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DISCRETIZED LIKELIHOOD METHODS—ASYMPTOTIC PROPERTIES OF DISCRETIZED LIKELIHOOD ESTIMATORS (DLE'S)[1)]

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Abstract

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a density $f(x, \theta)$. Let c_n be a maximum order of consistency. We consider a solution $\hat{\theta}_n$ of the discretized likelihood equation

$$\sum_{i=1}^{n} \log f(X_i, \hat{\theta}_n + rc_n^{-1}) - \sum_{i=1}^{n} \log f(X_i, \hat{\theta}_n) = a_n(\hat{\theta}_n, r)$$

where $a_n(\theta, r)$ is chosen so that $\hat{\theta}_n$ is asymptotically median unbiased (AMU). Then the solution $\hat{\theta}_n$ is called a discretized likelihood estimator (DLE). In this paper it is shown in comparison with DLE that a maximum likelihood estimator (MLE) is second order asymptotically efficient but not third order asymptotically efficient in the regular case. Further it is seen that the asymptotic efficiency (including higher order cases) may be systematically discussed by the discretized likelihood methods.

1. Introduction

Recently second order asymptotic efficiency has been studied by Chibisov [4], [5], Pfanzagl [8], [9], Takeuchi and Akahira [3], [11], Efron [6], Ghosh and Subramanyam [7] and others. Furthermore third order asymptotic efficiency was discussed in Takeuchi and Akahira [11], [12], [13] and Pfanzagl and Wefelmeyer [10]. In this paper using the discretized likelihood method we consider the asymptotic efficiency of estimators including higher order cases.

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a density $f(x, \theta)$. Let c_n be a maximum order of consistent estimator of θ. We have proposed a solution $\hat{\theta}_n$ of the discretized likeli-

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hood equation

(1.1) $$\sum_{i=1}^{n} \log f(X_i, \hat{\theta}_n + rc_n^{-1}) - \sum_{i=1}^{n} \log f(X_i, \hat{\theta}_n) = a_n(\hat{\theta}_n, r) ,$$

where $a_n(\theta, r)$ is chosen so that $\hat{\theta}_n$ is asymptotically median unbiased (AMU) (the possibility of which will be shown in the context) [3]. Then the solution $\hat{\theta}_n$ is called a discretized likelihood estimator (DLE). If for each real number r,

$$\sum_{i=1}^{n} \log f(X_i, \theta + rc_n^{-1}) - \sum_{i=1}^{n} \log f(X_i, \theta)$$

is locally monotone in θ, then the asymptotic distribution of the DLE $\hat{\theta}_n$ attains the bound of the asymptotic distributions (discussed below) of AMU estimators of θ at r. It is easily seen that there is at least one estimator which attain the bound. In regular cases with $c_n = \sqrt{n}$ the left-hand side of (1.1) is expanded as

(1.2) $$\frac{\partial}{\partial\theta} \sum_{i=1}^{n} \log f(X_i, \hat{\theta}_n) + \frac{r}{2\sqrt{n}} \frac{\partial^2}{\partial\theta^2} \sum_{i=1}^{n} \log f(X_i, \hat{\theta}_n) + \cdots = a_n(\hat{\theta}_n, r) .$$

We derive from (1.2) the order to which the maximum likelihood estimator (MLE) is asymptotically efficient. In this paper it is shown that an MLE is second order asymptotically efficient but not third order asymptotically efficient. The motivation for the definition of the DLE is that; when we test the hypothesis $\theta = \theta_0 + rc_n^{-1}$ against $\theta = \theta_0$, the most powerful test is given by rejecting the hypothesis if

$$\sum_{i=1}^{n} \log f(X_i, \theta_0 + rc_n^{-1}) - \sum_{i=1}^{n} \log f(X_i, \theta_0) < k_n$$

hence if an estimator $\hat{\theta}_r$ is defined so that the event $\hat{\theta}_r > \theta_0$ is equivalent to the above inequality (at least asymptotically up to some order), then $\hat{\theta}_r$ is efficient (asymptotically up to some order) for specified choice of r. Therefore if $\hat{\theta}_r$ can be defined independently of r, then it is shown to be efficient (asymptotically up to the above mentioned order), and if not, we can establish that there does not exist any efficient (in the same sense) estimator. It is also seen that the asymptotic efficiency (including higher order cases) may be systematically discussed by the discretized likelihood method.

2. Notations and definitions

Let $(\mathcal{X}, \mathcal{B})$ be a sample space. We consider a family of probability measures on $\mathcal{B}$, $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, where the index set Θ is called the parameter space. We assume that Θ is an open set in a Euclidean 1-space R^1. Consider n-fold direct products $(\mathcal{X}^{(n)}, \mathcal{B}^{(n)})$ of $(\mathcal{X}, \mathcal{B})$ and the corresponding product measures $P_{n,\theta}$ of P_θ. An estimator of θ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^{(n)}$-measurable functions $\hat{\theta}_n$ on $\mathcal{X}^{(n)}$ into Θ $(n=1, 2, \cdots)$. For simplicity we denote an estimator as $\hat{\theta}_n$ instead of $\{\hat{\theta}_n\}$. For increasing sequence of positive numbers $\{c_n\}$ (c_n tending to infinity) an estimator $\hat{\theta}_n$ is called consistent with order $\{c_n\}$ (or $\{c_n\}$-consistent for short) if for every $\varepsilon>0$ and every $\vartheta \in \Theta$ there exist a sufficiently small positive number δ and a sufficiently large number L satisfying the following:

$$\varlimsup_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} P_{n,\theta}\{c_n|\hat{\theta}_n-\theta|\geqq L\}<\varepsilon \qquad ([1]) .$$

For each $k=1, 2, \cdots$, a $\{c_n\}$-consistent estimator $\hat{\theta}_n$ is kth order asymptotically median unbiased (or kth order AMU) estimator if for any $\vartheta \in \Theta$, there exists a positive number δ such that

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} c_n^{k-1}\left|P_{n,\theta}\{\hat{\theta}_n\leqq\theta\}-\frac{1}{2}\right|=0 ;$$

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} c_n^{k-1}\left|P_{n,\theta}\{\hat{\theta}_n\geqq\theta\}-\frac{1}{2}\right|=0 .$$

For $\hat{\theta}_n$ kth order AMU, $G_0(t, \theta)+c_n^{-1}G_1(t, \theta)+\cdots+c_n^{-(k-1)}G_{k-1}(t, \theta)$ is called to be the kth order asymptotic distribution of $c_n(\hat{\theta}_n-\theta)$ (or $\hat{\theta}_n$ for short) if

$$\lim_{n\to\infty} c_n^{k-1}|P_{n,\theta}\{c_n(\hat{\theta}_n-\theta)<t\}-G_0(t, \theta)-c_n^{-1}(t, \theta)-\cdots-c_n^{-(k-1)}G_{k-1}(t, \theta)|=0 .$$

We note that $G_i(t, \theta)$ $(i=1, \cdots, k-1)$ may be generally absolute continuous functions, hence the asymptotic distributions for any fixed n may not be a distribution function.

Suppose that $\hat{\theta}_n$ is AMU and has the kth order asymptotic distribution $G_0(t, \theta)+c_n^{-1}G_1(t, \theta)+\cdots+c_n^{-(k-1)}G_{k-1}(t, \theta)$. Letting θ_0 $(\in\Theta)$ be arbitrary but fixed we consider the problem of testing hypothesis $H^+:\theta=\theta_0+tc_n^{-1}$ $(t>0)$ against $K:\theta=\theta_0$. Put $\Phi_{1/2}=\{\{\phi_n\}: \mathrm{E}_{n,\theta_0+tc_n^{-1}}(\phi_n)=1/2+o(c_n^{-(k-1)}),\ 0\leqq\phi_n(\tilde{x}_n)\leqq1$ for all $\tilde{x}_n \in \mathcal{X}^{(n)}$ $(n=1, 2, \cdots)\}$. Putting $A_{\hat{\theta}_n,\theta_0}=\{c_n(\hat{\theta}_n-\theta_0)\leqq t\}$, we have

$$\lim_{n\to\infty} P_{n,\theta_0+tc_n^{-1}}(A_{\hat{\theta}_n,\theta_0})=\lim_{n\to\infty} P_{n,\theta_0+tc_n^{-1}}\{\hat{\theta}_n\leqq\theta_0+tc_n^{-1}\}=\frac{1}{2} .$$

Hence it is seen that a sequence $\{\chi_{A_{\hat{\theta}_n,\theta_0}}\}$ of the indicators (or characteristic functions) of $A_{\hat{\theta}_n,\theta_0}$ $(n=1, 2, \cdots)$ belongs to $\Phi_{1/2}$. If

$$\sup_{\{\phi_n\}\in\Phi_{1/2}} \overline{\lim_{n\to\infty}}\, c_n^{k-1}\{\mathrm{E}_{n,\theta_0}(\phi_n)-H_0^+(t,\theta_0)-c_n^{-1}H_1^+(t,\theta_0)-\cdots-c_n^{-(k-1)}H_{k-1}^+(t,\theta_0)\}=0\ ,$$

then we have

$$G_0(t,\theta_0)\leqq H_0^+(t,\theta_0)\ ;$$

and for any positive integer j ($\leqq k$) if $G_i(t,\theta_0)=H_i^+(t,\theta_0)$ ($i=1,\cdots,j-1$), then

$$G_j(t,\theta_0)=H_j^+(t,\theta_0)\ .$$

Consider next the problem of testing hypothesis $H^-:\theta=\theta_0+tc_n^{-1}$ ($t<0$) against $K:\theta=\theta_0$. If

$$\inf_{\{\phi_n\}\in\Phi_{1/2}} \underline{\lim}_{n\to\infty}\, c_n^{k-1}\{\mathrm{E}_{n,\theta_0}(\phi_n)-H_0^-(t,\theta_0)-c_n^{-1}H_1^-(t,\theta_0)-\cdots-c_n^{-(k-1)}H_{k-1}^-(t,\theta_0)\}=0\ ,$$

then we have

$$G_0(t,\theta_0)\geqq H_0^-(t,\theta_0)\ ;$$

and for any positive integer j ($\leqq k$) if $G_i(t,\theta_0)=H_i^-(t,\theta_0)$ ($i=0,\cdots,j-1$), then $G_j(t,\theta_0)\geqq H_j^-(t,\theta_0)$.

$\hat{\theta}_n$ is called to be kth order asymptotically efficient if the kth order asymptotic distribution of it attains uniformly the bound of the kth order asymptotic distributions of kth order AMU estimators, that is, for each $\theta\in\Theta$

$$G_i(t,\theta)=\begin{cases} H_i^+(t,\theta) & \text{for } t>0\ ,\\ H_i^-(t,\theta) & \text{for } t<0\ ,\end{cases}$$

$i=0,\cdots,k-1$ ([2], [11]). (Note that for $t=0$ we have $G_i(0,\theta)=H_i^+(0,\theta)=H_i^-(0,\theta)$ ($i=0,\cdots,k-1$) from the condition of kth order asymptotically median unbiasedness.)

We assume that for each $\theta\in\Theta$ P_θ is absolutely continuous with respect to σ-finite measure μ.

We denote a density $dP_\theta/d\mu$ by $f(x,\theta)$. Let $L(\theta;\tilde{x}_n)$ be a likelihood function, that is, $L(\theta;\tilde{x}_n)=\prod_{i=1}^n f(x_i,\theta)$, where $\tilde{x}_n=(x_1,x_2,\cdots,x_n)$. For each $k=1,2,\cdots$, a $\{c_n\}$-consistent estimator $\hat{\theta}_n$ is called discretized likelihood estimator (DLE) if for each real number r, $\hat{\theta}_n$ satisfies the discretized likelihood equation

$$\log L(\hat{\theta}_n+rc_n^{-1};\tilde{x}_n)-\log L(\hat{\theta}_n;\tilde{x}_n)=a_n(\hat{\theta}_n,r)\ , \tag{2.1}$$

where $a_n(\theta, r)$ is a function in θ and r and it also depends on n. The function $a_n(\theta, r)$ is not defined for the moment but will be determined in the sequel so that the solution obtained from the above equation be asymptotically median unbiased up to kth order. It should be noted that DLE $\hat{\theta}_n$ is required to be $\{c_n\}$-consistent and we do not claim that the solution of the equation (2.1) be $\{c_n\}$-consistent. We implicitely claim that there exists a solution of the equation in the $O(c_n^{-1})$-neighborhood of the true value. In the practical situation we have to obtain the DLE first by finding a $\{c_n\}$-consistent estimator $\tilde{\theta}_n$ in some way or another then find a solution of the equation in the neighborhood of $\tilde{\theta}_n$. Suppose that for given function $a_n(\theta, r)$,

$$\log L(\theta+rc_n^{-1}; \tilde{x}_n)-\log L(\theta; \tilde{x}_n)-a_n(\theta, r) \tag{2.2}$$

is locally monotone in θ with probability larger than $1-o(c_n^{-(k-1)})$. For regular case the particular form of $a_n(\theta, r)$ will be given later (e.g. page 47 etc.). For the present it is only necessary to remark that $a_n(\theta, r)$ is of the magnitude of order smaller than the previous terms of (2.2). Then the kth order asymptotic distribution of the DLE $\hat{\theta}_{\mathrm{DL}}$ attains the bound of the kth order asymptotic distributions of kth order AMU estimators of θ at r. Indeed, it follows by the monotone of (2.2) that for any $\vartheta \in \Theta$, there exists a positive number δ such that

$$\lim_{n\to\infty} \sup_{\theta:\,|\theta-\vartheta|<\delta} c_n^{k-1}|P_{n,\theta}\{\hat{\theta}_n>\theta-rc_n^{-1}\}-P_{n,\theta}\{\log L(\theta, \tilde{x}_n) \\ -\log L(\theta-rc_n^{-1}, \tilde{x}_n)>a_n(\theta-rc_n^{-1}, r)\}|=0 . \tag{2.3}$$

Letting θ_0 ($\in \Theta$) be arbitrary but fixed we consider the problem of testing hypothesis H: $\theta=\theta_0-rc_n^{-1}$ ($r>0$) against alternative K: $\theta=\theta_0$. Putting $A_{\hat{\theta}_n,\theta}=\{c_n(\hat{\theta}_n-\theta)>-r\}$ we have $P_{n,\theta_0-rc_n^{-1}}(A_{\hat{\theta}_n,\theta_0})=1/2+o(c_n^{-(k-1)})$. Let $\mathcal{U}_k$ be the class of the all kth order AMU estimators. Set $\Phi_{1/2}=\{\{\phi_n\}$: $\mathrm{E}_{n,\theta_0-rc_n^{-1}}(\phi_n)=1/2+o(c_n^{-(k-1)})$, $0\leq\phi_n(\tilde{x}_n)\leq 1$ for all $\tilde{x}_n \in \mathcal{X}^{(n)}$ $(n=1, 2,\cdots)\}$. It is noted that every sequence $\{\chi_{A_{\hat{\theta}_n,\theta}}(\tilde{x}_n)\}$ of the indicators of the sets $A_{\hat{\theta}_n,\theta}$ with the estimators $\hat{\theta}_n$ in $\mathcal{U}_k$ is contained in $\Phi_{1/2}$. In order to obtain the upper bound of $\overline{\lim}_{n\to\infty} P_{n,\theta_0}(A_{\hat{\theta}_n,\theta_0})$ in $\mathcal{U}_k$, it is sufficient to find a sequence $\{\phi_n^*\}$ of the tests which maximize $\overline{\lim}_{n\to\infty} \mathrm{E}_{n,\theta_0}(\phi_n)$ in $\Phi_{1/2}$. It is shown by the Neyman-Pearson fundamental lemma that ϕ_n^* has the rejection S_n satisfying

$$\sum_{i=1}^{n} \log \frac{f(X_i, \theta_0-rc_n^{-1})}{f(X_i, \theta_0)}<k_n ,$$

where k_n is some constant. Then it follows from (2.3) that the upper bound of $\overline{\lim}_{n\to\infty} P_{n,\theta_0}(A_{\hat{\theta}_n,\theta_0})$ in $\mathcal{U}_k$ is given by $\overline{\lim}_{n\to\infty} \mathrm{E}_{n,\theta_0}(\phi_n^*)$. Hence the kth order asymptotic distribution of the DLE $\hat{\theta}_{\mathrm{DL}}$ attains the bound of the kth order asymptotic distributions of kth order AMU estimators at $-r$. In a similar way as the case when $r>0$, we also obtain for $r<0$ the upper bound of $\overline{\lim}_{n\to\infty} P_{n,\theta_0}(A^c_{\hat{\theta}_n,\theta_0})$ in $\mathcal{U}_k$ of the same form. Hence the desired result also holds for the case $r<0$. In later sections it will be seen that $\hat{\theta}_{\mathrm{DL}}$ is asymptotically efficient up to second order. Note that the DLE usually depends on r in cases more than third order.

In the subsequent discussion we shall deal with the case when $c_n = \sqrt{n}$.

3. Second order asymptotic efficiency

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a density $f(x, \theta)$ satisfying (i)~(iv).

(i) $\{x: f(x, \theta)>0\}$ does not depend on θ;

(ii) For almost all $x[\mu]$, $f(x, \theta)$ is three times continuously differentiable in θ;

(iii) For each $\theta \in \Theta$

$$0<I(\theta)=\mathrm{E}_\theta\left[\left\{\frac{\partial}{\partial\theta}\log f(X, \theta)\right\}^2\right]=-\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(X, \theta)\right]<\infty\ ;$$

(iv) There exist

$$J(\theta)=\mathrm{E}_\theta\left[\left\{\frac{\partial^2}{\partial\theta^2}\log f(X, \theta)\right\}\left\{\frac{\partial}{\partial\theta}\log f(X, \theta)\right\}\right]$$

and

$$K(\theta)=\mathrm{E}_\theta\left[\left\{\frac{\partial}{\partial\theta}\log f(X, \theta)\right\}^3\right]$$

and the following holds:

$$\mathrm{E}_\theta\left[\frac{\partial^3}{\partial\theta^3}\log f(X, \theta)\right]=-3J(\theta)-K(\theta)\ .$$

By the following way we have shown in [11] that an MLE is second order asymptotically efficient. Let $\hat{\theta}_{\mathrm{ML}}$ be a maximum likelihood estimator. By Taylor expansion we have

$$\begin{aligned}0&=\sum_{i=1}^n \frac{\partial}{\partial\theta}\log f(X_i, \hat{\theta}_{\mathrm{ML}})\\&=\sum_{i=1}^n \frac{\partial}{\partial\theta}\log f(X_i, \theta)+\sum_{i=1}^n\left\{\frac{\partial^2}{\partial\theta^2}\log f(X_i, \theta)\right\}(\hat{\theta}_{\mathrm{ML}}-\theta)\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\left\{\frac{\partial^3}{\partial\theta^3}\log f(X_i,\theta^*)\right\}(\hat\theta_{\mathrm{ML}}-\theta)^2,$$

where $|\theta^*-\theta|\leqq|\hat\theta_{\mathrm{ML}}-\theta|$. Putting $T_n=\sqrt{n}(\hat\theta_{\mathrm{ML}}-\theta)$ we obtain

$$0=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i,\theta)+\frac{1}{n}\left\{\sum_{i=1}^{n}\frac{\partial^2}{\partial\theta^2}\log f(X_i,\theta)\right\}T_n$$
$$+\frac{1}{2n\sqrt{n}}\left\{\sum_{i=1}^{n}\frac{\partial^3}{\partial\theta^3}\log f(X_i,\theta^*)\right\}T_n^2.$$

Set

$$Z_1(\theta)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i,\theta);$$

$$Z_2(\theta)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\frac{\partial^2}{\partial\theta^2}\log f(X_i,\theta)+I(\theta)\right\};$$

$$W(\theta)=\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^3}{\partial\theta^3}\log f(X_i,\theta).$$

Then it follows that $W(\theta)$ converges in probability to $-3J(\theta)-K(\theta)$. Hence the following theorem holds:

THEOREM 1 ([11]). *Under conditions* (i)~(iv)

$$\text{(3.1)}\qquad \sqrt{n}(\hat\theta_{\mathrm{ML}}-\theta)=\frac{Z_1(\theta)}{I(\theta)}+\frac{Z_1(\theta)Z_2(\theta)}{\sqrt{n}I(\theta)^2}-\frac{3J(\theta)+K(\theta)}{2\sqrt{n}I(\theta)^3}Z_1(\theta)^2+o_p\left(\frac{1}{\sqrt{n}}\right)$$

up to order $n^{-1/2}$ *as* $n\to\infty$.

Put

$$\hat\theta^*_{\mathrm{ML}}=\hat\theta_{\mathrm{ML}}+\frac{K(\hat\theta_{\mathrm{ML}})}{6nI(\hat\theta_{\mathrm{ML}})^2}.$$

Then $\hat\theta^*_{\mathrm{ML}}$ is second order AMU. From Theorem 1 we have established the following:

THEOREM 2 ([11]). *Under conditions* (i)~(iv), $\hat\theta_{\mathrm{ML}}$ *is second order asymptotically efficient.*

In the sequel we obtain the same result for DLE. We further assume the following:

(v) For given function $a_n(\theta, r)$

$$\log L(\theta+rn^{-1/2},\tilde{x}_n)-\log L(\theta,\tilde{x}_n)-a_n(\theta,r)$$

is locally monotone in θ with probability larger than $1-o(n^{-1})$.

Remark. In usual situation it is generally true since $(1/n)(\partial^2/\partial\theta^2)\cdot$

$\log L(\theta, \tilde{x}_n)$ is asymptotically equal to $-I(\theta)$ (<0) and $a_n(\theta, r)$ is smaller order than n^{-1}, and is usually of constant order ($O(1)$) as is shown below. Let $\hat{\theta}_n$ be an DLE. Since

$$\sum_{i=1}^{n} \log f(X_i, \hat{\theta}_n + rn^{-1/2}) - \sum_{i=1}^{n} \log f(X_i, \hat{\theta}_n) = a_n ,$$

it follows by Taylor expansion that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(X_i, \hat{\theta}_n) + \frac{r}{2n} \sum_{i=1}^{n} \frac{\partial^2}{\partial\theta^2} \log f(X_i, \hat{\theta}_n) + \frac{r^2}{6n\sqrt{n}} \sum_{i=1}^{n} \frac{\partial^3}{\partial\theta^3} \log f(X_i, \theta_n^*) = \frac{a_n}{r} ,$$

where

$$|\theta_n^* - \hat{\theta}_n| < \frac{r}{\sqrt{n}} .$$

Since $(1/n) \sum_{i=1}^{n} \{(\partial^3/\partial\theta^3) \log f(X_i, \theta)\}$ converges in probability to $-3J(\theta)-K(\theta)$, it is seen that

$$Z_1(\hat{\theta}_n) + \frac{r}{2}\left\{-I(\hat{\theta}_n) + \frac{1}{\sqrt{n}} Z_2(\hat{\theta}_n)\right\} - \frac{r^2}{6\sqrt{n}}\{3J(\hat{\theta}_n) + K(\hat{\theta}_n)\} + o_p\left(\frac{1}{\sqrt{n}}\right) = \frac{a_n}{r} . \tag{3.2}$$

On the other hand we have

$$Z_1(\hat{\theta}_n) = Z_1(\theta) + \frac{1}{\sqrt{n}}\{Z_2(\theta) - \sqrt{n}\, I(\theta)\}T_n - \frac{3J(\theta)+K(\theta)}{2\sqrt{n}} T_n^2 + o_p\left(\frac{1}{\sqrt{n}}\right) , \tag{3.3}$$

where $T_n = \sqrt{n}(\hat{\theta}_n - \theta)$. Since

$$Z_1(\hat{\theta}_n) = Z_1(\theta) + \frac{1}{\sqrt{n}} Z_2(\theta) T_n - I(\theta) T_n - \frac{3J(\theta)+K(\theta)}{2\sqrt{n}} T_n^2 + o_p\left(\frac{1}{\sqrt{n}}\right) ,$$

it follows that

$$T_n = \frac{1}{I(\theta)}\left\{-Z_1(\hat{\theta}_n) + Z_1(\theta) + \frac{1}{\sqrt{n}} Z_2(\theta) T_n - \frac{3J(\theta)+K(\theta)}{2\sqrt{n}} T_n^2\right\} + o_p\left(\frac{1}{\sqrt{n}}\right) . \tag{3.4}$$

Since

$$I(\hat{\theta}_n) = I(\theta) + \frac{1}{\sqrt{n}}\{2J(\theta) + K(\theta)\}T_n + o_p\left(\frac{1}{\sqrt{n}}\right) ;$$

$$J(\hat{\theta}_n)=J(\theta)+\frac{1}{\sqrt{n}}J'(\theta)T_n+o_p\left(\frac{1}{\sqrt{n}}\right) ;$$

$$K(\hat{\theta}_n)=K(\theta)+\frac{1}{\sqrt{n}}K'(\theta)T_n+o_p\left(\frac{1}{\sqrt{n}}\right) ;$$

$$Z_2(\hat{\theta}_n)=Z_2(\theta)-J(\theta)T_n+o_p(1) ,$$

it follows from (3.2) that

$$Z_1(\hat{\theta}_n)=\frac{a_n}{r}+\frac{r}{2}I(\theta)+\frac{r}{6\sqrt{n}}\{3rJ(\theta)+rK(\theta)-3Z_2(\theta)\}$$
$$+\frac{r}{2\sqrt{n}}\{3J(\theta)+K(\theta)\}T_n+o_p\left(\frac{1}{\sqrt{n}}\right)$$

up to order $n^{-1/2}$. From (3.4) we have

$$T_n=-\frac{a'_n}{rI}-\frac{r^2(3J+K)}{6I\sqrt{n}}+\frac{Z_1}{I}+\frac{Z_1Z_2}{I^2\sqrt{n}}-\frac{(3J+K)Z_1^2}{2I^3\sqrt{n}}+o_p\left(\frac{1}{\sqrt{n}}\right)$$

up to order $n^{-1/2}$, where $a_n=-(rI/2)+a'_n$ with $a'_n=o(1/\sqrt{n})$ so that $\hat{\theta}_n$ be AMU. In order to have second order asymptotic median unbiasedness of $\hat{\theta}_n$ we put

$$a'_n=-\frac{r^3(3J+K)}{6\sqrt{n}}-\frac{rK}{6I\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)$$

and we denote by T_n^* the corresponding T_n to this particular value of a_n. Then we may also denote $T_n^*=\sqrt{n}(\hat{\theta}_n^*-\theta)$. Then it is shown that $\hat{\theta}_n^*$ is second order AMU. Hence we have established the following theorem.

THEOREM 3. *Under conditions* (i)~(v), *the* DLE $\hat{\theta}_n^*$ *with* a_n *defined above satisfies the following*:

$$T_n^*=\sqrt{n}(\hat{\theta}_n^*-\theta)$$
$$=\frac{K}{6I^2\sqrt{n}}+\frac{Z_1(\theta)}{I(\theta)}+\frac{Z_1(\theta)Z_2(\theta)}{I(\theta)^2\sqrt{n}}-\frac{3J(\theta)+K(\theta)}{2I(\theta)^3\sqrt{n}}Z_1(\theta)^2$$
$$+\frac{r}{2I(\theta)\sqrt{n}}\left\{Z_2(\theta)-\frac{3J(\theta)+K(\theta)}{I(\theta)}Z_1(\theta)\right\}+o_p\left(\frac{1}{\sqrt{n}}\right)$$

up to order $n^{-1/2}$ *as* $n\to\infty$.

Remark. Since we have

$$\mathrm{E}_\theta\left[Z_1(\theta)\left\{Z_2(\theta)-\frac{3J(\theta)+K(\theta)}{I(\theta)}Z_1(\theta)\right\}\right]=0 ;$$

$$V_\theta(T_n^*)=\frac{1}{I(\theta)}+o\left(\frac{1}{\sqrt{n}}\right),$$

and the third order cumulant is equal to that of the MLE up to the order $n^{-1/2}$. Hence the asymptotic expansion of the distribution of T_n^* is equal to that of $\sqrt{n}(\hat{\theta}_{\mathrm{ML}}^*-\theta)$ up to the order $n^{-1/2}$. Therefore the DLE $\hat{\theta}_n^*$ is asymptotically equivalent to the MLE $\hat{\theta}_{\mathrm{ML}}^*$ up to that order.

4. Third order asymptotic efficiency

We proceed to the problem of third order asymptotic efficiency. We further assume the following:

(vi) For almost all $x[\mu]$, $f(x,\theta)$ is four times continuously differentiable in θ;

(vii) there exist

$$L(\theta)=\mathrm{E}_\theta\left[\left\{\frac{\partial^3}{\partial\theta^3}\log f(X,\theta)\right\}\left\{\frac{\partial}{\partial\theta}\log f(X,\theta)\right\}\right];$$

$$M(\theta)=\mathrm{E}_\theta\left[\left\{\frac{\partial^2}{\partial\theta^2}\log f(X,\theta)\right\}^2\right];$$

$$N(\theta)=\mathrm{E}_\theta\left[\left\{\frac{\partial^2}{\partial\theta^2}\log f(X,\theta)\right\}\left\{\frac{\partial}{\partial\theta}\log f(X,\theta)\right\}^2\right]$$

and

$$H(\theta)=\mathrm{E}_\theta\left[\left\{\frac{\partial}{\partial\theta}\log f(X,\theta)\right\}^4\right]$$

and the following holds:

$$\mathrm{E}_\theta\left[\frac{\partial^4}{\partial\theta^4}\log f(X,\theta)\right]=-4L(\theta)-3M(\theta)-6N(\theta)-H(\theta).$$

Let $\hat{\theta}_n$ be an DLE. Since

$$\sum_{i=1}^n\log f\left(X_i,\hat{\theta}_n+\frac{r}{\sqrt{n}}\right)-\sum_{i=1}^n\log f(X_i,\hat{\theta}_n)=a_n,$$

it follows that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n\frac{\partial}{\partial\theta}\log f(X_i,\hat{\theta}_n)+\frac{r}{2n}\sum_{i=1}^n\frac{\partial^2}{\partial\theta^2}\log f(X_i,\hat{\theta}_n)$$
$$+\frac{r^2}{6n\sqrt{n}}\sum_{i=1}^n\frac{\partial^3}{\partial\theta^3}\log f(X_i,\hat{\theta}_n)+\frac{r^3}{24n^2}\sum_{i=1}^n\frac{\partial^4}{\partial\theta^4}\log f(X_i,\theta^*)$$
$$=\frac{a_n}{r},$$

where $|\theta^*-\theta|<r/\sqrt{n}$. Since $(1/n)\sum_{i=1}^{n}(\partial^4/\partial\theta^4)\log f(X_i,\theta)$ converges in probability to $-4L(\theta)-3M(\theta)-6N(\theta)-H(\theta)$, it is seen that

$$Z_1(\hat{\theta}_n)+\frac{r}{2}\left\{-I(\hat{\theta}_n)+\frac{1}{\sqrt{n}}Z_2(\hat{\theta}_n)\right\}-\frac{r^2}{6\sqrt{n}}\{3J(\hat{\theta}_n)+K(\hat{\theta}_n)\}$$
$$+\frac{r^2}{6n}Z_3(\hat{\theta}_n)-\frac{r^3}{24n}\{4L(\hat{\theta}_n)+3M(\hat{\theta}_n)+6N(\hat{\theta}_n)+H(\hat{\theta}_n)\}\sim\frac{a_n}{r},$$

where

$$Z_3(\theta)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\frac{\partial^3}{\partial\theta^3}\log f(X_i,\theta)+3J(\theta)+K(\theta)\right\}.$$

Hence

$$(4.1)\qquad Z_1(\hat{\theta}_n)\sim\frac{a_n}{r}+\frac{1}{2}I(\hat{\theta}_n)-\frac{r}{2\sqrt{n}}Z_2(\hat{\theta}_n)+\frac{r^2}{6\sqrt{n}}\{3J(\hat{\theta}_n)+K(\hat{\theta}_n)\}$$
$$-\frac{r^2}{6n}Z_3(\hat{\theta}_n)+\frac{r^3}{24n}\{4L(\hat{\theta}_n)+3M(\hat{\theta}_n)+6N(\hat{\theta}_n)+H(\hat{\theta}_n)\}.$$

On the other hand we have

$$Z_1(\hat{\theta}_n)=Z_1(\theta)+\frac{1}{\sqrt{n}}\{Z_2(\hat{\theta}_n)-\sqrt{n}I(\theta)\}T_n-\frac{3J(\theta)+K(\theta)}{2\sqrt{n}}T_n^2$$
$$-\frac{1}{6n}\{4L(\theta)+3M(\theta)+6N(\theta)+H(\theta)\}T_n^3+o_p\left(\frac{1}{n}\right),$$

where $T_n=\sqrt{n}(\hat{\theta}_n-\theta)$. We obtain

$$(4.2)\qquad T_n=\frac{1}{I(\theta)}\left[-Z_1(\hat{\theta}_n)+Z_1(\theta)+\frac{1}{\sqrt{n}}Z_2(\theta)T_n-\frac{3J(\theta)+K(\theta)}{2\sqrt{n}}T_n^2\right.$$
$$\left.-\frac{1}{6n}\{4L(\theta)+3M(\theta)+6N(\theta)+H(\theta)\}T_n^3\right]+o_p\left(\frac{1}{n}\right).$$

Since

$$I(\hat{\theta}_n)=I(\theta)+\frac{1}{\sqrt{n}}\{2J(\theta)+K(\theta)\}T_n$$
$$+\frac{1}{2n}\{2L(\theta)+2M(\theta)+5N(\theta)+H(\theta)\}T_n^2+o_p\left(\frac{1}{n}\right);$$

$$J(\hat{\theta}_n)=J(\theta)+\frac{1}{\sqrt{n}}\{L(\theta)+M(\theta)+N(\theta)\}T_n+o_p\left(\frac{1}{\sqrt{n}}\right);$$

$$K(\hat{\theta}_n)=K(\theta)+\frac{1}{\sqrt{n}}\{3N(\theta)+H(\theta)\}T_n+o_p\left(\frac{1}{\sqrt{n}}\right);$$

$$L(\hat{\theta}_n)=L(\theta)+\frac{1}{\sqrt{n}}L'(\theta)T_n+o_p\left(\frac{1}{\sqrt{n}}\right);$$

$$M(\hat{\theta}_n)=M(\theta)+\frac{1}{\sqrt{n}}M'(\theta)T_n+o_p\left(\frac{1}{\sqrt{n}}\right);$$

$$N(\hat{\theta}_n)=N(\theta)+\frac{1}{\sqrt{n}}N'(\theta)T_n+o_p\left(\frac{1}{\sqrt{n}}\right);$$

$$H(\hat{\theta}_n)=H(\theta)+\frac{1}{\sqrt{n}}H'(\theta)T_n+o_p\left(\frac{1}{\sqrt{n}}\right);$$

$$Z_2(\hat{\theta}_n)=Z_2(\theta)-J(\theta)T_n-\frac{1}{2\sqrt{n}}\{2L(\theta)+M(\theta)+N(\theta)\}T_n^2+o_p\left(\frac{1}{\sqrt{n}}\right);$$

and

$$Z_3(\hat{\theta}_n)=\frac{1}{\sqrt{n}}Z_3'(\theta)T_n+o_p\left(\frac{1}{\sqrt{n}}\right),$$

it follows from (4.1) that

$$\begin{aligned}Z_1(\hat{\theta}_n)=&\frac{a_n}{r}+\frac{r}{2}I-\frac{r}{2\sqrt{n}}Z_2+\frac{r^2(3J+K)}{6\sqrt{n}}-\frac{r^2}{6n}Z_3+\frac{r}{2\sqrt{n}}(3J+K)T_n\\&+\frac{r^3}{24n}(4L+3M+6N+H)+\frac{r}{4n}(4L+3M+6N+H)T_n^2\\&+\frac{r^2}{6n}(3L+3M+6N+H)T_n+o_p\left(\frac{1}{n}\right)\end{aligned}$$

up to order n^{-1} as $n\to\infty$. Under conditions (i)~(vii) we have from (4.2)

$$\begin{aligned}(4.3)\quad T_n=&-\frac{a_n}{rI}-\frac{r}{2}-\frac{r^2(3J+K)}{6I\sqrt{n}}+\frac{Z_1}{I}+\frac{rZ_2}{2I\sqrt{n}}+\frac{r^2Z_3}{6In}-\frac{r^3}{24In}\\&-\frac{r}{2I\sqrt{n}}(3J+K)T_n-\frac{r^2\mathcal{L}'}{2In}T_n+\frac{Z_2}{I\sqrt{n}}T_n-\frac{3J+K}{2I\sqrt{n}}T_n^2\\&-\frac{r}{4In}\mathcal{L}T_n^2-\frac{1}{6In}\mathcal{L}T_n^3+o_p\left(\frac{1}{n}\right)\\=&-\frac{a_n''}{rI}+\frac{K}{6I^2\sqrt{n}}+\frac{Z_1}{I}+\frac{r}{2I\sqrt{n}}\left(Z_2-\frac{3J+K}{I}Z_1\right)+\frac{Z_1Z_2}{I^2\sqrt{n}}\\&-\frac{(3J+K)Z_1^2}{2I^3\sqrt{n}}+\frac{1}{I^3n}\left\{Z_1Z_2^2-(3J+K)Z_1^2Z_2-\frac{\mathcal{L}}{6}Z_1^3\right.\\&+\frac{(3J+K)^2Z_1^2}{2I^2}-\frac{(3J+K)Z_1Z_2}{2I}-\frac{(3J+K)KZ_1}{6I}+\frac{KZ_2}{6}\\&-\frac{r(3J+K)}{12}-\frac{r^2I(3J+K)Z_2}{4}+\frac{r^2(3J+K)Z_1}{4}\end{aligned}$$

$$-\frac{3r(3J+K)Z_1Z_2}{2}+\frac{r(3J+K)^2Z_1^2}{4I}+\frac{rIZ_2^2}{2}-\frac{r^2I\mathcal{L}'Z_1}{2}$$
$$+\frac{r(3J+K)^2Z_1^2}{2I}-\frac{r}{4}\mathcal{L}Z_1^2\Big\}+o_p\Big(\frac{1}{n}\Big)$$

up to order n^{-1} as $n\to\infty$, where $a_n=-(rI/2)-r^3(3J+K)/6I\sqrt{n}+rK/6I\sqrt{n}+a_n''$ with $a_n''=O(1/n)$ and $\mathcal{L}=4L+3M+6N+H$ and $\mathcal{L}'=3L+3M+6N+H$. We denote by T_n^* the corresponding T_n to this particular value of a_n. Then we may also denote $T_n^*=\sqrt{n}(\hat{\theta}_n^*-\theta)$. On the other hand under conditions (i)~(vii) we have obtained in [11]

$$(4.4)\quad \sqrt{n}(\hat{\theta}_{\mathrm{ML}}-\theta)=\frac{Z_1}{I}+\frac{1}{I^2\sqrt{n}}\Big\{Z_1Z_2-\frac{(3J+K)Z_1^2}{2I}\Big\}$$
$$+\frac{1}{I^3n}\Big\{Z_1Z_2^2+\frac{1}{2}Z_1^2Z_3-\frac{3J+K}{2I}Z_1^2Z_2$$
$$+\frac{(3J+K)^2}{2I^2}Z_1^3-\frac{4L+3M+6N+H}{6I}Z_1^3\Big\}+o_p\Big(\frac{1}{n}\Big).$$

Let $\hat{\theta}_{\mathrm{ML}}^*$ be a modified MLE so that it is third order AMU. Comparing (4.3) with (4.4), we see that $\sqrt{n}(\hat{\theta}_{\mathrm{ML}}^*-\theta)$ and T_n^* are essentially different in the order n^{-1}. We put $T_{\mathrm{ML}}^*=\sqrt{n}(\hat{\theta}_{\mathrm{ML}}^*-\theta)$. Note that the difference of T_{ML}^* and T_n^* appears in the linear term of order $n^{-1/2}$ and other terms of order n^{-1}. It is shown in [12], [13] and [14] that the asymptotic cumulants are determined up to order n^{-1} by the terms of order up to $n^{-1/2}$ if the first term is equal to $Z_1(\theta)/I(\theta)$; and the fourth order cumulant is identical in the first term of order n^{-1} for all asymptotically efficient estimators. In the third order cumulants we have

$$\kappa_3(T_{\mathrm{ML}}^*-\mathrm{E}_\theta(T_{\mathrm{ML}}^*))=\frac{\beta_3(\theta)}{\sqrt{n}}+o\Big(\frac{1}{n}\Big);$$

$$\kappa_3(T_n^*-\mathrm{E}_\theta(T_n^*))=\frac{\beta_3(\theta)}{\sqrt{n}}+\frac{\gamma_3(\theta)}{n}+o\Big(\frac{1}{n}\Big).$$

Therefore there is a difference of asymptotic distributions of $\sqrt{n}(\hat{\theta}_{\mathrm{ML}}^*-\theta)$ and T_n^* in the term of n^{-1} if $\gamma_3(\theta)\neq 0$. If we denote by Theorem 3 and (4.3)

$$T_n^*=\frac{Z_1}{I}-\frac{J+K}{6I^2\sqrt{n}}+\frac{r}{\sqrt{n}}L^*+\frac{1}{\sqrt{n}}(Q-c)+\frac{1}{n}R+o_p\Big(\frac{1}{n}\Big),$$

where

$$L^*=\frac{1}{2I}\Big(Z_2-\frac{3J+K}{I}Z_1\Big);$$

$$Q=\frac{1}{I^2}\left(Z_1Z_2-\frac{3J+K}{2I}Z_1^2\right) ;$$

$$c=-\frac{J+K}{2I^2} .$$

We decompose that

$$\text{(4.5)} \qquad \frac{\beta_3(\theta)}{\sqrt{n}}+\frac{\gamma_3(\theta)}{n}+o\left(\frac{1}{n}\right)=-\frac{1}{2I^3}\,\mathrm{E}_\theta\,(Z_1^3)+\frac{3}{2I}\,\mathrm{E}_\theta\,[Z_1(T_n^*-\mathrm{E}_\theta\,(T_n^*))^2] \\ +\frac{3}{2In}\,\mathrm{E}_\theta\,[Z_1(rL^*+Q-c)^2]+o\left(\frac{1}{n}\right) .$$

Note that $\mathrm{E}_\theta(Q)=c$. We have $\mathrm{E}_\theta(Z_1^3)=K/\sqrt{n}$. For the second term of the right-hand side of (4.5) we use the following lemma (see [12], Lemma 4.1).

LEMMA. *Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a density $f(x, \theta)$ which is differentiable in θ for almost all $x[\mu]$. Suppose further that T_θ is a $\mathscr{B}^{(n)}$-measurable function on $\mathscr{X}^{(n)}$ into R^1 and differentiable function of θ and that*

$$\frac{\partial}{\partial\theta}\,\mathrm{E}_\theta\,(T_\theta)=\int_{\mathscr{X}^{(n)}}\frac{\partial}{\partial\theta}\left\{T_\theta\prod_{i=1}^n f(x_i, \theta)\right\}\prod_{i=1}^n d\mu(x_i)$$

holds. Then we have

$$\mathrm{E}_\theta\,(Z_1T_\theta)=\frac{1}{\sqrt{n}}\frac{\partial}{\partial\theta}\,\mathrm{E}_\theta\,(T_\theta)-\frac{1}{\sqrt{n}}\,\mathrm{E}_\theta\left(\frac{\partial T_\theta}{\partial\theta}\right) .$$

By the lemma we obtain

$$\text{(4.6)} \qquad \mathrm{E}_\theta\,[Z_1(T_n^*-\mathrm{E}_\theta\,(T_n^*))^2] \\ =\frac{1}{\sqrt{n}}\frac{\partial}{\partial\theta}\,\mathrm{V}_\theta\,(T_n^*)-\frac{2}{\sqrt{n}}\,\mathrm{E}_\theta\left[(T_n^*-\mathrm{E}_\theta\,(T_n^*)) \cdot\left\{\frac{\partial}{\partial\theta}(T_n^*-\mathrm{E}_\theta\,(T_n^*))\right\}\right] \\ =\frac{1}{\sqrt{n}}\frac{\partial}{\partial\theta}\left(\frac{1}{I}\right)+o\left(\frac{1}{n}\right) \\ =-\frac{2J+K}{I^2\sqrt{n}}+o\left(\frac{1}{n}\right) .$$

We also obtain

$$\text{(4.7)} \qquad \mathrm{E}_\theta\,[Z_1(rL^*+Q-c)^2] \\ =2r\,\mathrm{E}_\theta\,[Z_1L^*(Q-c)]+o(1)=\frac{r}{I}\left\{\frac{M}{I}-\frac{(3J+K)K}{2I^2}\right\}+o(1) .$$

From (4.5), (4.6) and (4.7) we have

$$\gamma_3(\theta)=\frac{3r}{2I(\theta)^2}\left[\frac{M(\theta)}{I(\theta)}-\frac{\{3J(\theta)+K(\theta)\}K(\theta)}{2I(\theta)^2}\right].$$

Since the third order asymptotic distribution of T_n^* attains the bound of the third order asymptotic distributions of third order AMU estimators at r, there is no third order AMU estimator which uniformly attains the bound, if $\gamma_3(\theta)$ is not equal to zero. Hence we have established:

THEOREM 4. *Under conditions* (i)$\sim$(vii), $\hat{\theta}^*_{ML}$ *is not third order asymptotically efficient if* $\gamma_3(\theta)\neq 0$.

5. Maximum log-likelihood estimator

Instead of the equation (1.1) we may take a solution $\hat{\theta}_n$ of the discretized likelihood equation:

$$\sum_{i=1}^n \log f(X_i, \hat{\theta}_n+rc_n^{-1})-\sum_{i=1}^n \log f(X_i, \hat{\theta}_n-rc_n^{-1})=0 . \tag{5.1}$$

The solution $\hat{\theta}_n$ is θ which maximizes

$$\int_{-rc_n^{-1}}^{rc_n^{-1}} \sum_{i=1}^n \log f(X_i, \theta+t)dt=\int_{-rc_n^{-1}}^{rc_n^{-1}} \log L(\theta+t)dt$$

where $L(\theta)$ denotes the likelihood function. Then $\hat{\theta}_n$ is called maximum log-likelihood estimator (MLLE) ([3]). If $\log L(\hat{\theta}_n+t)$ is locally linearized, $\hat{\theta}_n$ agrees with maximum probability estimator by Weiss and Wolfowitz ([15]).

Now we shall consider a location parameter case when the density function f satisfies the following assumption:

(viii) $f(x, \theta)=f(x-\theta)$ and $f(x)$ is symmetric w.r.t. the origin.

It follows by the symmetricity of f that the solution $\hat{\theta}_n$ of (5.1) is AMU. We also have

$$J(\theta)=K(\theta)=0 .$$

Let $\hat{\theta}_n$ be an MLLE. Then

$$\sum_{i=1}^n \log f(X_i-\hat{\theta}_n-(r/\sqrt{n}))-\sum_{i=1}^n \log f(X_i-\hat{\theta}_n+(r/\sqrt{n}))=0 .$$

Without loss of generality we assume that $\theta_0=0$. Since

$$\sum_{i=1}^n \log f(X_i-\hat{\theta}_n-(r/\sqrt{n}))-\sum_{i=1}^n \log f(X_i)$$

$$-\left\{\sum_{i=1}^{n}\log f(X_i-\hat{\theta}_n+(r/\sqrt{n}))-\sum_{i=1}^{n}\log f(X_i)\right\}=0\ ,$$

it follows by Taylor expansion around $\theta=0$ that

$$\frac{2r}{\sqrt{n}}\left[\frac{\partial}{\partial\theta}\sum_{i=1}^{n}\log f(X_i-\theta)\right]_{\theta=0}+\frac{2r\hat{\theta}_n}{\sqrt{n}}\left[\frac{\partial^2}{\partial\theta^2}\sum_{i=1}^{n}\log f(X_i-\theta)\right]_{\theta=0}$$
$$+\left(\frac{r\hat{\theta}_n^2}{\sqrt{n}}+\frac{r^3}{3n\sqrt{n}}\right)\left[\frac{\partial^3}{\partial\theta^3}\sum_{i=1}^{n}\log f(X_i-\theta)\right]_{\theta=0}$$
$$+\frac{1}{3}\left(\frac{r\hat{\theta}_n}{\sqrt{n}}+\frac{r^3\hat{\theta}_n}{n\sqrt{n}}\right)\left[\frac{\partial^4}{\partial\theta^4}\log f(X_i-\theta)\right]_{\theta=0}\sim 0\ ,$$

where $|\theta^*|\leqq|\theta|$. Putting $T_n=\sqrt{n}\,\hat{\theta}_n$, we obtain

$$2rZ_1+2r\left(\frac{Z_2}{\sqrt{n}}-I(0)\right)T_n+\{Z_3-\sqrt{n}(3J(0)+K(0))\}$$
$$\cdot\left(\frac{rT_n^2}{n}+\frac{r^3}{3n}\right)-\frac{1}{3}\{4L(0)+3M(0)+6N(0)+H(0)\}$$
$$\cdot\left(\frac{r}{n}T_n^3+\frac{r^3}{n}T_n\right)\sim 0\ .$$

Hence it follows that

$$T_n=\frac{Z_1}{I}+\frac{Z_2}{I\sqrt{n}}\left(\frac{Z_1}{I}+\frac{Z_1Z_2}{I^2n}\right)+\frac{r^2}{6In}Z_3+\frac{1}{2In}Z_3\frac{Z_1^2}{I^2}$$
$$-\frac{1}{6In}(4L+3M+6N+H)\left(\frac{Z_1^3}{I^3}+r^2\frac{Z_1}{I}\right)+o_p\left(\frac{1}{n}\right)\ .$$

Under conditions (i)$\sim$(vii) we have

$$(5.2)\quad T_n=\frac{Z_1}{I}+\frac{Z_1Z_2}{I^2\sqrt{n}}+\frac{1}{I^3n}\left\{Z_1Z_2^2+\frac{1}{2}Z_1^2Z_3-\frac{1}{6I}(4L+3M+6N+H)Z_1^3\right\}$$
$$+\frac{r^2Z_3}{6In}-\frac{r^2(4L+3M+6N+H)}{6I^2n}Z_1+o_p\left(\frac{1}{n}\right)$$

up to order n^{-1} as $n\to\infty$. As was stated preciously the difference in the order n^{-1} term between (4.4) and (5.2) does not affect the asymptotic distribution up to the order n^{-1}. Hence we have established:

THEOREM 5. *Under conditions* (i)$\sim$(iv) *and* (vi)$\sim$(viii), *the MLLE* $\hat{\theta}_n$ *is asymptotically equivalent to the MLE* $\hat{\theta}_{ML}$ *up to order* n^{-1}.

It is shown in [12], [13] and [14] that $\hat{\theta}_{ML}$ maximizes the symmetric probability

$$P_{n,\theta}\{\sqrt{n}\,|\hat{\theta}_{ML}-\theta|<u\}$$

up to the order n^{-1} among all regular AMU estimators. Therefore it is seen that

$$\lim_{n\to\infty} n[P_{n,\theta}\{\sqrt{n}\,|\hat{\theta}_{\mathrm{ML}}-\theta|<u\}-P_{n,\theta}\{\sqrt{n}\,|\hat{\theta}_{\mathrm{DL}}-\theta|<u\}]\geqq 0$$

for all u, where $\hat{\theta}_{\mathrm{DL}}$ denotes DLE. The asymptotic distribution of the MLLE $\hat{\theta}_{\mathrm{MLL}}$ is equal to that of $\hat{\theta}_{\mathrm{ML}}$ up to the order n^{-1}, but not in the $n^{-3/2}$ and we can show that

$$\lim_{n\to\infty} n^{3/2}[P_{n,\theta}\{\sqrt{n}\,|\hat{\theta}_{\mathrm{MLL}}-\theta|<r\}-P_{n,\theta}\{\sqrt{n}\,|\hat{\theta}_{\mathrm{ML}}-\theta|<r\}]\geqq 0$$

in general situations. Hence for symmetric intervals $\hat{\theta}_{\mathrm{MLL}}$ is not fourth order asymptotically efficient.

6. Conclusion remarks on discretized likelihood methods[1)]

If we also define as asymptotic efficient estimator as $\hat{\theta}_n$ which maximize $\lim_{n\to\infty} P_{n,\theta}\{c_n|\hat{\theta}_n-\theta|<r\}$ among AMU estimators $\hat{\theta}_n$, then $\hat{\theta}_n^*$ satisfying the following equation (6.1) is asymptotically efficient:

$$\prod_{i=1}^{n}\frac{f(X_i,\hat{\theta}_n^*+rc_n^{-1})}{f(X_i,\hat{\theta}_n^*)}-\prod_{i=1}^{n}\frac{f(X_i,\hat{\theta}_n^*-rc_n^{-1})}{f(X_i,\hat{\theta}^*)}=a_n\,, \tag{6.1}$$

where a_n is chosen so that $\hat{\theta}_n^*$ is AMU. Then (6.1) is also expressed as

$$\begin{aligned}\exp\{\log L(\hat{\theta}_n^*+rc_n^{-1})-\log L(\hat{\theta}_n^*)\}\\ -\exp\{\log L(\hat{\theta}_n^*-rc_n^{-1})-\log L(\hat{\theta}_n^*)\}=a_n\,.\end{aligned} \tag{6.2}$$

If $\exp(\log L)$ is linearize, then (6.2) is reduced to the type of (5.1), where in this case the right-hand side of (5.1) is not always zero.

Hence it is seen from the above that the asymptotic efficiency (including higher order cases) may be systematically discussed by the discretized likelihood methods.

Further the discretized likelihood methods may be also applied to the statistical estimation theory of the fixed sample size.

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