a_{\nu,\eta}((n_1+n_2)/2)_\eta C_\eta(\boldsymbol{P}^2)}{(n_1/2)_\eta C_\eta(\boldsymbol{I})}$$
$$\cdot|\boldsymbol{I}-\boldsymbol{P}^2|^{(n_1+n_2)/2}.$$

$$(4.2)\quad \text{(ii)}\quad \mathrm{E}(\exp(tV^{(p)}))=C_2|\boldsymbol{\Lambda}|^{-n_1/2}|\boldsymbol{I}-\boldsymbol{P}^2|^{(n_1+n_2)/2}$$
$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}\sum_{i=0}^{\infty}\frac{t^i}{i!}$$
$$\cdot\sum_{\lambda,\eta}\frac{g_{\lambda,\kappa}^{\eta}(n_1/2)_\eta C_\eta(\boldsymbol{I})}{((n_1+n_2)/2)_\eta}\sum_{d=0}^{k}\sum_{\delta}a_{\kappa,\delta}\frac{C_\delta(-\boldsymbol{\Lambda}^{-1})}{C_\delta(\boldsymbol{I})}$$
$$\cdot\sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu(\boldsymbol{P}^2)}{(n_1/2)_\nu C_\nu(\boldsymbol{I})},$$

where λ is a partition of i in (ii) *and* $C_2=\pi^{p^2/2}/\Gamma_p(p/2)$.

PROOF. (i) follows from the result of [9]

$$\mathrm{E}(\exp(tV^{(p)}))=\exp(-\mathrm{tr}\,\boldsymbol{\Omega})|\boldsymbol{\Lambda}|^{-n_1/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n_1/2)_\kappa C_\kappa(\boldsymbol{I})}{k!(a/2)_\kappa}$$
$$\cdot\sum_{d=0}^{k}\sum_{\delta}a_{\kappa,\delta}\left(\frac{1}{2}a\right)_\delta t^{k-d}\sum_{n=0}^{d}\sum_{\nu}\frac{a_{\kappa,\nu}C_\nu(-\boldsymbol{\Lambda}^{-1})L_\nu^m(\boldsymbol{\Omega})}{(n_1/2)_\nu C_\nu(\boldsymbol{I})C_\nu(\boldsymbol{I})}$$

where $a=n_1+n_2$ and $m=(n_1-p-1)/2$. We get the result as before by considering $\boldsymbol{\Omega}=(1/2)\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{M}\boldsymbol{Y}\boldsymbol{Y}'\boldsymbol{M}'\boldsymbol{\Sigma}_1^{-1/2}$ and integrating $\boldsymbol{Y}\boldsymbol{Y}'$ as we did before. To prove (ii), use the joint density function of $r_1,\cdots,r_p$ given in Equation (3.20) of [7] and let $\boldsymbol{L}=\boldsymbol{R}(\boldsymbol{I}+\boldsymbol{R})^{-1}$, we get the density function of L as

$$(4.3)\quad C(p,m,n)|\boldsymbol{\Lambda}|^{-n_1/2}|\boldsymbol{I}-\boldsymbol{P}^2|^{(n_1+n_2)/2}$$
$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\left(\frac{1}{2}(n_1+n_2)\right)_\kappa\Big/k!\sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta(-\boldsymbol{\Lambda}^{-1})}{C_\delta(\boldsymbol{I})}$$
$$\cdot\sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu(\boldsymbol{P}^2)}{(n_1/2)_\nu C_\nu(\boldsymbol{I})}$$
$$\cdot C_\kappa(\boldsymbol{L})|\boldsymbol{L}|^{(n_1-p-1)/2}|\boldsymbol{I}-\boldsymbol{L}|^{(n_2-p-1)/2}.$$

Now, consider $\exp(t\,\mathrm{tr}\,\boldsymbol{L})C_\kappa(\boldsymbol{L})|\boldsymbol{L}|^{(n_1-p-1)/2}|\boldsymbol{I}-\boldsymbol{L}|^{(n_2-p-1)/2}$

$$(4.4)\quad =\sum_{i=0}^{\infty}\frac{t^i}{i!}\sum_{\lambda}C_\lambda(\boldsymbol{L})C_\kappa(\boldsymbol{L})|\boldsymbol{L}|^{(n_1-p-1)/2}|\boldsymbol{I}-\boldsymbol{L}|^{(n_2-p-1)/2}$$

$$=\sum_{i=0}^{\infty}\frac{t^i}{i!}\sum_{\lambda,\eta}g_{\lambda,\kappa}^{\eta}C_\eta(\boldsymbol{L})|\boldsymbol{L}|^{(n_1-p-1)/2}|\boldsymbol{I}-\boldsymbol{L}|^{(n_2-p-1)/2}.$$

Now using Equation (22) of [1] to integrate $\boldsymbol{L}$ from $\boldsymbol{0}$ to $\boldsymbol{I}$ in (4.4) then substituting in (4.3) we get (4.2).

Now considering the moment of $U^{(p)}=\lambda\,\mathrm{tr}\,\boldsymbol{S}_1\boldsymbol{S}_2^{-1}$, by [9] we have

$$\mathrm{E}\,[(U^{(p)})^k]=(-1)^k\sum_{\kappa}\frac{C_\kappa(\lambda\boldsymbol{\Lambda})(n_1/2)_\kappa}{((p+1-n_2)/2)_\kappa}\sum_{n=0}^{k}\sum_{\nu}\frac{(-1)^n a_{\kappa,\nu}C_\nu(-\boldsymbol{\Omega})}{(n_1/2)_\nu C_\nu(\boldsymbol{I})}. \tag{4.5}$$

Let $\boldsymbol{\Omega}=(1/2)\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{MYY'M'\Sigma}_1^{-1/2}$ with $\boldsymbol{YY'}$ distributed as (2.1). Multiplying (4.5) by (2.1) and integrating out $\boldsymbol{YY'}$ we get the following theorem:

THEOREM 4.2. *The moment of $U^{(p)}$ for canonical correlation under violations is given by*

$$\mathrm{E}\,[(U^{(p)})^k]=(-1)^k\sum_{\kappa}\frac{C_\kappa(\lambda\boldsymbol{\Lambda})(n_1/2)_\kappa}{((p+1-n_2)/2)_\kappa}\cdot\sum_{n=0}^{k}\sum_{\nu}\frac{(-1)^n a_{\kappa,\nu}((n_1+n_2)/2)_\nu C_\nu(\boldsymbol{A})}{(n_1/2)_\nu C_\nu(\boldsymbol{I})} \tag{4.6}$$

where $\boldsymbol{A}=(\boldsymbol{I}-\boldsymbol{P}^2)^{-1}-\boldsymbol{I}$.

5. Non-central distributions of four statistics for $p=2$

In this section we derive the distributions of $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and r_2 starting from (2.2).

Distribution of $U^{(2)}$. Putting $\lambda=1$ and $p=2$ in (2.2) we have the joint density of r_1, r_2 $(r_1<r_2)$ in the form

$$C(2,m,n)(\lambda_1\lambda_2)^{-n_1/2}|\boldsymbol{I}+\boldsymbol{\Omega}_1|^{-(n_1+n_2)/2}(r_1r_2)^m[(1+r_1)(1+r_2)]^{-(n_1+n_2)/2}$$
$$\cdot(r_2-r_1)\sum_{k=0}^{\infty}\sum_{\kappa}\left(\frac{1}{2}(n_1+n_2)\right)_\kappa\frac{C_\kappa\begin{pmatrix}r_1/(1+r_1) & 0\\ 0 & r_2/(1+r_2)\end{pmatrix}}{k!}$$
$$\cdot\sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta\begin{pmatrix}-1/\lambda_1 & 0\\ 0 & -1/\lambda_2\end{pmatrix}}{C_\delta(\boldsymbol{I})}$$
$$\cdot\sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu[(\boldsymbol{I}+\boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Omega}_1]}{(n_1/2)_\nu C_\nu(\boldsymbol{I})}. \tag{5.1}$$

Now, denoting the characteristic roots of $(\boldsymbol{I}+\boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Omega}_1$ by ρ_1^2 and ρ_2^2, (5.1) becomes

$$C(2,m,n)(\lambda_1\lambda_2)^{-n_1/2}[(1-\rho_1^2)(1-\rho_2^2)]^{(n_1+n_2)/2} \tag{5.2}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{((n_1+n_2)/2)_\kappa}{k!} C_\kappa \begin{pmatrix} r_1/(1+r_1) & 0 \\ 0 & r_2/(1+r_2) \end{pmatrix}$$

$$\cdot \sum_{d=0}^{k} \sum_{\delta} \frac{a_{\kappa,\delta} C_\delta \begin{pmatrix} -1/\lambda_1 & 0 \\ 0 & -1/\lambda_2 \end{pmatrix}}{C_\delta(\boldsymbol{I})}$$

$$\cdot \sum_{n=0}^{d} \sum_{\nu} \frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu \begin{pmatrix} \rho_1^2 & 0 \\ 0 & \rho_2^2 \end{pmatrix}}{(n_1/2)_\nu C_\nu(\boldsymbol{I})}$$

$$\cdot (r_1 r_2)^m [(1+r_1)(1+r_2)]^{-(n_1+n_2)/2}(r_2-r_1) .$$

Further, let

$$F=(r_1r_2)^m[(1+r_1)(1+r_2)]^{-(n_1+n_2)/2}(r_2-r_1)C_\kappa \begin{pmatrix} r_1/(1+r_1) & 0 \\ 0 & r_2/(1+r_2) \end{pmatrix} , \tag{5.3}$$

and $C_\kappa(A)$ be written in the form

$$C_\kappa(\boldsymbol{A}) = \sum_{r+2s=k} b_\kappa(r, s) a_1^r a_2^s , \tag{5.4}$$

where a_1, a_2 are the first and second elementary symmetric functions of the latent roots of the 2×2 matrix $\boldsymbol{A}$ and $b_\kappa(r, s)$ can be found in [8]. Then using (5.4) in (5.3) we get

$$F = \sum_{r+2s=k} b_\kappa(r, s) \left(\frac{r_1}{1+r_1} + \frac{r_2}{1+r_2}\right)^r \left[\frac{r_1 r_2}{(1+r_1)(1+r_2)}\right]^s \tag{5.5}$$

$$\cdot (r_1 r_2)^m [(1+r_1)(1+r_2)]^{-(n_1+n_2)/2}(r_2-r_1) .$$

Now transform $x=r_1+r_2$, $y=r_1r_2$, integrate y from 0 to $x^2/4$ and x from 0 to U, we get

$$\int_0^u \int_0^{x^2/4} F dy dx = \sum_{r+2s=k} b_\kappa(r, s) H_{rs}(u) , \tag{5.6}$$

where $H_{rs}(u) = \sum_{i=0}^{r} \binom{r}{i} 2^{r-i} \int_0^u \int_0^{x^2/4} \frac{x^i y^{m+r+s-i}}{(1+x+y)^{m+n+r+s+3}} dy dx$. Hence we have the following theorem:

THEOREM 5.1. *The exact C.D.F. of $U^{(2)}$ under violations in the canonical correlation case is given by*

$$C(2, m, n)(\lambda_1\lambda_2)^{-n_1/2}[(1-\rho_1^2)(1-\rho_2^2)]^{(n_1+n_2)/2} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{((n_1+n_2)/2)_\kappa}{k!} \tag{5.7}$$

$$\cdot \sum_{r+2s=k} b_\kappa(r, s) H_{rs}(U) \sum_{d=0}^{k} \sum_{\delta} \frac{a_{\kappa,\delta} C_\delta \begin{pmatrix} -1/\lambda_1 & 0 \\ 0 & -1/\lambda_2 \end{pmatrix}}{C_\delta(\boldsymbol{I})}$$

$$\cdot \sum_{n=0}^{d} \sum_{\nu} \frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu \begin{pmatrix} \rho_1^2 & 0 \\ 0 & \rho_2^2 \end{pmatrix}}{(n_1/2)_\nu C_\nu(\boldsymbol{I})} ,$$

where λ_1, λ_2 *are the latent roots of* $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$, ρ_1^2, ρ_2^2 *are those of* $(\boldsymbol{I}+\boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Omega}_1$ *and* $b_\kappa(r, s)$, $H_{rs}(U)$ *are given in* (5.4), (5.6) *respectively.*

Distribution of $V^{(2)}$. Start from (5.5), and make the transformation $X=V^{(2)}=(r_1/(1+r_1))+(r_2/(1+r_2))$ and $y=(r_1/(1+r_1))(r_2/(1+r_2))$. To find the C.D.F. of $V^{(2)}$, we have to consider two cases:
(i) When $v\leqq 1$, then integrate y from 0 to $x^2/4$ and x from 0 to v, we get

$$(5.8)\qquad \int_0^v\int_0^{x^2/4} Fdydx=\sum_{r+2s=k} b(r, s)F_{rs}(v)\ ,$$

where $F_{rs}(v)=\int_0^v\int_0^{x^2/4} x^r y^{m+s}(1-x+y)^n dydx$.
(ii) When $v>1$, let $x'=(1/(1+r_1))+(1/(1+r_2))=2-x$, $y'=(1/(1+r_1))(1/(1+r_2))=1-x+y$, then integrate y' from 0 to $(x')^2/4$ and x' from $2-v$ to 1, we have

$$(5.9)\qquad \int_{2-v}^1\int_0^{(x')^2/4} Fdy'dx'=\sum_{r+2s=k} b_\kappa(r, s)F'_{rs}(v)\ ,$$

where $F'_{rs}(v)=\int_{2-v}^1\int_0^{(x')^2/4}(2-x')^r(1-x'+y')^{m+s}y'^n dy'dx'$. Hence we have the following theorem:

THEOREM 5.2. *The exact C.D.F. of* $V^{(2)}$ *under violations in the canonical correlation case is given by*

$$(5.10)\qquad C(2, m, n)(\lambda_1\lambda_2)^{-n_1/2}[(1-\rho_1^2)(1-\rho_2^2)]^{(n_1+n_2)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}$$
$$\cdot\sum_{r+2s=k} b_\kappa(r, s)K_{rs}(v)\sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta\begin{pmatrix}-1/\lambda_1 & 0\\ 0 & -1/\lambda_2\end{pmatrix}}{C_\delta(\boldsymbol{I})}$$
$$\cdot\sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu\begin{pmatrix}\rho_1^2 & 0\\ 0 & \rho_2^2\end{pmatrix}}{(n_1/2)_\nu C_\nu(\boldsymbol{I})}\ ,$$

where λ_1, λ_2, ρ_1^2, ρ_2^2 *are the same as in Theorem* 5.1 *and*

$$K_{rs}(V^{(2)})=\begin{cases}F_{rs}(v)\ , & \text{if } v\leqq 1\\ F_{rs}(1)+F'_{rs}(v)\ , & \text{if } v>1\ .\end{cases}$$

Distribution of $W^{(2)}$. Now, if we make the transformation $x=1/(1+r_1)(1+r_2)$ and $y=r_1r_2/(1+r_1)(1+r_2)$ in (5.5) and then integrate y from 0 to $(1-\sqrt{x})^2$ and x from 0 to w, we get

$$(5.11)\qquad \int_0^w\int_0^{(1-\sqrt{x})^2} Fdydx=\sum_{r+2s=k} b_\kappa(r, s)G_{rs}(w)\ ,$$

where $G_{rs}(w)=\int_0^w \int_0^{(1-\sqrt{x})^2} x^n y^{m+s}(1-x+y)^r dydx$. Now we state the following theorem:

THEOREM 5.3. *The exact C.D.F. of $W^{(2)}$ under violations in the canonical correlation case is given by*

$$(5.12)\qquad C(2,m,n)(\lambda_1\lambda_2)^{-n_1/2}[(1-\rho_1^2)(1-\rho_2^2)]^{(n_1+n_2)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}$$
$$\cdot\sum_{r+2s=k} b_\kappa(r,s)G_{rs}(w)\sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta\begin{pmatrix}-1/\lambda_1 & 0\\ 0 & -1/\lambda_2\end{pmatrix}}{C_\delta(I)}$$
$$\cdot\sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu\begin{pmatrix}\rho_1^2 & 0\\ 0 & \rho_2^2\end{pmatrix}}{(n_1/2)_\nu C_\nu(I)},$$

where λ_1, λ_2, ρ_1^2, ρ_2^2 and $b_\kappa(r,s)$ are the same as in Theorem 5.1 *and $G_{rs}(w)$ is given in* (5.11).

Distribution of $L_{(2)}=r_2/(1+r_2)$. Again, transform $l_1=r_1/(1+r_1)$ and $l_2=r_2/(1+r_2)$ in (5.5), and integrate l_1 from 0 to l_2 and l_2 from 0 to l. We get

$$(5.13)\qquad \int_0^l\int_0^{l_2} Fdydx=\sum_{r+2s=k} b_\kappa(r,s)P_{rs}(l),$$

where $P_{rs}(l)=\left[\sum_{i=0}^{r}\binom{r}{i}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\int_0^l l_2^{2m+2s+r+i+2}(1-l_2)^n dl_2\right]\Big/[(m+s+t+i+1)(m+s+t+i+2)]$. Now, we state the following theorem:

THEOREM 5.4. *The exact C.D.F. of the largest root $L_{(2)}$ under violations in the canonical correlation case is given by*

$$(5.14)\qquad C(2,m,n)(\lambda_1\lambda_2)^{-n_1/2}[(1-\rho_1^2)(1-\rho_2^2)]^{(n_1+n_2)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}$$
$$\cdot\sum_{r+2s=k} b_\kappa(r,s)P_{rs}(l)\sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta\begin{pmatrix}-1/\lambda_1 & 0\\ 0 & -1/\lambda_2\end{pmatrix}}{C_\delta(I)}$$
$$\cdot\sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}((n_1+n_2)/2)_\nu C_\nu\begin{pmatrix}\rho_1^2 & 0\\ 0 & \rho_2^2\end{pmatrix}}{(n_1/2)_\nu C_\nu(I)},$$

where λ_1, λ_2, ρ_1^2, ρ_2^2 and $b_\kappa(r,s)$ are the same as in Theorem 5.1 *and $P_{rs}(L_{(2)})$ is given in* (5.13).

6. Numerical results

In order to study the robustness against non-normality of the tests of independence between two sets of variates based on the four criteria above, we can evaluate the powers of the test by using the distributions obtained in Equations (5.6), (5.9), (5.12) and (5.14). Let

$$B=C(2, m, n)(\lambda_1\lambda_2)^{-n_1/2}[(1-\rho_1^2)(1-\rho_2^2)]^{(n_1+n_2)/2},$$

then we can write the C.D.F.'s of $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and $L_{(2)}$ in the following forms respectively:

$$\Pr\{U^{(2)}\leqq u\}=B\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}\sum_{r+2s=k}b_\kappa(r, s)H_{rs}(u)\mathcal{U}_{k,\kappa}, \tag{6.1}$$

$$\Pr\{V^{(2)}\leqq v\}=B\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}\sum_{r+2s=k}b_\kappa(r, s)K_{rs}(v)\mathcal{U}_{k,\kappa}, \tag{6.2}$$

$$\Pr\{W^{(2)}\leqq w\}=B\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}\sum_{r+2s=k}b_\kappa(r, s)G_{rs}(w)\mathcal{U}_{k,\kappa}, \tag{6.3}$$

and

$$\Pr\{L_{(2)}\leqq l\}=B\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{k!}\sum_{r+2s=k}b_\kappa(r, s)P_{rs}(l)\mathcal{U}_{k,\kappa}, \tag{6.4}$$

where the expression $\mathcal{U}_{k,\kappa}$ is available in the appendix and $H_{rs}(u)$, $K_{rs}(v)$, $G_{rs}(w)$ and $P_{rs}(l)$ can be expressed in terms of incomplete beta functions [9] as follows:

Let $B(i, j, a)=\int_0^a x^{i-1}(1-x)^{j-1}dx$, then

$$\begin{aligned} H_{rs}(u)=\sum_{i=0}^{r}\binom{r}{i}&\frac{2^{r-i}(m+r+s-i)!(n+i+1)!}{(m+n+r+s+2)!} \\ &\cdot\left\{B(a_2, n+1, A_1)-2^{i+1}\sum_{k=0}^{m+r+s-i}\binom{n+i+k+1}{k}\right. \\ &\left.\cdot B(a_2+2k, a_1, A_2)\right\}, \end{aligned} \tag{6.5}$$

where $a_1=2n+i+3$, $a_2=i+1$, $A_1=u/(1+u)$ and $A_2=u/(2+u)$. For $0\leqq v\leqq 1$,

$$\begin{aligned} K_{rs}(v)=F_{rs}(v)=\frac{2^{r+1}}{m+s+1}\sum_{i=0}^{n}(-1)^i R_i B\Big(&2(m+s+i)+r+3, \\ &2(n-i)+1, \frac{v}{2}\Big) \end{aligned} \tag{6.6}$$

and for $1<v\leqq 2$,

$$\begin{aligned} K_{rs}(v)&=F_{rs}(1)+F'_{rs}(v) \\ &=\frac{2^{r+1}}{m+s+1}\sum_{i=0}^{n}(-1)^i R_i B\Big(2(m+n+i)+r+3, \end{aligned}$$

$$2(n-i)+1, \frac{1}{2}\Big)+\frac{2^{r+1}}{n+1}\sum_{i=0}^{m+s}(-1)^i P_i$$

$$\cdot\Big[B(2(m+n-i)+r+1, 2(n+i)+3, v)$$

$$-B\Big(2(m+n-i)+r+1, 2(n+i)+3, \frac{1}{2}\Big)\Big],$$

where $R_i=\prod_{j=1}^{i}[(n+1-j)/(m+s+j+1)]$, $R_0=1$ and $P_i=\prod_{j=1}^{i}[(m+s-j+1)/(n+j+1)]$, $P_0=0$

(6.7) $$G_{rs}(w)=\sum_{i=0}^{r}\frac{(-1)^i 2^{r-i+1}}{m+s+1}Q_i B(2n+2, 2m+2s+r+i+3, \sqrt{w}),$$

where $Q_i=\prod_{j=1}^{i}[(r-j+1)/(m+s+j+1)]$, $Q_0=1$ and

(6.8) $$P_{rs}(l)=\sum_{i=0}^{r}\binom{r}{i}\sum_{j=0}^{n}(-1)^j\binom{n}{j}S_j B(2(m+s)+r+i+3, n+1, l),$$

where $S_j=1/[(m+s+t+j+1)(m+s+t+j+2)]$. Using (6.1) to (6.4), powers of $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and $L_{(2)}$ respectively have been computed to the study of 3), namely, the robustness against non-normality of the test of independence between a p-set and a q-set in a $(p+q)$-variate normal population. In computing the powers, for $\alpha=.05$, lower tail probabilities for $W^{(2)}$ and upper for others were considered; $m=0, 2$, $n=5, 15, 40$, and various values of (ρ_1^2, ρ_2^2) and (f_1, f_2) were taken where $f_i=\lambda_i-1$, $i=1, 2$. These powers are presented in Table 1 (not presented here but available in Mimeograph report No. 419, Department of Statistics, Purdue University). Further, Table 2 gives the values of the ratio $e=(p_1-p_0)/(p_0-\alpha)$, for m and n as above and selected (ρ_1^2, ρ_2^2) and (f_1, f_2), where p_1 is the power under violation of assumptions and p_0, power without violation of assumptions. The latter table serves for a better comparison of the performance of the four criteria. Some findings follow.

1. For small values of f_1 and f_2, the changes in powers are considerably small and it appears that slight non-normality does not affect seriously the test of independence between the two sets of variables, based on any of the four criteria.

2. For larger values of f_1 and f_2, the changes in powers are no longer small, indicating that the test of independence based on any of the four criteria is affected by serious departure from normality.

3. From the tabulations, especially from the values of e in Table 2, it may be seen that $V^{(2)}$ has the smallest value for e among the four criteria in 20 out of 28 cases (15 of 17 cases where $f_1\neq f_2$ and 5 cases where $f_1=f_2$) and $L_{(2)}$ has smallest in 7 cases, (all, except one, when

Table 2 Values of the ratio $e=(p_1-p_0)(p_0-\alpha)$. (p_1=power under violation of assumptions; p_0=power without violation; α=.05)

ρ_1^2	ρ_2^2	f_1	f_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$L_{(2)}$	ρ_1^2	ρ_2^2	f_1	f_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$L_{(2)}$
$n=5$, $m=0$								$n=5$, $m=2$							
0	.0001	0	.1	198	192	194	200	.0015	.0015	0	.1	14.9	12.2	12.3	12.6
		.05	.05	196	194	194	196			.05	.05	14.8	12.5	12.4	12.3
.0015	.0055	0	.1	2.89	2.83	2.87	2.90	0	.01	0	.03	.5216	.5219	.5220	.5206
		.05	.05	2.86	2.86	2.87	2.85	.01	.01	0	.1	1.95	1.89	1.93	1.97
0	.01	0	.1	2.04	2.01	2.03	2.04			.05	.05	1.94	1.94	1.94	1.92
		.05	.05	2.02	2.04	2.03	2.00								
.025	.025	0	.1	4.76	4.58	4.70	4.80								
		.05	.05	4.73	4.63	4.70	4.72								
$n=15$, $m=0$								$n=15$, $m=2$							
0	.0001	0	.03	25.4	25.3	25.5	25.6	0	.0001	0	.1	1898	1860	1890	1948
.0015	.0015	0	.1	1.36	1.35	1.36	1.38			.05	.05	1880	1876	1888	1872
.0015	.0035	0	.03	.540	.539	.540	.543	.025	.025	0	.1	.499	.485	.510	.508
.025	.025	0	.1	.267	.260	.261	.276			.05	.05	.495	.487	.512	.489
		.05	.05	.266	.260	.261	.271								
$n=40$, $m=0$								$n=40$, $m=2$							
.0015	.0015	0	.1	1.35	1.34	1.35	1.37	0	.0001	0	.03	.4575	.4568	.4564	.4595
		.05	.05	1.34	1.33	1.33	1.34	.00125	.00125	0	.1	3.55	3.53	3.54	3.65
.0015	.0055	0	.03	.1561	.1558	.1560	.1564			.05	.05	3.52	3.52	3.52	3.49

$f_1=f_2$). $V^{(2)}$ has largest value of e in one case and tied for largest in two cases (all when $f_1=f_2$) while $L_{(2)}$ has largest e in 17 cases (13 when $f_1\neq f_2$). $W^{(2)}$ has second smallest e 17 times, has one smallest and tied for smallest two times and largest e four times. $U^{(2)}$ has second largest e, 15 times and largest three times and tied for largest 5 times. Since a smaller value of e is indicative of better performance against non-normality, it is clear that $V^{(2)}$ is the most robust of the four criteria and $W^{(2)}$ is the second best. $L_{(2)}$ is very inconsistent and generally weak while $U^{(2)}$ is more steady and performs apparently as third best behind $W^{(2)}$.

4. The findings above are in general agreement with those obtained from asymptotic studies on Hotelling's trace [3] and those of Gayen in the bivariate case [3] and those of Olson [6] from the Monte Carlo study in multivariate analysis of variance.

Appendix

$$\mathcal{U}_{0,(0)}=1$$

$$\mathcal{U}_{1,(1)}=1+A_1$$

$$\mathcal{U}_{2,(2)}=1+2A_1+A_2$$

$$\mathcal{U}_{2,(1^2)}=1+2A_1+A_{11}$$

$$\mathcal{U}_{3,(3)}=1+3A_1+3A_2+A_3$$

$$\mathcal{U}_{3,(2,1)}=1+3A_1+(4/3)A_2+A_{21}+(5/3)A_{11}$$

$$\mathcal{U}_{4,(4)}=1+4A_1+6A_2+4A_3+A_4$$

$$\mathcal{U}_{4,(3,1)}=1+4A_1+(11/3)A_2+(7/3)A_{11}+(6/5)A_3+(14/5)A_{21}+A_{31}$$

$$\mathcal{U}_{4,(2^2)}=1+4A_1+(8/3)A_2+(10/3)A_{11}+4A_{21}+A_{22}$$

$$\mathcal{U}_{5,(5)}=1+5A_1+10A_2+10A_3+5A_4+A_5$$

$$\mathcal{U}_{5,(4,1)}=1+5A_1+7A_2+3A_{11}+(23/5)A_3+(27/5)A_{21}+(8/7)A_4 \\ +(27/7)A_{31}+A_{41}$$

$$\mathcal{U}_{5,(3,2)}=1+5A_1+(16/3)A_2+(14/3)A_{11}+(8/5)A_3+(42/5)A_{21} \\ +(8/3)A_{31}+(7/3)A_{22}+A_{32}$$

$$\mathcal{U}_{6,(6)}=1+6A_1+15A_2+20A_3+15A_4+6A_5+A_6$$

$$\mathcal{U}_{6,(5,1)}=1+6A_1+(34/3)A_2+(11/3)A_{11}+(56/5)A_3+(44/5)A_{21} \\ +(39/7)A_4+(66/7)A_{31}+(10/9)A_5+(44/9)A_{41}+A_{51}$$

$$\mathcal{U}_{6,(4,2)}=1+6A_1+9A_2+6A_{11}+(28/5)A_3+(72/5)A_{21}+(48/35)A_4 \\ +(66/7)A_{31}+(147/35)A_{22}+(12/5)A_{41}+(18/5)A_{32}+A_{42}$$

$$\mathcal{U}_{6,(3^2)}=1+6A_1+8A_2+7A_{11}+(16/5)A_3+(84/5)A_{21}+8A_{31}+7A_{22} \\ +6A_{32}+A_{33}$$

where A_i and A_{ij} are defined as

$$A_1=-\frac{1}{2}a_1\left(1-\frac{1}{2}e_{(1)}c_1\right)$$

$$A_2=\frac{1}{8}(3a_2+2b_1)[1-e_{(1)}c_1+e_{(2)}(3c_2+2d_1)/8]$$

$$A_{11}=b_1(1-e_{(1)}c_1+e_{(1^2)}d_1)$$

$$A_3=-\frac{1}{16}(5a_3+3a_1b_1)\Big[1-\frac{3}{2}e_{(1)}c_1+3e_{(2)}(3c_2+2d_1)/8 \\ -e_{(3)}(5c_3+3c_1d_1)/16\Big]$$

$$A_{21}=-\frac{1}{2}a_1b_1\left[1-\frac{3}{2}e_{(1)}c_1+e_{(2)}(3c_2+2d_1)/2+\frac{5}{3}e_{(1^2)}d_1-e_{(2,1)}c_1d_1/2\right]$$

$$A_4=\frac{1}{128}(35a_4+20a_2b_1+18b_2)[1-2e_{(1)}c_1+6e_{(2)}(3c_2+2d_1)/8$$
$$-e_{(3)}(5c_3+3c_1d_1)/4+c_{(4)}(35c_4+20c_2d_1+18d_2)/128]$$

$$A_{31}=\frac{1}{8}(3a_2b_1+2b_2)\Big[1-2e_{(1)}c_1+\frac{11}{24}e_{(2)}(3c_2+2d_1)+\frac{7}{3}e_{(1^2)}d_1$$
$$-\frac{3}{40}e_{(3)}(5c_3+2c_1d_1)-\frac{7}{5}e_{(2,1)}c_1d_1+e_{(3,1)}(3c_2d_1+2d_2)/8\Big]$$

$$A_{22}=b_2\Big[1-2e_{(1)}c_1+e_{(2)}(3c_2+2d_1)/3+\frac{10}{3}e_{(1^2)}d_1-2e_{(2,1)}c_1d_1+e_{(2^2)}d_2\Big]$$

$$A_5=-\frac{1}{256}(63a_5+35a_3b_1+30a_1b_2)\Big[1-\frac{5}{2}e_{(1)}c_1+\frac{5}{4}e_{(2)}(3c_2+2d_1)$$
$$-\frac{5}{8}e_{(3)}(5c_3+3c_1d_1)+5e_{(4)}(35c_4+20c_2d_1+18d_2)/128$$
$$-e_{(5)}(63c_5+35c_3d_1+30c_1d_2)/256\Big]$$

$$A_{41}=-\frac{1}{16}(5a_3b_1+3a_1b_2)\Big[1-\frac{5}{2}e_{(1)}c_1+7e_{(2)}(3c_2+2d_1)/8+3e_{(1^2)}d_1$$
$$-23e_{(3)}(5c_3+3c_1d_1)/80-27e_{(2,1)}c_1d_1/10+e_{(4)}(35c_4+20c_2d_1$$
$$+18d_2)/112+27e_{(3,1)}(3c_2d_1+2d_2)/56-e_{(4,1)}(5c_3d_1+3c_1d_2)/16\Big]$$

$$A_{32}=-\frac{1}{2}a_1b_2\Big[1-\frac{5}{2}e_{(1)}c_1+2e_{(2)}(3c_2+2d_1)/3+\frac{14}{3}e_{(1^2)}d_1$$
$$-e_{(3)}(5c_3+3c_1d_1)/10-21e_{(2,1)}c_1d_1/5+e_{(3,1)}(3c_2d_1+2d_2)/3$$
$$+\frac{7}{3}e_{(2^2)}d_2-e_{(3,2)}c_1d_2/2\Big]$$

$$A_6=\frac{1}{1024}(231a_6+126a_4b_1+105a_2b_2+100b_3)\Big[1-3e_{(1)}c_1$$
$$+\frac{15}{8}e_{(2)}(3c_2+2d_1)-\frac{5}{4}e_{(3)}(5c_3+3c_1d_1)+\frac{15}{128}e_{(4)}(35c_4$$
$$+20c_2d_1+18d_2)-\frac{3}{128}e_{(5)}(63c_5+35c_3d_1+30c_1d_2)$$
$$+\frac{1}{1024}e_{(6)}(231c_6+126c_4d_1+105c_2d_2+100d_3)\Big]$$

$$A_{51}=\frac{1}{128}(35a_4b_1+20a_2b_2+18b_3)\Big[1-3e_{(1)}c_1+\frac{17}{12}e_{(2)}(3c_2+2d_1)$$
$$+\frac{11}{3}e_{(1^2)}d_1-\frac{7}{10}e_{(3)}(5c_3+3c_1d_1)-\frac{22}{5}e_{(2,1)}c_1d_1+\frac{39}{896}e_{(4)}(35c_4$$

$$+20c_2d_1+18d_2)+\frac{33}{28}e_{(3,1)}(3c_2d_1+2d_2)-\frac{5}{1152}e_{(5)}(63c_5$$
$$+35c_3d_1+30c_1d_2)-\frac{11}{36}e_{(4,1)}(5c_3d_1+3c_1d_2)$$
$$+\frac{1}{128}e_{(5,1)}(35c_4d_1+20c_2d_2+18d_3)\Big]$$

$$A_{42}=\frac{1}{8}(3a_2b_2+2b_3)\Big[1-3e_{(1)}c_1+\frac{9}{8}e_{(2)}(3c_2+2d_1)+6e_{(1^2)}d_1$$
$$-\frac{36}{5}e_{(2,1)}c_1d_1-\frac{7}{20}e_{(3)}(5c_3+3c_1d_1)+\frac{3}{280}e_{(4)}(35c_4+20c_2d_1$$
$$+18d_2)+\frac{147}{35}e_{(2^2)}d_2+\frac{33}{28}e_{(3,1)}(3c_2d_1+2d_2)-\frac{3}{20}e_{(4,1)}(5c_3d_1$$
$$+3c_1d_2)-\frac{9}{5}e_{(3,2)}c_1d_2+e_{(4,2)}(3c_2d_2+2d_3)/8\Big]$$

$$A_{33}=b_3[1-3e_{(1)}c_1+e_{(2)}(3c_2+2d_1)+7e_{(1^2)}d_1-e_{(3)}(5c_3+3c_1d_1)/5$$
$$-42e_{(2,1)}c_1d_1/5+e_{(3,1)}(3c_2d_1+2d_2)+7e_{(2^2)}d_2-3e_{(3,2)}c_1d_2+e_{(3^2)}d_3]$$

where $a_i=\lambda_1^{-i}+\lambda_2^{-i}$
$b_i=(\lambda_1\lambda_2)^{-i}$
$c_i=\rho_1^i+\rho_2^i$
$d_i=(\rho_1\rho_2)^i$
$e_\nu=\left(\frac{n_1+n_2}{2}\right)_\nu\Big/\left(\frac{n_1}{2}\right)_\nu$
i's are positive integers.

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