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ASYMPTOTIC EXPANSION OF THE LOG-LIKELIHOOD FUNCTION BASED ON STOPPING TIMES DEFINED ON A MARKOV PROCESS

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Abstract

Consider the parameter space Θ which is an open subset of R^k, $k\geqq 1$, and for each $\theta\in\Theta$, let the r.v.'s Y_n, $n=0, 1,\cdots$ be defined on the probability space $(X, \mathcal{A}, P_\theta)$ and take values in a Borel set S of a Euclidean space. It is assumed that the process $\{Y_n\}$, $n\geqq 0$, is Markovian satisfying certain suitable regularity conditions. For each $n\geqq 1$, let ν_n be a stopping time defined on this process and have some desirable properties. For $0<\tau_n\to\infty$ as $n\to\infty$, set $\theta_{\tau_n}=\theta+h_n\tau_n^{-1/2}$, $h_n\to h\in R^k$, and consider the log-likelihood function $\Lambda_{\nu_n}(\theta)$ of the probability measure $\tilde{P}_{n,\theta_{\tau_n}}$ with respect to the probability measure $\tilde{P}_{n,\theta}$. Here $\tilde{P}_{n,\theta}$ is the restriction of P_θ to the σ-field induced by the r.v.'s $Y_0, Y_1,\cdots, Y_{\nu_n}$. The main purpose of this paper is to obtain an asymptotic expansion of $\Lambda_{\nu_n}(\theta)$ in the probability sense. The asymptotic distribution of $\Lambda_{\nu_n}(\theta)$, as well as that of another r.v. closely related to it, is obtained under both $\tilde{P}_{n,\theta}$ and $\tilde{P}_{n,\theta_{\tau_n}}$.

1. Introduction

Let the parameter space Θ be an open subset of R^k, $k\geqq 1$, and for each $\theta\in\Theta$, let Y_j, $j=0, 1,\cdots, n$ be the first $n+1$ r.v.'s from a (strictly) stationary Markov process. Each Y_j is defined on the probability space $(X, \mathcal{A}, P_\theta)$ and takes values in $(S, \mathcal{S})$, where S is a Borel subset of a Euclidean space and $\mathcal{S}$ is the σ-field of Borel subsets of S. Let $\mathcal{A}_n$ be the σ-field induced by the first $n+1$ r.v.'s, $\mathcal{A}_n=\sigma(Y_0, Y_1,\cdots, Y_n)$, and let $P_{n,\theta}$ be the restriction of P_θ to $\mathcal{A}_n$. It will be assumed that, for each θ and θ^* in Θ, $P_{n,\theta}$ and P_{n,θ^*} are mutually absolutely continuous

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$(P_{n,\theta} \approx P_{n,\theta^*})$, $n \geqq 0$. Then set

$$(1.1) \qquad (dP_{0,\theta^*}/dP_{0,\theta}) = q(Y_0; \theta, \theta^*) ; \qquad (dP_{1,\theta^*}/dP_{1,\theta}) = q(Y_0, Y_1; \theta, \theta^*) ,$$

and, by means of these, define

$$(1.2) \qquad q(Y_j | Y_{j-1}; \theta, \theta^*) = q(Y_{j-1}, Y_j; \theta, \theta^*)/q(Y_{j-1}; \theta, \theta^*) .$$

Also, define

$$(1.3) \qquad \varphi_j(\theta, \theta^*) = \varphi(Y_{j-1}, Y_j; \theta, \theta^*) = [q(Y_j | Y_{j-1}; \theta, \theta^*)]^{1/2} , \qquad j \geqq 1 .$$

Then the log-likelihood function of the measure P_{n,θ^*} with respect to the measure $P_{n,\theta}$ is well-defined with $P_{\bar\theta}$-probability one for all $\bar\theta \in \Theta$ and is given by the following expression (suppressing the r.v.'s involved)

$$(1.4) \qquad \Lambda_n(\theta, \theta^*) = \log [dP_{n,\theta^*}/dP_{n,\theta}] ,$$

where

$$(1.5) \qquad [dP_{n,\theta^*}/dP_{n,\theta}] = q(Y_0; \theta, \theta^*) \prod_{j=1}^{n} \varphi_j^2(\theta, \theta^*) .$$

From (1.4) and (1.5), it follows then that

$$(1.6) \qquad \Lambda_n(\theta, \theta^*) = \log q(Y_0; \theta, \theta^*) + \sum_{j=1}^{n} \log \varphi_j^2(\theta, \theta^*) .$$

Next, for each $\theta \in \Theta$, let $\{\nu_n\}$, $n \geqq 1$, be an increasing sequence of non-negative integer-valued r.v.'s tending to ∞ a.s. $[P_\theta]$ as $n \to \infty$, and such that, for each n, $(\nu_n = m) \in \mathcal{A}_m$, $m \geqq 0$. Replacing n by ν_n in (1.6), the log-likelihood is based on a random number of r.v.'s, namely,

$$(1.7) \qquad \Lambda_{\nu_n}(\theta, \theta^*) = \log q(Y_0; \theta, \theta^*) + \sum_{j=1}^{\nu_n} \log \varphi_j^2(\theta, \theta^*) .$$

(See also Proposition 3.1 below.) The basic purpose of this paper is that of obtaining an asymptotic expansion as well as the asymptotic normality of the log-likelihood, given by (1.7), for specific θ^*'s. These results, when obtained, will be a nontrivial generalization of similar results in Roussas [7].

A more complete outline of what is done in this paper is the following. It is shown (Theorem 3.1) that an asymptotic expansion of the log-likelihood function, given in (1.7), holds true when θ^* has the form θ_{τ_n} given in (2.5). This is carried out in Section 5 by first obtaining (Proposition 5.1) a similar expansion for the log-likelihood function given in (5.1) and then establishing relation (5.4). The proof of the convergence stated in (5.4) as well as the proof of Theorem 3.2 require certain results referring to random sums of r.v.'s and conditions under which such sums converge in some sense. These results appear in Section 4 while Section 5 contains the proof of Theorem 3.1. The asymptotic normality of the log-likelihood function given in (3.1) as well as certain other asymptotic results that follow from Theorems 3.1, 3.2 and standard contiguity-type arguments are presented in Section 3. Finally, our assumptions appear in Section 2. These assumptions are of non-standard (non-Cramér) type and involve the differentiability in quadratic mean of certain random functions. The underlying basic idea throughout is that of contiguity which was introduced and developed by LeCam [5].

In the sequel, all limits will be taken as $n\to\infty$ unless otherwise explicitly stated.

2. Assumptions and some comments

In this section, the assumptions under which the results in this paper are obtained are gathered together. Although the first four assumptions are the same as those utilized in Roussas [7], for the sake of completeness they are herein restated. Due to the existence of a random number of r.v.'s in forming the log-likelihood function in (1.7), there is a need for an additional assumption. This is formulated as Assumption (A5) below. Some illustrative comments follow its formulation.

ASSUMPTIONS

(A1) For each $\theta\in\Theta$, the Markov process $\{Y_n\}$, $n\geqq 0$, is (strictly) stationary and metrically transitive (ergodic). (See Doob [4], p. 191 and p. 460.)

(A2) For each $n\geqq 0$, the probability measures $\{P_{n,\theta};\ \theta\in\Theta\}$ are mutually absolutely continuous.

(A3) (i) For each $\theta\in\Theta$, the random function $\varphi_1(\theta,\theta^*)$, defined by (1.3), is differentiable in quadratic mean (q.m.) $[P_\theta]$ with respect to θ^* at θ with q.m. derivative $\dot{\varphi}_1(\theta)$. (See, for example, Roussas [7], p. 43.)

Let the covariance function $\Gamma(\theta)$ be defined by

$$\Gamma(\theta)=4\mathcal{E}_\theta[\dot{\varphi}_1(\theta)\dot{\varphi}_1'(\theta)]\,. \tag{2.1}$$

Then

(ii) $\Gamma(\theta)$ is positive definite for every $\theta\in\Theta$.

(A4) (i) For each $\theta\in\Theta$, $q(Y_0;\theta,\theta^*)\to 1$ in $P_{0,\theta}$-probability as $\theta^*\to\theta$.
(ii) For each fixed $\theta\in\Theta$, $q(Y_0;\theta,\theta^*)$ is $\mathcal{A}_0\times C$-measurable and $q(Y_0,Y_1;\theta,\theta^*)$ is $\mathcal{A}_1\times C$-measurable, where C is the σ-field of Borel sub-sets of Θ.
(iii) $\dot{\varphi}_1(\theta)$ is $\mathcal{A}_1\times C$-measurable.

Now, let $\{\tau_n\}$, $n\geqq 1$, be a sequence of positive real numbers tending increasingly to ∞. The τ_n's may depend on the parameter θ. If we set then

(2.2) $\alpha_n = [\tau_n]$ (the greatest integer number in τ_n),

it will clearly hold that

(2.3) $$0 \leqq \alpha_n \uparrow \infty$$

and

(2.4) $$\frac{\alpha_n}{\tau_n} \to 1 .$$

Next, for an arbitrary but fixed $\theta \in \Theta$, set

(2.5) $$\theta_{\tau_n} = \theta + \frac{h_n}{\tau_n^{1/2}} , \qquad h_n \to h \in R^k ,$$

so that $\theta_{\tau_n} \in \Theta$ for all sufficiently large n. Also, set

(2.6) $$\varphi_{\tau_n j}(\theta) = \varphi_j(\theta, \theta_{\tau_n}) ,$$

where $\varphi_j(\theta, \theta^*)$ is defined by (1.3). Then the last assumption to be made is the following.

(A5) For each $\theta \in \Theta$, $\alpha_n[\mathcal{E}_\theta \varphi^4_{\tau_n 1}(\theta) - 1] = O(1)$, where α_n and $\varphi_{\tau_n 1}(\theta)$ are defined by (2.2) and (2.6), respectively.

Comments on Assumption (A5)

Assumption (A5) is used only in proving Lemma 5.4 below. It is felt that it may be replaced by a weaker assumption or dispensed with it altogether. We have, however, not yet been successful in doing this. At any rate, this assumption has been checked and found to be satisfied in the following two examples.

Example 2.1. The independent r.v.'s X_n, $n \geqq 1$, have the double exponential density

$$p(x; \theta) = \frac{1}{2} \exp(-|x - \theta|) .$$

Then, for $h \geqq 0$, it is seen that

$$\alpha_n[\mathcal{E}_\theta \varphi^4_{\tau_n 1}(\theta) - 1] = \frac{\alpha_n}{\tau_n} \frac{h_n^2}{3} [\exp(h^*_{1n}) + 2 \exp(h^*_{2n})] ,$$

where h^*_{1n} and $\varphi^4_{\tau_n 1}$ lie between 0 and $h_n/\tau_n^{1/2}$. Thus,

$$\alpha_n[\mathcal{E}_\theta \varphi^4_{\tau_n 1}(\theta) - 1] \to h^2 .$$

The same convergence is established for $h < 0$, and therefore Assumption (A5) is satisfied. (See also Example 3.3, p. 49, in Roussas [7].)

Example 2.2. Here the r.v.'s X_n, $n \geq 0$, are centered at their expectations and constitute a Gaussian process with covariance given by

$$\mathcal{E}_\theta(X_m X_n) = \exp(-\theta|m-n|), \qquad \theta > 0.$$

This process is a stationary and metrically transitive Markov process. (See also Example 3.4, p. 50, in Roussas [7].) After rather long calculations, it is seen that the expression

$$\mathcal{E}_\theta[\varphi_1^4(\theta, \theta+h) - 1]$$

is equal to

$$1 - \exp(-2\theta) - \{1 - \exp[-2(\theta+h)]\}^{1/2} \cdot \{1 - \exp[-2(\theta+h)] + 4[\exp(-2\theta-h) - \exp(-2\theta)]\}^{1/2}$$

divided by

$$\{1 - \exp[-2(\theta+h)]\}^{1/2}\{1 - \exp[-2(\theta+h)] + 4[\exp(-2\theta-h) - \exp(-2\theta)]\}^{1/2}.$$

Replacing in these expressions h by $h_n/\tau_n^{1/2}$, it may be shown that

$$\alpha_n[\mathcal{E}_\theta \varphi_{\tau_n 1}^4(\theta) - 1] \to h^2[1 - \exp(-2\theta)]^{-1} - 2h^2 \exp(-4\theta)[1 - \exp(-2\theta)]^{-2}.$$

Thus, Assumption (A5) is fulfilled also in this example.

Remark 2.1. It is noted in passing that Assumptions (A1)–(A4) are known to be satisfied for the examples discussed above. (See Examples 3.1–3.4, pp. 47–52, in Roussas [7]. Incidentally, Assumption (A5) is also satisfied for Examples 3.1, 3.2 in the reference just cited.)

3. Main results and their implications

Let $\{\nu_n\}$, $n \geq 1$, be a sequence of stopping times on the sequence $\{Y_r\}$, $r \geq 0$ (so that, for each n, $(\nu_n = m) \in \mathcal{A}_m$, $m \geq 0$) and, by employing the notation introduced in (2.6), define $\Lambda_{\nu_n}(\theta)$ by

$$\Lambda_{\nu_n}(\theta) = \Lambda_{\nu_n}(\theta, \theta_{\tau_n}) = \log q(Y_0; \theta, \theta_{\tau_n}) + \sum_{j=1}^{\nu_n} \log \varphi_{\tau_n j}^2(\theta). \tag{3.1}$$

The fact that $\Lambda_{\nu_n}(\theta)$ is the log-likelihood function corresponding to the r.v.'s $Y_0, Y_1, \cdots, Y_{\nu_n}$ is shown formally in the following

PROPOSITION 3.1. For a stopping time ν_n on $\{Y_r\}$, $r \geq 0$, define

$$\mathcal{F}_n = \sigma(Y_0, Y_1, \cdots, Y_{\nu_n}) \tag{3.2}$$

and let

$$\tilde{P}_{n,\theta}=P_{\theta}|\mathcal{F}_n \tag{3.3}$$

denote the restriction of P_θ on $\mathcal{F}_n$. Then, under Assumptions (A1)–(A2), the probability measures $\{\tilde{P}_{n,\theta};\ \theta\in\Theta\}$ are mutually absolutely continuous and furthermore

$$L_{\nu_n}=\frac{d\tilde{P}_{n,\theta^*}}{d\tilde{P}_{n,\theta}}=q(Y_0;\theta,\theta^*)\prod_{j=1}^{\nu_n}q(Y_j|Y_{j-1};\theta,\theta^*) . \tag{3.4}$$

PROOF. The r.v. L_{ν_n} is $\mathcal{F}_n$-measurable. In fact, for $x\in R$,

$$\begin{aligned}(L_{\nu_n}\leqq x)&=\sum_{m=0}^{\infty}(L_{\nu_n}\leqq x)\cap(\nu_n=m)\\&=\sum_{m=0}^{\infty}(L_m\leqq x)\cap(\nu_n=m)\\&=\sum_{m=0}^{\infty}A_{nm}=B_n ,\end{aligned}$$

where $A_{nm}=(L_m\leqq x)\cap(\nu_n=m)\in\mathcal{A}_m$ since $(\nu_n=m)\in\mathcal{A}_m$, by assumption, and also $(L_m\leqq x)\in\mathcal{A}_m$. Thus, $B_n\in\mathcal{A}$. Next, for every $m\geqq 0$, $B_n\cap(\nu_n=m)=A_{nm}\in\mathcal{A}_m$ and this shows that $B_n\in\mathcal{F}_n$. Next, for $A\in\mathcal{F}_n$, we clearly have the following series of equalities, namely,

$$\begin{aligned}\int_A L_{\nu_n}d\tilde{P}_{n,\theta}&=\int_{\bigcup_{j=0}^{\infty}[A\cap(\nu_n=j)]}L_{\nu_n}d\tilde{P}_{n,\theta}\\&=\sum_{j=0}^{\infty}\int_{A\cap(\nu_n=j)}L_{\nu_n}d\tilde{P}_{n,\theta}\\&=\sum_{j=0}^{\infty}\int_{A\cap(\nu_n=j)}L_j d\tilde{P}_{n,\theta}\\&=\sum_{j=0}^{\infty}\int_{A\cap(\nu_n=j)}L_j dP_{j,\theta}\\&=\sum_{j=0}^{\infty}\int_{A\cap(\nu_n=j)}dP_{j,\theta^*}\\&=\sum_{j=0}^{\infty}P_{j,\theta^*}[A\cap(\nu_n=j)]\\&=\sum_{j=0}^{\infty}P_{\theta^*}[A\cap(\nu_n=j)]=P_{\theta^*}(A)=\tilde{P}_{n,\theta^*}(A) .\end{aligned}$$

This completes the proof of the proposition.

Next, define $\Delta_{\nu_n}(\theta)$ by

$$\Delta_{\nu_n}(\theta)=2\nu_n^{-1/2}\sum_{j=1}^{\nu_n}\varphi_j(\theta) . \tag{3.5}$$

Remark 3.1. It is worth noting that the r.v.'s Λ_{ν_n} and $h'\Delta_{\nu_n}$ are $\mathcal{F}_n$-measurable. This measurability may be established in the same way as that of the r.v. L_{ν_n} in the previous proposition, and is implicitly utilized in Theorems 3.1–3.4 below.

The basic results of this paper are the following two theorems.

THEOREM 3.1. *Let h_n, $h \in R^k$ be such that $h_n \to h$, let $\{\tau_n\}$, $n \geqq 1$, be a sequence of positive real numbers (possibly depending on θ) tending increasingly to ∞, and let θ_{τ_n} be defined by* (2.5). *For each $\theta \in \Theta$, let $\{\nu_n\}$, $n \geqq 1$, be a sequence of non-negative integer-valued r.v.'s increasingly tending to ∞* a.s. $[P_\theta]$, *$\nu_n/\tau_n \to 1$ in P_θ-probability and $(\nu_n = m) \in \mathcal{A}_m$, $m \geqq 0$, for each n. Finally, let $\Lambda_{\nu_n}(\theta)$, $\Delta_{\nu_n}(\theta)$, and $\Gamma(\theta)$ be defined by* (3.1), (3.5) *and* (2.1), *respectively. Then, under Assumptions* (A1)–(A5) *and for each $\theta \in \Theta$,*

$$\Lambda_{\nu_n}(\theta) - h'\Delta_{\nu_n}(\theta) \to -(1/2)h'\Gamma(\theta)h \qquad \text{in } P_\theta\text{-probability}. \tag{3.6}$$

THEOREM 3.2. *In the notation of Theorem* 3.1, *under Assumptions* (A1)–(A5) *and for each $\theta \in \Theta$,*

$$\mathcal{L}[\Delta_{\nu_n}(\theta) \mid P_\theta] \Rightarrow N(\theta, \Gamma(\theta)). \tag{3.7}$$

A direct implication of Theorems 3.1 and 3.2 is

THEOREM 3.3. *In the notation of Theorem* 3.1, *under the same Assumptions employed there, and for each $\theta \in \Theta$,*

$$\mathcal{L}[\Lambda_{\nu_n}(\theta) \mid P_\theta] \Rightarrow N(-(1/2)h'\Gamma(\theta)h, h'\Gamma(\theta)h). \tag{3.8}$$

Theorem 3.3 has the following

COROLLARY 3.1. *In the notation of Theorem* 3.1, *and under the same assumptions employed there, the sequences $\{\tilde{P}_{n,\theta}\}$ and $\{\tilde{P}_{n,\theta_{\tau_n}}\}$ are contiguous.*

PROOF. It is immediate by Theorem 6.1 (iv), p. 33, in Roussas [7].

Finally, on the basis of Corollary 3.1, it may be shown that there exist versions of the above three theorems when θ is replaced by θ_{τ_n}. (See Theorems 4.4, 4.5, and 4.6, pages 54, 66 in Roussas [7].)

The remaining of the paper is concerned with the proof of Theorems 3.1 and 3.2. This will be done after several auxiliary results have been stated and proved.

4. Some auxiliary results

In this section, some auxiliary results are established which will allow us to obtain the proof of Theorem 3.1. Some of these results are of considerable interest on their own right.

We first consider a slightly modified version of Theorem 1 of Anscombe [2]. For its formulation, two conditions to be stated below are required. To this end, let W_n, $n=1, 2, \cdots$, be real-valued r.v.'s defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $\{\beta_n\}$, $n\geqq 1$, be a non-decreasing sequence of positive integers tending to ∞. Then suppose that the following conditions are satisfied, namely,

(C1) $\mathcal{L}(W_{\beta_n}|P)\Rightarrow Q$ a probability measure in R.

(C2) For every $\varepsilon>0$ and some $\delta>0$,

$$P(|W_{n'}-W_{\beta_n}|<\varepsilon;\ \text{for all } n' \text{ such that } |n'-\beta_n|<\delta\beta_n)\rightarrow 1\,;$$

equivalently,

$$P[\max(|W_{n'}-W_{\beta_n}|;\ n' \text{ such that } |n'-\beta_n|<\delta\beta_n)\geqq\varepsilon]\rightarrow 0\,.$$

We may now formulate the result referred to above.

THEOREM 4.1. *Consider the sequence of r.v.'s* $\{W_n\}$, $n\geqq 1$, *defined on the probability space* $(\Omega, \mathcal{F}, P)$ *and satisfying conditions* (C1) *and* (C2) *above. Let also* $\{r_n\}$, $n\geqq 1$, *be a sequence of positive integer-valued r.v.'s such that* $(r_n/\beta_n)\rightarrow 1$ *in probability. Then*

$$\mathcal{L}(W_{r_n}|P)\Rightarrow Q\,. \tag{4.1}$$

PROOF. For reasons of completeness the proof of the result (4.1) is included here. For an arbitrary $\varepsilon>0$ and some $\delta>0$ (see condition (C2)), define the events A_n and B_n by

$$A_n=(|r_n-\beta_n|<\delta\beta_n)\,,$$

$$B_n=(|W_{n'}-W_{\beta_n}|<\varepsilon \text{ for all } n' \text{ such that } |n'-\beta_n|<\delta\beta_n)$$

and set

$$C_n=A_n\cap(|W_{r_n}-W_{\beta_n}|<\varepsilon)=(|r_n-\beta_n|<\delta\beta_n)\cap(|W_{r_n}-W_{\beta_n}|<\varepsilon)\,.$$

Then, by condition (C2) and the assumption that $(r_n/\beta_n)\rightarrow 1$ in probability, we have that there exists a positive integer $N=N(\varepsilon, \delta)$ such that

$$P(A_n)>1-\varepsilon \quad\text{and}\quad P(B_n)>1-\varepsilon \qquad \text{for } n\geqq N\,. \tag{4.2}$$

Furthermore,

$$A_n\cap B_n\subseteq C_n \tag{4.3}$$

because, for $\omega\in(A_n\cap B_n)$, $\omega\in A_n$ and hence $|r_n(\omega)-\beta_n|<\delta\beta_n$; also, $\omega\in B_n$ which implies that $|W_{n'}(\omega)-W_{\beta_n}(\omega)|<\varepsilon$ for all n' such that $|n'-\beta_n|<\delta\beta_n$, and therefore $r_n(\omega)$ is such an n'. From (4.2) and (4.3), we have then that

$$\mathrm{P}(C_n)>1-2\varepsilon \text{ or, equivalently, } \mathrm{P}(C_n^c)\leqq 2\varepsilon\ , \qquad n\geqq N. \tag{4.4}$$

Next,

$$(W_{r_n}\leqq x)\subseteq[(W_{r_n}\leqq x)\cap C_n]\cup C_n^c$$

and on C_n, $W_{\beta_n}-\varepsilon<W_{r_n}$, so that

$$(W_{r_n}\leqq x)\subseteq(W_{\beta_n}\leqq x+\varepsilon)\cup C_n^c$$

and hence

$$\mathrm{P}(W_{r_n}\leqq x)\leqq \mathrm{P}(W_{\beta_n}\leqq x+\varepsilon)+2\varepsilon\ , \qquad n\geqq N. \tag{4.5}$$

Also,

$$(W_{r_n}>x)\subseteq[(W_{r_n}>x)\cap C_n]\cup C_n^c$$

and on C_n, $W_{r_n}<W_{\beta_n}+\varepsilon$, so that

$$(W_{r_n}>x)\subseteq(W_{\beta_n}>x-\varepsilon)\cup C_n^c$$

and hence

$$\mathrm{P}(W_{r_n}>x)\leqq \mathrm{P}(W_{\beta_n}>x-\varepsilon)+2\varepsilon\ , \qquad n\geqq N.$$

Equivalently,

$$\mathrm{P}(W_{\beta_n}\leqq x-\varepsilon)-2\varepsilon\leqq \mathrm{P}(W_{r_n}\leqq x)\ , \qquad n\geqq N,$$

so that, by means of this inequality and relation (4.5),

$$\mathrm{P}(W_{\beta_n}\leqq x-\varepsilon)-2\varepsilon\leqq \mathrm{P}(W_{r_n}\leqq x)\leqq \mathrm{P}(W_{\beta_n}\leqq x+\varepsilon)+2\varepsilon\ , \qquad n\geqq N. \tag{4.6}$$

Consider now $x\in C(F)$, where F is the d.f. corresponding to the probability measure Q and let $\varepsilon\downarrow 0$ in (4.6). Then the desired result follows.

The result presented below will, in some instances, help check the validity of condition (C2) of Theorem 4.1. More specifically, the r.v.'s W_{β_n}, $n\geqq 1$, in the applications of Theorem 4.1 will often be of the form

$$W_{\beta_n}=\sum_{j=1}^{\beta_n}X_{nj} \quad \text{with} \quad X_{nj}\geqq 0\ , \qquad j\geqq 1.$$

Then condition (C2) takes on a specific form given below. More precisely, we have the following result.

PROPOSITION 4.1. For each $n=1,2,\cdots$, let X_{nj}, $j\geqq 1$, be r.v.'s defined on the probability space $(\Omega,\mathcal{F},P)$ and such that $X_{nj}\geqq 0$, $j\geqq 1$, and, for each n, X_{nj}, $j \geqq 1$, are stationary. Let $\{\beta_n\}$, $n \geqq 1$, be a non-decreasing sequence of positive integers tending to ∞ and define W_n by

$$W_n = \sum_{j=1}^{n} X_{nj} .$$

Then, for every $\varepsilon > 0$ and some $\delta > 0$,

$$\text{(4.7)} \quad \mathrm{P}(|W_{n'} - W_{\beta_n}| > \varepsilon \text{ for all } n' \text{ such that } |n' - \beta_n| < \delta\beta_n) \geqq 1 - \mathrm{P}\left(\sum_{j=1}^{[\delta\beta_n]} X_{nj} \geqq \varepsilon\right),$$

where, we recall that, for $x > 0$, $[x]$ denotes the largest integer $\leqq x$. (Thus, condition (C2) will be satisfied, if

$$\mathrm{P}\left(\sum_{j=1}^{[\delta\beta_n]} X_{nj} \geqq \varepsilon\right) \to 0 .)$$

PROOF. The inequality $|n' - \beta_n| < \delta\beta_n$ implies the inequality

$$\text{(4.8)} \quad [(1-\delta)\beta_n] < n' \leqq [(1+\delta)\beta_n] .$$

Then

$$\begin{aligned} \text{(4.9)} \quad & \mathrm{P}(|W_{n'} - W_{\beta_n}| < \varepsilon \text{ for all } n' \text{ such that } |n' - \beta_n| < \delta\beta_n) \\ & = 1 - \mathrm{P}(|W_{n'} - W_{\beta_n}| \geqq \varepsilon \text{ for at least one } n' \\ & \qquad\qquad \text{such that } |n' - \beta_n| < \delta\beta_n) \\ & = 1 - \mathrm{P}[\max(|W_{n'} - W_{\beta_n}| \geqq \varepsilon;\ n' \text{ such that } |n' - \beta_n| < \delta\beta_n)] . \end{aligned}$$

But, by (4.8),

$$\begin{aligned} & \mathrm{P}[\max(|W_{n'} - W_{\beta_n}| \geqq \varepsilon;\ n' \text{ such that } |n' - \beta_n| < \delta\beta_n)] \\ & \leqq \mathrm{P}(\max\{|W_{n'} - W_{\beta_n}| \geqq \varepsilon;\ [(1-\delta)\beta_n] \leqq n' \leqq [(1+\delta)\beta_n]\}) \\ & \leqq \mathrm{P}(\max\{|W_{n'} - W_{\beta_n}| \geqq \varepsilon;\ [(1-\delta)\beta_n] < n' < \beta_n\}) \\ & \quad + \mathrm{P}(\max\{|W_{n'} - W_{\beta_n}| \geqq \varepsilon;\ \beta_n \leqq n' \leqq [(1+\delta)\beta_n]\}) \\ & = \mathrm{P}\left\{\sum_{j=[(1-\delta)\beta_n]+2}^{\beta_n} X_{nj} \geqq \varepsilon\right\} + \mathrm{P}\left\{\sum_{j=\beta_n+1}^{[(1+\delta)\beta_n]} X_{nj} \geqq \varepsilon\right\} . \end{aligned}$$

Each one of these last terms involves a number of successive X_{nj}'s which is $\leqq [\delta\beta_n]$. Then, by stationarity, the sum of these last two terms is bounded by

$$2\,\mathrm{P}\left(\sum_{j=1}^{[\delta\beta_n]} X_{nj} \geqq \varepsilon\right) .$$

Thus, (4.9) becomes

$$\mathrm{P}(|W_{n'} - W_{\beta_n}| < \varepsilon \text{ for all } n' \text{ such that } |n' - \beta_n| < \delta\beta_n) \geqq 1 - 2\,\mathrm{P}\left(\sum_{j=1}^{[\delta\beta_n]} X_{nj} \geqq \varepsilon\right)$$

as asserted in (4.7).

The following two properties will prove useful in establishing certain result in the next section. In addition, they are of some independent interest.

PROPOSITION 4.2. For each $n=1,2,\cdots$, let X_{nj}, $j\geqq 1$, be r.v.'s defined on the probability space $(\Omega, \mathscr{F}, P)$ and let r_n, $n\geqq 1$, be positive integer-valued r.v.'s such that $(r_n/\beta_n)\to 1$ in probability, where $\{\beta_n\}$, $n\geqq 1$, in a nondecreasing sequence of positive integers tending to ∞. Then, for every $\varepsilon>0$, there exists a positive integer $N=N(\varepsilon)$ such that $n>N$ implies that

$$\text{(4.10)}\qquad \mathrm{P}\left(\left|\sum_{j=1}^{\beta_n} X_{nj}-\sum_{j=1}^{r_n} X_{nj}\right|\geqq\varepsilon\right)=\mathrm{P}(A_n)<\varepsilon+\varepsilon^{-1}\sum_{j=[(1-\varepsilon^2)\beta_n]+2}^{[(1+\varepsilon^2)\beta_n]}\mathscr{E}|X_{nj}|\;.$$

In particular, if, for each n, the $\mathscr{E}|X_{nj}|$, $j\geqq 1$, are all equal, then the above inequality becomes

$$\text{(4.11)}\qquad \mathrm{P}\left(\left|\sum_{j=1}^{\beta_n} X_{nj}-\sum_{j=1}^{r_n} X_{nj}\right|\geqq\varepsilon\right)=\mathrm{P}(A_n)<\varepsilon(1+2\beta_n\mathscr{E}|X_{n1}|)\;.$$

PROOF. In the first place,

$$\text{(4.12)}\qquad \mathrm{P}\left(\left|\frac{r_n}{\beta_n}-1\right|\geqq\varepsilon^2\right)=\mathrm{P}(|r_n-\beta_n|\geqq\varepsilon^2\beta_n)<\varepsilon\;,\qquad n>N=N(\varepsilon)\;.$$

Next,

$$\left(\left|\frac{r_n}{\beta_n}-1\right|<\varepsilon^2\right)=(|r_n-\beta_n|<\varepsilon^2\beta_n)\subseteqq\{[(1-\varepsilon^2)\beta_n]<r_n\leqq[(1+\varepsilon^2)\beta_n]\}$$
$$\subseteqq\{[(1-\varepsilon^2)\beta_n]<r_n<\beta_n\}\cup\{\beta_n\leqq r_n\leqq[(1+\varepsilon^2)\beta_n]\}\;.$$

Therefore, for $n>N$,

$$\begin{aligned}\text{(4.13)}\qquad \mathrm{P}(A_n)&=\mathrm{P}[A_n\cap(|r_n-\beta_n|\geqq\varepsilon^2\beta_n)]+\mathrm{P}[A_n\cap(|r_n-\beta_n|<\varepsilon^2\beta_n)]\\&\leqq\mathrm{P}(|r_n-\beta_n|\geqq\varepsilon^2\beta_n)+\mathrm{P}(A_n\cap\{[(1-\varepsilon^2)\beta_n]<r_n<\beta_n\})\\&\quad+\mathrm{P}(A_n\cap\{\beta_n\leqq r_n\leqq[(1+\varepsilon^2)\beta_n]\})<\varepsilon+p_{n1}+p_{n2}\;,\end{aligned}$$

where the last inequality holds true because of (4.12) and where p_{n1} and p_{n2} denote the last two terms on the right-hand side of the inequality before relation (4.13). But

$$p_{n1}=\mathrm{P}\left\{A_n\cap\bigcup_{i=[(1-\varepsilon^2)\beta_n]+1}^{\beta_n-1}(r_n=i)\right\}$$
$$=\mathrm{P}\left\{\bigcup_{i=[(1-\varepsilon^2)\beta_n]+1}^{\beta_n-1}A_n\cap(r_n=i)\right\}$$

$$=\mathrm{P}\left\{\bigcup_{m=[(1-\varepsilon^2)\beta_n]+2}^{\beta_n}\left(\left|\sum_{j=m}^{\beta_n} X_{nj}\right|\geqq\varepsilon\right)\right\}$$

$$=\mathrm{P}\left(\max\left\{\left|\sum_{j=m}^{\beta_n} X_{nj}\right|\geqq\varepsilon;\ [(1-\varepsilon^2)\beta_n]+2\leqq m\leqq\beta_n\right\}\right)$$

$$\leqq\mathrm{P}\left(\max\left\{\sum_{j=m}^{\beta_n}|X_{nj}|\geqq\varepsilon;\ [(1-\varepsilon^2)\beta_n]+2\leqq m\leqq\beta_n\right\}\right)$$

$$=\mathrm{P}\left\{\sum_{j=[(1-\varepsilon^2)\beta_n]+2}^{\beta_n}|X_{nj}|\geqq\varepsilon\right\}\leqq\varepsilon^{-1}\sum_{j=[(1-\varepsilon^2)\beta_n]+2}^{\beta_n}\mathcal{E}|X_{nj}|\ ;$$

that is,

$$p_{n1}\leqq\varepsilon^{-1}\sum_{j=[(1-\varepsilon^2)\beta_n]+2}^{\beta_n}\mathcal{E}|X_{nj}|\ . \tag{4.14}$$

Next,

$$p_{n2}=\mathrm{P}\left\{A_n\cap\bigcup_{i=\beta_n}^{[(1+\varepsilon^2)\beta_n]}(r_n=i)\right\}$$

$$=\mathrm{P}\left\{\bigcup_{i=\beta_n}^{[(1+\varepsilon^2)\beta_n]}A_n\cap(r_n=i)\right\}$$

$$=\mathrm{P}\left\{\bigcup_{m=\beta_n+1}^{[(1+\varepsilon^2)\beta_n]}\left(\left|\sum_{j=\beta_n+1}^{m}X_{nj}\right|\geqq\varepsilon\right)\right\}$$

$$=\mathrm{P}\left(\max\left\{\left|\sum_{j=\beta_n+1}^{m}X_{nj}\right|\geqq\varepsilon;\ \beta_n+1\leqq m\leqq[(1+\varepsilon^2)\beta_n]\right\}\right)$$

$$\leqq\mathrm{P}\left(\max\left\{\sum_{j=\beta_n+1}^{m}|X_{nj}|\geqq\varepsilon;\ \beta_n+1\leqq m\leqq[(1+\varepsilon^2)\beta_n]\right\}\right)$$

$$=\mathrm{P}\left\{\sum_{j=\beta_n+1}^{[(1+\varepsilon^2)\beta_n]}|X_{nj}|\geqq\varepsilon\right\}\leqq\varepsilon^{-1}\sum_{j=\beta_n+1}^{[(1+\varepsilon^2)\beta_n]}\mathcal{E}|X_{nj}|\ ;$$

that is,

$$p_{n2}\leqq\varepsilon^{-1}\sum_{j=\beta_n+1}^{[(1+\varepsilon^2)\beta_n]}\mathcal{E}|X_{nj}|\ . \tag{4.15}$$

Then, by means of (4.14)–(4.15), relation (4.13) becomes

$$\mathrm{P}(A_n)<\varepsilon+\varepsilon^{-1}\sum_{j=[(1-\varepsilon^2)\beta_n]+2}^{[(1+\varepsilon^2)\beta_n]}\mathcal{E}|X_{nj}|\ ,\qquad n>N\ ,$$

which is what relation (4.10) asserts. Relation (4.11) follows from relation (4.10) because

$$[(1+\varepsilon^2)\beta_n]-([(1-\varepsilon^2)\beta_n]+2)+1\leqq 2\varepsilon^2\beta_n\ .$$

PROPOSITION 4.3. For each $n=1,2,\cdots$, let X_{nj}, $j\geqq 1$, be r.v.'s defined on the probability space $(\Omega,\mathcal{F},P)$ and such that

$$\mathcal{E}(X_{n,j+1}|X_{n1},\cdots,X_{nj})=0\ ,\qquad \text{a.s. } [P],\ j\geqq 1,\ \mathcal{E}X_{n1}=0\ .$$

Also, let $\{\beta_n\}$, $n\geqq 1$, be a nondecreasing sequence of positive integers tending to ∞. Furthermore, let $(r_n/\beta_n)\to 1$ in probability, where r_n, $n\geqq 1$, are positive integer-valued r.v.'s. Then, for every $\varepsilon>0$, there exists a positive integer $N=N(\varepsilon)$ such that $n>N$ implies

$$\text{(4.16)}\quad \mathrm{P}\left(\left|\sum_{j=1}^{\beta_n} X_{nj}-\sum_{j=1}^{r_n} X_{nj}\right|\geqq\varepsilon\right)=\mathrm{P}(A_n)<\varepsilon+\varepsilon^{-2}\sum_{j=[(1-\varepsilon^3)\beta_n]+2}^{[(1+\varepsilon^3)\beta_n]}\sigma^2(X_{nj})\ .$$

In particular, if the r.v.'s X_{nj}, $j\geqq 1$, have the same variance, then the above inequality becomes

$$\text{(4.17)}\qquad \mathrm{P}\left(\left|\sum_{j=1}^{\beta_n} X_{nj}-\sum_{j=1}^{r_n} X_{nj}\right|\geqq\varepsilon\right)=\mathrm{P}(A_n)<\varepsilon[1+2\beta_n\sigma^2(X_{n1})]\ .$$

PROOF. Working as in Proposition 4.2, we obtain, for $n>N$,

$$\text{(4.18)}\qquad \mathrm{P}(A_n)<\varepsilon+p_{n1}+p_{n2}\ ,\qquad n>N\ ,$$

where p_{n1} and p_{n2} are the same quantities as those in the above cited proposition. Next, again as in Proposition 4.2, with ε^2 replaced by ε^3,

$$\text{(4.19)}\qquad p_{n1}=\mathrm{P}\left(\max\left\{\left|\sum_{j=m}^{\beta_n} X_{nj}\right|\geqq\varepsilon;\ [(1-\varepsilon^3)\beta_n]+2\leqq m\leqq\beta_n\right\}\right)$$

$$\text{(4.20)}\qquad \leqq\varepsilon^{-2}\sum_{j=[(1-\varepsilon^3)\beta_n]+2}^{\beta_n}\sigma^2(X_{nj})\ ,$$

where the inequality follows from the extended Kolmogorov inequality (see, for example, Loève [6], p. 386). Also, by the same proposition and the inequality mentioned above,

$$\text{(4.21)}\qquad p_{n2}=\mathrm{P}\left(\max\left\{\left|\sum_{j=\beta_n+1}^{m} X_{nj}\right|\geqq\varepsilon;\ \beta_n+1\leqq m\leqq[(1+\varepsilon^3)\beta_n]\right\}\right)$$
$$\leqq\varepsilon^{-2}\sum_{j=\beta_n+1}^{[(1+\varepsilon^3)\beta_n]}\sigma^2(X_{nj})\ .$$

Then, by means of (4.19)-(4.21), relation (4.18) becomes

$$\mathrm{P}(A_n)<\varepsilon+\varepsilon^{-2}\sum_{j=[(1-\varepsilon^3)\beta_n]+2}^{[(1+\varepsilon^3)\beta_n]}\sigma^2(X_{nj})\ ,\qquad n\geqq N\ ,$$

as asserted in (4.16). Relation (4.17) follows from (4.16) and the fact that

$$[(1+\varepsilon^3)\beta_n]-\{[(1-\varepsilon^3)\beta_n]+2\}+1\leqq 2\varepsilon^3\beta_n\ .$$

5. Proof of main results

In our understanding there is no direct way of establishing Theorem 3.1. What can be done instead is to prove a version of this theorem when ν_n is replaced by α_n (see Proposition 5.1 below). Then, on the basis of this and relation (5.4), one may arrive at the desired result.

To start with, let α_n be defined by (2.2) and let

$$\Lambda_{\alpha_n}(\theta)=\Lambda_{\alpha_n}(\theta, \theta_{\tau_n})=\log q(Y_0; \theta, \theta_{\tau_n})+\sum_{j=1}^{\alpha_n} \log \varphi_{\tau_n j}^2(\theta) \tag{5.1}$$

and

$$\Delta_{\alpha_n}(\theta)=2\alpha_n^{-1/2} \sum_{j=1}^{\alpha_n} \dot{\varphi}_j(\theta) . \tag{5.2}$$

Then one has

PROPOSITION 5.1. Let h_n, $h \in R^k$ be such that $h_n \to h$, let $\{\tau_n\}$, $n \geqq 1$, be a sequence of positive real numbers (possibly depending on θ) tending increasingly to ∞, and let θ_{τ_n} be defined by (2.5). Let α_n, $\Lambda_{\alpha_n}(\theta)$, $\Delta_{\alpha_n}(\theta)$ and $\Gamma(\theta)$ be defined by (2.2), (5.1), (5.2) and (2.1), respectively. Then, under Assumptions (A1)-(A4) and for each $\theta \in \Theta$,

$$\Lambda_{\alpha_n}(\theta)-h'\Delta_{\alpha_n}(\theta) \to -(1/2)h'\Gamma(\theta)h \quad \text{in } P_\theta\text{-probability} . \tag{5.3}$$

PROOF. Set

$$\theta_n=\theta+h_n n^{-1/2} , \qquad h_n \to h, \ h \in R^k .$$

Then it is well known that

$$\Lambda_n(\theta, \theta_n)-h'\Delta_n(\theta) \to -(1/2)h'\Gamma(\theta)h \quad \text{in } P_\theta\text{-probability} .$$

Furthermore, since $\{\alpha_n\} \subseteqq \{n\}$,

$$\Lambda_{\alpha_n}(\theta, \theta_{\alpha_n})-h'\Delta_{\alpha_n}(\theta) \to -(1/2)h'\Gamma(\theta)h \quad \text{in } P_\theta\text{-probability} .$$

Next,

$$\theta_{\tau_n}=\theta+h_n\tau_n^{-1/2}=\theta+(h_n\alpha_n^{1/2}\tau_n^{-1/2})\alpha_n^{-1/2}=\theta+h_n^*\alpha_n^{-1/2} , \qquad h_n^* \to h, \ h \in R$$

so that $\Lambda_{\alpha_n}(\theta)-h'\Delta_{\alpha_n}(\theta) \to -(1/2)h'\Gamma(\theta)h$.

On the basis of Proposition 5.1, Theorem 3.1 will follow provided the following result is established, namely,

$$(\Lambda_{\nu_n}-h'\Delta_{\nu_n})-(\Lambda_{\alpha_n}-h'\Delta_{\alpha_n}) \to 0 \quad \text{in } P_\theta\text{-probability} , \tag{5.4}$$

where the parameter θ has been omitted from $\Lambda_{\nu_n}(\theta)$, $\Delta_{\nu_n}(\theta)$, $\Lambda_{\alpha_n}(\theta)$ and $\Delta_{\alpha_n}(\theta)$. In order to show relation (5.4), it suffices to show

$$\left(\sum_{j=1}^{\nu_n} \log \varphi_{nj}^2-h'\Delta_{\nu_n}\right)-\left(\sum_{j=1}^{\alpha_n} \log \varphi_{nj}^2-h'\Delta_{\alpha_n}\right) \to 0 \tag{5.5}$$

in P_θ-probability, where again the parameter θ has been omitted from $\varphi_{nj}^2(\theta)$. From a purely mathematical point of view, the $\log \varphi_{nj}^2$ in (5.5) must be replaced by expressions not involving log's. This can be done provided $\log \varphi_{nj}^2 = 2\log \varphi_{nj} = 2\log[1+(\varphi_{nj}-1)]$ may be expanded. This can be done provided that $|\varphi_{nj}-1|$ is bounded away from 1. Thus we are led to the following lemma.

LEMMA 5.1. *Consider the sequence* $\{\alpha_n\}$, $n \geqq 1$, *defined by* (2.2), *and for each* $\theta \in \Theta$, *let* $\{\nu_n\}$, $n \geqq 1$, *be as in Theorem* 3.1 (*so that* $(\nu_n/\alpha_n) \to 1$ *in* P_θ-*probability*). *For* $n \geqq 1$, *let* l_n *stand for either* α_n *or* ν_n *and define* $W_{l_n} = \max(|\varphi_{nj}-1|;\ 1 \leqq j \leqq l_n)$. *Then, under Assumptions* (A1)-(A4),

(i) $W_{\alpha_n} \to 0$ *in* P_θ-*probability*,

(ii) $W_{\nu_n} \to 0$ *in* P_θ-*probability*.

PROOF. (i) It follows from Lemma 5.2, p. 56 in Roussas [7] and an argument similar to that employed in proving Proposition 5.1.

(ii) The r.v.'s W_{α_n} satisfy condition (C1) of Theorem 4.1 by means of part (i); they are also seem to satisfy condition (C2) after some algebra. Hence the result follows.

Lemma 5.1 implies that for any $\varepsilon > 0$ and for all sufficiently large n the set $A_n = (W_{\alpha_n} > \varepsilon) \cup (W_{\nu_n} > \varepsilon)$ satisfies the inequality $P_\theta(A_n) < \varepsilon$. Thus, on the set A_n^c (of $P(A_n^c) > 1-\varepsilon$) and for all sufficiently large n,

$$
(5.6)\quad
\begin{aligned}
&\sum_j \log \varphi_{nj}^2 = 2\sum_j (\varphi_{nj}-1) - \sum_j (\varphi_{nj}-1)^2 + 2\sum_j c_{nj}(\varphi_{nj}-1)^3 \qquad |c_{nj}| \leqq 3\,,\\
&\tilde{\sum_j} \log \varphi_{nj}^2 = 2\tilde{\sum_j} (\varphi_{nj}-1) - \tilde{\sum_j} (\varphi_{nj}-1)^2 + 2\tilde{\sum_j} \tilde{c}_{nj}(\varphi_{nj}-1) \qquad |\tilde{c}_{nj}| \leqq 3\,,
\end{aligned}
$$

where $\sum_j$, $\tilde{\sum_j}$ denote summations for $1 \leqq j \leqq \alpha_n$ and $1 \leqq j \leqq \nu_n$, respectively. From (5.6), it follows that the relevant algebra would be easier, if the last term could be disregarded. This is possible by the following lemma.

LEMMA 5.2. *Let* $\{\alpha_n\}$ *and* $\{\nu_n\}$, $n \geqq 1$, *be as in Lemma* 5.1, *let* l_n *stand for either* α_n *or* ν_n *and define*

$$
V_{l_n} = \sum_{j=1}^{l_n} |\varphi_{nj}-1|^3\,, \qquad n \geqq 1\,.
$$

Then, under Assumptions (A1)-(A4),

(i) $V_{\alpha_n} \to 0$ *in* P_θ-*probability*,

(ii) $V_{\nu_n} \to 0$ *in* P_θ-*probability*.

PROOF. (i) It follows from the proof of Lemma 5.3, p. 57, in Roussas [7] and an argument similar to the one employed in the proof of Proposition 5.1.

(ii) The r.v.'s V_{α_n} satisfy condition (C1) of Theorem 4.1 by means of part (i). To show that they also satisfy condition (C2), consider Proposition 4.1 with $X_{nj}=|\varphi_{nj}-1|^3$.

Next, by means of relationship (5.6) and Lemma 5.2, relation (5.5) reduces to

$$(5.7)\qquad \left[2\tilde{\sum_j}(\varphi_{nj}-1)-\tilde{\sum_j}(\varphi_{nj}-1)^2-h'\Delta_{\nu_n}\right] - \left[2\sum_j(\varphi_{nj}-1)-\sum_j(\varphi_{nj}-1)^2-h'\Delta_{\alpha_n}\right]\to 0 \quad \text{in } P_\theta\text{-probability}\,.$$

This, however, is clearly a consequence of Lemmas 5.3–5.5 below. Detailed proof is given only for Lemma 5.5. Details on all other proofs may be found in Akritas and Roussas [1].

LEMMA 5.3. *Let* $\{\alpha_n\}$, $n\geqq 1$, *and* $\{\nu_n\}$, $n\geqq 1$, *be as in Lemma* 5.1. *Then, under Assumptions* (A1)–(A4),

$$(5.8)\qquad 2\sum_{j=1}^{\alpha_n}(\varphi_{nj}-1)^2-2\sum_{j=1}^{\nu_n}(\varphi_{nj}-1)^2\to 0 \qquad \textit{in } P_\theta\textit{-probability}\,.$$

PROOF. It follows by considering Proposition 4.2 with $X_{nj}=(\varphi_{nj}-1)^2$.

LEMMA 5.4. *Let* $\{\alpha_n\}$ *and* $\{\nu_n\}$, $n\geqq 1$, *be as in Lemma* 5.1. *Then, under Assumptions* (A1)–(A5),

$$(5.9)\qquad \sum_{j=1}^{\nu_n}[2(\varphi_{nj}-1)+(\varphi_{nj}-1)^2-2\alpha_n^{-1/2}h'\dot{\varphi}_j] - \sum_{j=1}^{\alpha_n}[2(\varphi_{nj}-1)+(\varphi_{nj}-1)^2-2\alpha_n^{-1/2}h'\dot{\varphi}_j]\to 0 \quad \textit{in } P_\theta\textit{-probability}\,.$$

PROOF. It follows by considering Proposition 4.3 with $X_{nj}=2(\varphi_{nj}-1)+(\varphi_{nj}-1)^2-2\alpha_n^{-1/2}h'\dot{\varphi}_j$, and making use of Assumption (A5). (It should be remarked in passing that this is the only instance where Assumption (A5) is used.)

LEMMA 5.5. *Let* $\{\alpha_n\}$ *and* $\{\nu_n\}$, $n\geqq 1$, *be as in Lemma* 5.1. *Then, under Assumptions* (A1)–(A4),

$$(5.10)\qquad 2\alpha_n^{-1/2}\sum_{j=1}^{\nu_n}h'\dot{\varphi}_j-2\nu_n^{-1/2}\sum_{j=1}^{\nu_n}h'\dot{\varphi}_j\to 0 \qquad \textit{in } P_\theta\textit{-probability}\,.$$

PROOF. Retaining the notation $\sum_j$ and $\tilde{\sum_j}$ for summation over j from 1 to α_n and from 1 to ν_n, respectively, and setting

$$X_{nj}=2\alpha_n^{-1/2}h'\dot{\varphi}_j\,, \qquad j\geqq 1\,,$$

we have

$$\mathcal{E}_\theta(X_{n,j+1}|X_{n1},\cdots,X_{nj})=0 \qquad \text{a.s. } [P_\theta],\ j\geqq 1,\ \mathcal{E}_\theta X_{n1}=0\,.$$

Furthermore, for each n, the r.v.'s X_{nj}, $j\geqq 1$, have the same variance. Then relation (4.17) in Proposition 4.3, with β_n and r_n replaced by α_n and ν_n, respectively, becomes

$$P_\theta\left(\left|\sum_{j=1}^{\alpha_n}X_{nj}-\sum_{j=1}^{\nu_n}X_{nj}\right|\geqq\varepsilon\right)\leqq\varepsilon[1+2\alpha_n\mathcal{E}_\theta(2\alpha_n^{-1/2}h'\dot{\varphi}_1)^2]\,. \tag{5.11}$$

But

$$\alpha_n\mathcal{E}_\theta(2\alpha_n^{-1/2}h'\dot{\varphi}_1)^2=4\mathcal{E}_\theta(h'\dot{\varphi}_1)^2 \qquad (<\infty)$$

and therefore the right-hand side of (5.11) implies that

$$2\alpha_n^{-1/2}\sum_j h'\dot{\varphi}_j-2\alpha_n^{-1/2}\tilde{\sum_j} h'\dot{\varphi}_j\to 0 \qquad \text{in } P_\theta\text{-probability}\,.$$

By Theorem 3.2, with β_n replaced by α_n,

$$\mathcal{L}\left(2\alpha_n^{-1/2}\sum_j h'\dot{\varphi}_j\,|\,P_\theta\right)\Rightarrow N(0,h'\Gamma(\theta)h)\,.$$

Thus,

$$\mathcal{L}\left(2\alpha_n^{-1/2}\tilde{\sum_j} h'\dot{\varphi}_j\,|\,P_\theta\right)\Rightarrow N(0,h'\Gamma(\theta)h)\,. \tag{5.12}$$

Furthermore,

$$2\alpha_n^{-1/2}\tilde{\sum_j} h'\dot{\varphi}_j-2\nu_n^{-1/2}\tilde{\sum_j} h'\dot{\varphi}_j=[1-(\alpha_n/\nu_n)^{1/2}]\left(2\alpha_n^{-1/2}\tilde{\sum_j} h'\dot{\varphi}_j\right)\,. \tag{5.13}$$

By assumption, the first term on the right-hand side of (5.13) tends to 0 in P_θ-probability, and the second term by, (5.12), is bounded in P_θ-probability. The desired result follows.

We may now proceed with the proof of Theorems 3.1, 3.2.

PROOF OF THEOREM 3.1. As it was shown above, the proof of Theorem 3.1 reduces to showing that relation (5.7) holds true. This, however, is obtained by adding up the expressions (5.8)–(5.10).

PROOF OF THEOREM 3.2. It follows from relations (5.12), (5.13).

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