Ann. Inst. Statist. Math.
31 (1979), Part A, 57-65

ALIAS BALANCED AND ALIAS PARTIALLY BALANCED FRACTIONAL 2^m FACTORIAL DESIGNS OF RESOLUTION $2l+1$

TERUHIRO SHIRAKURA

(Received Sept. 25, 1978; revised Feb. 13, 1979)

Abstract

As a generalization of alias balanced designs due to Hedayat, Raktoe and Federer [5], we introduce the concept of alias partially balanced designs for fractional 2^m factorial designs of resolution $2l+1$. All orthogonal arrays of strength $2l$ yield alias balanced designs. Some balanced arrays of strength $2l$ yield alias balanced and alias partially balanced designs. In particular, simple arrays which are a special case of balanced arrays yield alias partially balanced designs. At most 2^m-1 alias balanced (or alias partially balanced) designs are generated from an alias balanced (or alias partially balanced) design by level permutations. This implies that alias balanced or alias partially balanced designs need not be orthogonal arrays or balanced arrays of strength $2l$.

1. Introduction

Consider an experiment with m factors each at two levels. An assembly or treatment combination is represented by $(j_1, j_2, \cdots, j_m)$ where j_k, the level of the kth factor, equals 0 or 1. As unknown effects, θ_ϕ, θ_t and in general $\theta_{t_1\cdots t_k}$ denote the general mean, main effect of tth factor and k-factor interaction of corresponding factors, respectively. For a fixed integer l $(1\leqq l\leqq m/2)$, let $\boldsymbol{\theta}$ be the $\nu_l\times 1$ vector composed of the effects up to l-factor interactions and let $\boldsymbol{\theta}^*$ be the $\nu_l^*\times 1$ vector of the remaining effects, where $\nu_l=\sum_{\beta=0}^{l}\binom{m}{\beta}$ and $\nu_l^*=2^m-\nu_l$, i.e.,

$$\boldsymbol{\theta}'=(\theta_\phi;\theta_1,\theta_2,\cdots,\theta_m;\theta_{12},\cdots,\theta_{m-1m};\cdots;\theta_{12\cdots l},\cdots,\theta_{m-l+1\cdots m})$$

$$\boldsymbol{\theta}^{*\prime}=(\theta_{12\cdots l+1},\cdots,\theta_{m-l\cdots m};\cdots;\theta_{12\cdots m}) .$$

As usual, $\boldsymbol{\theta}$ is to be estimated and $\boldsymbol{\theta}^*$ is not of interest for estimation. The expected value of the observation $y(j_1,\cdots,j_m)$ for an assembly $(j_1,\cdots,j_m)$ can then be expressed as

(1.1) $$\mathcal{E}(y(j_1, \cdots, j_m)) = e'\boldsymbol{\theta} + e^{*\prime}\boldsymbol{\theta}^* ,$$

where the elements of $(e', e^{*\prime})$ corresponding to $\theta_{t_1 t_2 \cdots t_k}$ are given by $d(j_{t_1}) \cdot d(j_{t_2}) \cdots d(j_{t_k})$, and in particular the element to θ_ϕ is given by 1 (see, e.g., Yamamoto, Shirakura and Kuwada [14]). Here $d(j) = -1$ or 1 according as $j=0$ or 1.

Let T be a fraction with N assemblies. (Note that T can be considered as a (0, 1) matrix of size $m \times N$ whose columns denote assemblies.) Consider the $N \times 1$ observation vector $\boldsymbol{y}_T$ of T whose elements are independent random variables with common variance σ^2. Then from (1.1), the expected value of $\boldsymbol{y}_T$ can be expressed as

(1.2) $$\mathcal{E}(\boldsymbol{y}_T) = E\boldsymbol{\theta} + E^*\boldsymbol{\theta}^* ,$$

where E and E^* denote the $N \times \nu_l$ and $N \times \nu_l^*$ design matrices of T relative to $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^*$, respectively, whose elements are -1 or 1. A fraction T is called a fractional 2^m factorial (simply, 2^m-FF) design of resolution $2l+1$ if $\boldsymbol{\theta}$ is estimable ignoring $\boldsymbol{\theta}^*$. For a 2^m-FF design T of resolution $2l+1$, the best linear unbiased estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is given by $\hat{\boldsymbol{\theta}} = M^{-1}E'\boldsymbol{y}_T$, where $M = E'E$ is called the information matrix of T. However under model (1.2), the expected value of $\hat{\boldsymbol{\theta}}$ becomes

(1.3) $$\mathcal{E}(\hat{\boldsymbol{\theta}}) = \boldsymbol{\theta} + A\boldsymbol{\theta}^* ,$$

where $A = M^{-1}E'E^*$ is called the alias matrix of T. This matrix A constitutes an aliasing relation of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^*$ in a 2^m-FF design of resolution $2l+1$. In view of (1.3), Hedayat, Raktoe and Federer [5] have introduced the concept of alias balanced (AB) designs in order to classify designs. In this paper, we introduce the concept of alias partially balanced (APB) designs as a generalization of AB designs. As will be seen from Yamamoto, Shirakura and Kuwada [14], this concept is similar to a generalization of orthogonal fractional designs to balanced fractional designs. We also discuss what designs are AB or APB designs.

2. Definitions

Let $a(t_1 \cdots t_u; t_1' \cdots t_v')$, $(0 \leq u \leq l, l+1 \leq v \leq m)$, be the elements of A corresponding to effects $\theta_{t_1 \cdots t_u}$ and $\theta_{t_1' \cdots t_v'}$. Then (1.3) is equivalent to that for each $\hat{\theta}_{t_1 \cdots t_u}$ in $\hat{\boldsymbol{\theta}}$,

$$\mathcal{E}(\hat{\theta}_{t_1 \cdots t_u}) = \theta_{t_1 \cdots t_u} + \sum_{v=l+1}^{m} \sum_{\{t_1', \cdots, t_v'\} \in \mathfrak{M}_v} a(t_1 \cdots t_u; t_1' \cdots t_v')\theta_{t_1' \cdots t_v'} ,$$

where $\mathfrak{M}_k$ denotes the collection of all subsets of $\{1, 2, \cdots, m\}$ with cardinality k. Note that $\theta_{t_1\cdots t_u}$, $\hat{\theta}_{t_1\cdots t_u}$ and $a(t_1\cdots t_u; t'_1\cdots t'_v)$ correspond to θ_ϕ, $\hat{\theta}_\phi$ and $a(\phi; t'_1\cdots t'_v)$, respectively, when $u=0$. We first define an AB design due to Hedayat, Raktoe and Federer [5].

DEFINITION 1. A 2^m-FF design of resolution $2l+1$ is said to be AB if

$$\left\{\sum_{v=l+1}^{m} \sum_{\{t'_1,\cdots,t'_v\}\in\mathfrak{M}_v} a^2(t_1\cdots t_u; t'_1\cdots t'_v)\right\}^{1/2}$$

is constant for all subsets $\{t_1,\cdots,t_u\}$, $(0\leqq u\leqq l)$.

As a natural generalization of Definition 1, we make the following

DEFINITION 2. A 2^m-FF design of resolution $2l+1$ is said to be APB if

$$\left\{\sum_{v=l+1}^{m} \sum_{\{t'_1,\cdots,t'_v\}\in\mathfrak{M}_v} a^2(t_1\cdots t_u; t'_1\cdots t'_v)\right\}^{1/2}$$

are dependent only on u $(0\leqq u\leqq l)$.

We next define orthogonal arrays, balanced arrays and simple arrays which may constitute AB and APB designs.

DEFINITION 3. A (0, 1) matrix T of size $m\times N$ is called a balanced array of size N, m constraints, strength t $(\leqq m)$ and index set $\mathcal{M}=\{\mu_i | i=0, 1,\cdots, t\}$ if for every $t\times N$ submatrix T_0 of T, every vector with weight (or number of nonzero elements) i occurs exactly μ_i times as a column of T_0.

DEFINITION 4. The matrix T of Definition 3 is called an orthogonal array of size N, m constraints, strength t and index λ if all μ_i's are equal, i.e., $\lambda=\mu_0=\mu_1=\cdots=\mu_t$.

DEFINITION 5. Let $\Omega(j; m)$ be a (0, 1) matrix of size $m\times\binom{m}{j}$ composed of all distinct column vectors with weight j. A matrix T obtained by juxtaposing each $\Omega(j; m)$, $(0\leqq j\leqq m)$, λ_j $(\geqq 0)$ times is called a simple array with parameters $(m; \lambda_0, \lambda_1,\cdots, \lambda_m)$.

For the above arrays, we write simply B-array $[N, m, t; \mathcal{M}]$, O-array $[N, m, t; \lambda]$ and S-array $[m, \lambda_0, \lambda_1,\cdots, \lambda_m]$, respectively. From Definitions 3 and 5, it is easy to check that an S-array $[m; \lambda_0,\cdots, \lambda_m]$ is a B-array $[N, m, t; \mathcal{M}]$, where

$$(2.1)\qquad N=\sum_{j=0}^{m}\lambda_j\binom{m}{j}, \qquad \mu_i=\sum_{j=0}^{m}\lambda_j\binom{m-t}{j-i}, \qquad \text{for } i=0, 1,\cdots, t.$$

Note that $\binom{a}{b}=0$ if and only if $b>a\geqq 0$ or $b<0$.

3. AB and APB designs

To avoid repetition, note throughout Sections 3 and 4 that a fraction T is assumed to be a 2^m-FF design of resolution $2l+1$ (i.e., the information matrix M is nonsingular). In this section, it is shown that some arrays defined in Section 2 yield AB and APB designs as fractions.

Let T be a fraction with N assemblies which is composed of n distinct assemblies $\boldsymbol{j}_q'=(j_{1q}, j_{2q}, \cdots, j_{mq})$, $(q=1,\cdots,n)$, with each multiplicity r_q $\left(N=\sum_{q=1}^{n} r_q\right)$. Further let $\boldsymbol{e}_q$ be the $\nu_l\times 1$ coefficient vector for $\boldsymbol{\theta}$ obtained from (1.1) according to the assembly $\boldsymbol{j}_q'$. Then we have

LEMMA 3.1. *For the above fraction T,*

$$AA'=2^m\{M^{-1}+M^{-1}HM^{-1}\}-I\,, \tag{3.1}$$

where I is the identity matrix of order ν_l and

$$H=\sum_{q=0}^{n} r_q(r_q-1)\boldsymbol{e}_q\boldsymbol{e}_q'\,. \tag{3.2}$$

PROOF. From (1.1) and (1.2), we have $[E:E^*][E:E^*]'=EE'+E^*E^{*\prime}=2^m\operatorname{diag}(G_{r_1}, G_{r_2},\cdots, G_{r_n})$, where G_r denotes the $r\times r$ matrix with all elements 1. Hence by an argument similar to the proof of Theorem 1 in Shirakura [8], it is easy to show that (3.1) holds.

Remark. In Shirakura [8], a $2^m\times 1$ vector $\boldsymbol{w}=(\boldsymbol{e}':\boldsymbol{e}^{*\prime})'$ in (1.1) is assumed to be normalized, i.e., $\boldsymbol{w}'\boldsymbol{w}=1$. As far as the problem of this paper is concerned, however, it may be assumed without loss of generality that such a vector $\boldsymbol{w}$ satisfies $\boldsymbol{w}'\boldsymbol{w}=2^m$.

THEOREM 3.1. *Let T be an O-array $[N, m, 2l; \lambda]$. Then T is an AB design.*

PROOF. It is well known (cf. [14]) that $M=NI=2^{2l}\lambda I$ holds for an O-array $[N, m, 2l; \lambda]$. Since the elements of $\boldsymbol{e}_q$ are -1 or 1, the diagonal elements of $M^{-1}HM^{-1}$ are all $(2^{2l}\lambda)^{-2}\sum_{q=0}^{n} r_q(r_q-1)$ $(=a$, say). Hence it follows from Lemma 3.1 that every diagonal element of AA' is equal to $2^m\{(2^{2l}\lambda)^{-1}+a\}-1$. This means that T is an AB design.

Recall an $(l+1)$ sets triangular type multidimensional partially balanced association algebra $\mathfrak{A}$ defined in [14]. Then we have

LEMMA 3.2. *For a fraction T, if $AA' \in \mathfrak{A}$, then T is an APB design.*

PROOF. The proof follows immediately from properties of $\mathfrak{A}$.

THEOREM 3.2. *Let T be a B-array $[N, m, 2l; \mathcal{M}]$ whose columns are all distinct. Then T is an APB design.*

PROOF. It has been shown in [14] that the information matrix M and its inverse M^{-1} belong to $\mathfrak{A}$. From Lemma 3.1, we also have $AA' = 2^m M^{-1} - I \in \mathfrak{A}$, since H in (3.2) vanishes. This completes the proof, because of Lemma 3.2.

Let $\mathbf{0}$ and $\mathbf{1}$ be the $m \times 1$ vectors with elements 0 and 1, respectively.

THEOREM 3.3. *For any nonnegative integers r_1 and r_2, let*

$$T = [\underbrace{\mathbf{0} : \mathbf{0} : \cdots : \mathbf{0}}_{r_1} : \underbrace{\mathbf{1} : \mathbf{1} : \cdots : \mathbf{1}}_{r_2} : T^*] , \tag{3.3}$$

where T^* is a B-array $[N^*, m, 2l; \mathcal{M}^* = \{\mu_0^*, \mu_1, \cdots, \mu_{2l-1}, \mu_{2l}^*\}]$ whose columns are all distinct and exclusive of $\mathbf{0}$ and $\mathbf{1}$. Then T is an APB design.

PROOF. It is clear that T is a B-array $[N = N^* + r_1 + r_2, m, 2l; \mathcal{M} = \{\mu_0 = \mu_0^* + r_1, \mu_1, \cdots, \mu_{2l-1}, \mu_{2l} = \mu_{2l}^* + r_2\}]$. Thus $M^{-1} \in \mathfrak{A}$. From (1.1), the $\nu_l \times 1$ coefficient vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ for $\boldsymbol{\theta}$ according to the assemblies $\mathbf{0}'$ and $\mathbf{1}'$, respectively, are given by

$$\boldsymbol{e}_1' = (1; -1, -1, \cdots, -1; 1, 1, \cdots, 1; \cdots; (-1)^l, (-1)^l, \cdots, (-1)^l) ,$$

$$\boldsymbol{e}_2' = (1; \underbrace{1, \ 1, \cdots, \ 1}_{m}; \underbrace{1, 1, \cdots, 1}_{\binom{m}{2}}; \cdots; \underbrace{1 \ , \ 1 \ , \cdots, \ 1}_{\binom{m}{l}}) .$$

This means that $F = \boldsymbol{e}_1 \boldsymbol{e}_1' \in \mathfrak{A}$ and $G = \boldsymbol{e}_2 \boldsymbol{e}_2' \in \mathfrak{A}$ hold. Thus $H = r_1(r_1 - 1)F + r_2(r_2 - 1)G \in \mathfrak{A}$ in (3.2). Hence $M^{-1}HM^{-1} \in \mathfrak{A}$, so that $AA' \in \mathfrak{A}$. From Lemma 3.2, the proof is completed.

THEOREM 3.4. *In Theorem 3.3, suppose $\mu_1 = \mu_2 = \cdots = \mu_{2l-1}$ $(= \lambda$, say$)$, $\lambda - 1 \leqq \mu_0^* \leqq \lambda$ and $r_1 = 0$ or 1 according as $\mu_0^* = \lambda$ or $\lambda - 1$. Then T of (3.3) is an AB design for any nonnegative integer r_2.*

PROOF. Clearly T is a B-array $[N = N^* + r_1 + r_2, m, 2l; \mathcal{M}]$ such that $\mu_0 = \mu_1 = \cdots = \mu_{2l-1} = \lambda$ and $\mu_{2l} = \mu_{2l}^* + r_2$. By (1.1) and (1.2), therefore, it can be shown that M is expressed as $M = 2^{2l}\lambda I + (\mu_{2l} - \lambda)G$. Thus M^{-1} can also be written as the form $M^{-1} = bI + cG$, where b and c are some real numbers. Let $\mathfrak{B}=[I, G]$ be an algebra generated by the matrices I and G. Then we have $I \in \mathfrak{B}$, $M^{-1} \in \mathfrak{B}$ and $M^{-1}e_2e_2'M^{-1}=M^{-1}GM^{-1} \in \mathfrak{B}$. Hence it follows from Lemma 3.1 that $AA' \in \mathfrak{B}$ holds. This means that the diagonal elements of AA' are the same, which completes the proof.

THEOREM 3.5. *Let T be an S-array $[m; \lambda_0, \lambda_1, \cdots, \lambda_m]$. Then T is an APB design.*

PROOF. Let $E_{(k)}$ be the $\binom{m}{k} \times \nu_l$ submatrix of E corresponding to $\Omega(k; m)$ in T (i.e., $E_{(k)}$ is the design matrix of the fraction $\Omega(k; m)$ relative to $\boldsymbol{\theta}$). By Definition 5, every assembly with weight k occurs λ_k times in T for each $k=0, 1, \cdots, m$. Therefore it is easy to see that H in (3.2) reduces to

$$H=\sum_{k=0}^{m} \lambda_k(\lambda_k-1)E'_{(k)}E_{(k)} .$$

Since T is a B-array $[N, m, 2l; \mathcal{M}]$ where N and μ_i's are given by (2.1), $M^{-1} \in \mathfrak{A}$ holds. Again from (2.1), $\Omega(k; m)$, $(k=0, \cdots, m)$, are themselves B-arrays $\left[\binom{m}{k}, m, 2l; \mathcal{M}^{(k)}=\{\mu_i^{(k)}\}\right]$ where $\mu_i^{(k)}=\binom{m-2l}{k-i}$, $(i=0, \cdots, 2l)$. Hence $E'_{(k)}E_{(k)} \in \mathfrak{A}$. Hence $H \in \mathfrak{A}$, so that $A'A \in \mathfrak{A}$. By Lemma 3.2, the proof is completed.

Remark. It is well known (cf. [14]) that an O-array $[N, m, 2l; \lambda]$ is equivalent to an orthogonal 2^m-FF design of resolution $2l+1$ with a desirable property that the covariance matrix $\mathrm{Var}[\hat{\boldsymbol{\theta}}]=M^{-1}\sigma^2$ is diagonal. Also it has been shown in [11] and [14] that a B-array $[N, m, 2l; \mathcal{M}]$ under the nonsingularity of M is equivalent to a balanced 2^m-FF design of resolution $2l+1$ with the second desirable property that $\mathrm{Var}[\hat{\boldsymbol{\theta}}]$ is invariant under any permutation of m factors. Furthermore, the results obtained in this section imply that the above two arrays have other desirable properties that they may yield AB and APB designs. Moreover, Theorm 3.4 means that AB designs are not always O-arrays of strength $2l$. For a general B-array $[N, m, 2l; \mathcal{M}]$ T, it is difficult to show whether T is an APB design, since the diagonal elements of H in (3.2) can not be explicitly expressed. However, it will be seen from the results of Shirakura [7], [9], Srivastava and/or Chopra [1], [2], [3], [4], [12], etc. that for practical values of m and N for $l=2$ or 3, most of B-arrays $[N, m, 2l; \mathcal{M}]$ are S-arrays $[m; \lambda_0, \cdots, \lambda_m]$ where a connection between the μ_i's and λ_j's is given by (2.1). Moreover for such given m and N, one of optimal balanced 2^m-FF designs of resolution V or VII with respect to the trace criterion can be obtained from such an S-array.

4. AB and APB designs generated by level permutations

In this section, we observe that a fraction which is not an O-array and a B-array may also constitute an AB or APB design. Consider a set $\boldsymbol{\Omega}=\{\boldsymbol{\omega}'=(\omega_1, \omega_2, \cdots, \omega_m) \mid \omega_i=0 \text{ or } 1;\ i=1, 2, \cdots, m\}$. For a fraction T with N assemblies and any $\boldsymbol{\omega}$ in $\boldsymbol{\Omega}$, define

$$T(\boldsymbol{\omega})=T+J(\boldsymbol{\omega}), \qquad (\text{mod } 2)\,,$$

where $J(\boldsymbol{\omega})$ denotes the $m\times N$ matrix whose columns are all $\boldsymbol{\omega}$. Then an element $\boldsymbol{\omega}$ in $\boldsymbol{\Omega}$ is called a level permutation and $T(\boldsymbol{\omega})$ is called a generated fraction by $\boldsymbol{\omega}$. We first prove the following lemma:

LEMMA 4.1. *Let T and $T(\boldsymbol{\omega})$ be a fraction with N assemblies and its generated fraction, respectively. Let $E(\boldsymbol{\omega})$ and $E^*(\boldsymbol{\omega})$ be the design matrices of $T(\boldsymbol{\omega})$ relative to $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^*$, respectively. Then*

$$E(\boldsymbol{\omega})=ED(\boldsymbol{\omega}) \qquad \textit{and} \qquad E^*(\boldsymbol{\omega})=E^*D^*(\boldsymbol{\omega}) \tag{4.1}$$

hold where $D(\boldsymbol{\omega})$ and $D^*(\boldsymbol{\omega})$ are respectively the $\nu_l\times\nu_l$ and $\nu_l^*\times\nu_l^*$ diagonal matrices given by

$$D(\boldsymbol{\omega})=\operatorname{diag}\,(1\,;(-1)^{\omega_1},\cdots,(-1)^{\omega_m};(-1)^{\omega_1+\omega_2},\cdots,(-1)^{\omega_{m-1}+\omega_m};\\ \cdots;(-1)^{\omega_1+\cdots+\omega_l},\cdots,(-1)^{\omega_{m-l+1}+\cdots+\omega_m})\,,$$

$$D^*(\boldsymbol{\omega})=\operatorname{diag}\,((-1)^{\omega_1+\cdots+\omega_{l+1}},\cdots,(-1)^{\omega_{m-l}+\cdots+\omega_m};\cdots;(-1)^{\omega_1+\cdots+\omega_m})\,.$$

PROOF. From (1.1), the expected value of observation for an assembly $(j_1+\omega_1,\cdots,j_m+\omega_m)$ is expressed as

$$\mathcal{E}(y(j_1+\omega_1,\cdots,j_m+\omega_m))=\boldsymbol{e}'(\boldsymbol{\omega})\boldsymbol{\theta}+\boldsymbol{e}^{*\prime}(\boldsymbol{\omega})\boldsymbol{\theta}^*\,,$$

where the elements of $(\boldsymbol{e}'(\boldsymbol{\omega}), \boldsymbol{e}^{*\prime}(\boldsymbol{\omega}))$ corresponding to $\theta_{t_1\cdots t_k}$ are $d(j_{t_1}+\omega_{t_1})\cdots d(j_{t_k}+\omega_{t_k})$, and the element to θ_ϕ is 1. Since $d(j_t+\omega_t)=(-1)^{\omega_t}d(j_t)$, (mod 2), for $t=1,\cdots,m$, it is clear that $\boldsymbol{e}'(\boldsymbol{\omega})=\boldsymbol{e}'D(\boldsymbol{\omega})$ and $\boldsymbol{e}^{*\prime}(\boldsymbol{\omega})=\boldsymbol{e}^{*\prime}\cdot D^*(\boldsymbol{\omega})$. From (1.2), we have (4.1).

As a matter of fact, it has been shown by Srivastava, Raktoe and Pesotan [13] that there exist orthogonal matrices P and P^* satisfying $E(\boldsymbol{\omega})=EP$ and $E^*(\boldsymbol{\omega})=E^*P^*$ for a more general asymmetric fractional design. However we have given here another proof of the lemma for explicit expressions of P and P^*. Similar expressions of P and P^* have also been given by Raktoe [6].

THEOREM 4.1. *If T is an AB (or APB) design, then for every $\boldsymbol{\omega}$ in $\boldsymbol{\Omega}$, $T(\boldsymbol{\omega})$ is also an AB (or APB) design.*

PROOF. Let $M(\boldsymbol{\omega})$ and $A(\boldsymbol{\omega})$ be the information and alias matrices of $T(\boldsymbol{\omega})$, respectively. Then from Lemma 4.1, we have $M(\boldsymbol{\omega})=D(\boldsymbol{\omega})\cdot MD(\boldsymbol{\omega})$ and therefore, $A(\boldsymbol{\omega})=D(\boldsymbol{\omega})M^{-1}E'E^*D^*(\boldsymbol{\omega})$. Hence

$$A(\boldsymbol{\omega})A'(\boldsymbol{\omega})=D(\boldsymbol{\omega})M^{-1}E'E^*E^{*\prime}EM^{-1}D(\boldsymbol{\omega})=D(\boldsymbol{\omega})AA'D(\boldsymbol{\omega}) .$$

This means that the diagonal elements of $A(\boldsymbol{\omega})A'(\boldsymbol{\omega})$ are the same as those of AA'. This completes the proof.

It is easy to verify that if T is an O-array $[N, m, 2l; \lambda]$, then $T(\boldsymbol{\omega})$ is also an O-array $[N, m, 2l; \lambda]$. In this case, therefore, Theorem 4.1 results in Theorem 3.1. However for any B-array (or S-array) T of Theorems 3.2–3.5, $T(\boldsymbol{\omega})$ can not be a B-array of strength $2l$ for every $\boldsymbol{\omega}$ in $\Omega-\{0, 1\}$ as long as T is neither an O-array $[N, m, 2l; \lambda]$ nor O-array $[N, m=2l, 2l-1; \lambda']$, (see Shirakura [10]). This means that (2^m-2) distinct AB designs (or APB designs) which are not B-arrays of strength $2l$ can be generated from the B-array T by level permutations. (For any two fractions T_1 and T_2 with N assemblies, T_1 is distinct from T_2 if $T_1\neq T_2Q$ for any permutation matrix Q of order N.) Note that for a B-array $[N, m, 2l; \mathcal{M}]$ T, $T(\mathbf{1})$ is also a B-array $[N, m, 2l; \bar{\mathcal{M}}=\{\bar{\mu}_i=\mu_{2l-i}\,|\,i=0, \cdots, 2l\}]$, it being called the complement of T. Thus as a corollary of Theorem 4.1, we have

COROLLARY 4.1. *In Theorem* 3.3, *suppose* $\mu_1=\mu_2=\cdots=\mu_{2l-1}$ $(=\lambda$, *say*), $\lambda-1\leqq\mu_{2l}^*\leqq\lambda$ *and* $r_2=0$ *or* 1 *according as* $\mu_{2l}^*=\lambda$ *or* $\lambda-1$. *Then* T *of* (3.3) *is an AB design for any nonnegative integer* r_1.

KOBE UNIVERSITY

REFERENCES

[1] Chopra, D. V. (1975). Balanced optimal 2^8 fractional factorial designs of resolution V, $52\leqq N\leqq 59$, *A Survey of Statistical Designs and Linear Models*, ed., J. N. Srivastava, North-Holland Publishing Co., Amsterdam, 91-100.

[2] Chopra, D. V. and Srivastava, J. N. (1973). Optimal balanced 2^7 fractional factorial designs of resolution V, with $N\leqq 42$, *Ann. Inst. Statist. Math.*, **25**, 587-604.

[3] Chopra, D. V. and Srivastava, J. N. (1973). Optimal balanced 2^7 fractional factorial designs of resolution V, $49\leqq N\leqq 55$, *Commun. Statist.*, **2**, 59-84.

[4] Chopra, D. V. and Srivastava, J. N. (1974). Optimal balanced 2^8 fractional factorial designs of resolution V, $37\leqq N\leqq 51$, *Sankhyā*, A, **36**, 41-52.

[5] Hedayat, A., Raktoe, B. L. and Federer, W. T. (1974). On a measure of aliasing due to fitting an incomplete model, *Ann. Statist.*, **2**, 650-660.

[6] Raktoe, B. L. (1976). On alias matrices and generalized defining relationships of equi-information factorial arrangements, *J. Roy. Statist. Soc.* B, **38**, 279-283.

[7] Shirakura, T. (1976). Optimal balanced fractional 2^m factorial designs of resolution VII, $6\leqq m\leqq 8$, *Ann. Statist.*, **4**, 515-531.

[8] Shirakura, T. (1976). A note on the norm of alias matrices in fractional replication, *Austral. J. Statist.*, 18, 158-160.

[9] Shirakura, T. (1977). Contributions to balanced fractional 2^m factorial designs derived from balanced arrays of strength $2l$, *Hiroshima Math. J.*, **7**, 217-285.

[10] Shirakura, T. (1978). Quasi-balanced fractional 2^m factorial designs, submitted for publication.

[11] Srivastava, J. N. (1970). Optimal balanced 2^m fractional factorial designs, *S. N. Roy Memorial Volume*, Univ. of North Carolina and Indian Statistical Institute, 689-706.

[12] Srivastava, J. N. and Chopra, D. V. (1971). Balanced optimal 2^m fractional factorial designs of resolution V, $m \leqq 6$, *Technometrics*, **13**, 257-269.

[13] Srivastava, J. N., Raktoe, B. L. and Pesotan, H. (1976). On invariance and randomization in fractional replication, *Ann. Statist.*, **4**, 423-430.

[14] Yamamoto, S., Shirakura, T. and Kuwada, M. (1975). Balanced arrays of strength $2l$ and balanced fractional 2^m factorial designs, *Ann. Inst. Statist. Math.*, **27**, 143-157.