Ann. Inst. Statist. Math.
30 (1978), Part A, 465-472

# ON HOMOGENEOUS STOCHASTIC PROCESSES ON COMPACT ABELIAN GROUPS

P. A. MORETTIN

(Received Sept. 26, 1978)


## Abstract

This article discusses the extension to general compact Abelian groups of some results previously established by R. Roy for the case of the circle and the sphere. Estimators of the covariance function and spectral parameters for a homogeneous stochastic process defined on a compact Abelian group are considered and their properties are derived.


## 1. Introduction

In this paper we extend to general compact Abelian groups some results of Roy (see [7], [8], [9] and [10]) given for the case of the circle and the sphere. Let $G$ be a compact Abelian group (CAG) with multiplication as operation and let $(\Omega, \mathcal{A}, P)$ be a probability space. Consider a stochastic process $\{X(g, \omega): g \in G, \omega \in \Omega\}$, that is, for each $g \in G$, $X(g, \omega)$, is an $\mathcal{A}$-measurable complex-valued function over $\Omega$. Denote by $\mathrm{E}[X(g, \omega)]$ the mean function of the process and by $R(g_1, g_2) = \mathrm{Cov}\{X(g_1, \omega), X(g_2, \omega)\}$ the autocovariance function of $X(g, \omega)$. From now on we omit the dependence on $\omega$ and write simply $\{X(g), g \in G\}$.

We say that $X(g)$ is *weakly homogeneous* if

(i) $\mathrm{E}[X(g)] = c_x =$ constant, for each $g \in G$;

(ii) $X(g) \in L_2(\Omega, \mathcal{A}, P)$, for each $g \in G$;

(iii) $\mathrm{Cov}\{X(g_1), X(g_2)\}$ is a function only of $g_1 g_2^{-1}$.

Here, $L_2(\Omega, \mathcal{A}, P)$ is the Hilbert space of all square integrable random variables on $(\Omega, \mathcal{A}, P)$, with inner product given by $(U, V) = \mathrm{E}(U\bar{V})$. Without loss of generality we assume thet $c_x = 0$, hence we can write

$$R(h) = \mathrm{E}[X(gh)\overline{X(g)}], \tag{1.1}$$

$g, h \in G$.

AMS Classification: Primary-60G10, Secondary-62M15.

In Section 2 we discuss homogeneous stochastic processes $\{X(g), g \in G\}$ on a CAG $G$ which are time-independent and derive properties for the estimates of $R(h)$ and for the spectral parameters to be defined. In Section 3 we consider the case of homogeneous processes which are time-dependent.

Homogeneous stochastic processes defined on locally compact Abelian groups are discussed by Morettin [5]. A basic reference on the subject is Yaglom [11].

## 2. Time-independent processes

Let $\{X(g), g \in G\}$ be a homogeneous, real-valued stochastic process on a CAG $G$, with finite second-order moments and continuous in quadratic mean (q.m.).

According with the representation theory for compact groups (see Pontryagin [6]) there exists at most a denumerable number of homomorphisms from $G$ on the group of matrices with finite orders,

$$g \in G \to A^{(n)}(g) = [A_{ij}^{(n)}(g)] ,$$

$1 \leqq i, j \leqq d_n < \infty$, $n=1, 2, 3, \cdots$, such that

$$A^{(n)}(g_1 g_2) = A^{(n)}(g_1) A^{(n)}(g_2) , \tag{2.1}$$

$$A^{(n)}(g^{-1}) = [A^{(n)}(g)]^{-1} = \overline{A^{(n)}(g)} . \tag{2.2}$$

The elements $A_{ij}^{(n)}$ satisfy

$$\int_G A_{ij}^{(n)}(g) A_{kl}^{(m)}(g) dg = \delta_{nm} \delta_{ik} \delta_{jl} d_n^{-1} \tag{2.3}$$

where $\delta_{ij}$ is the Kronecker delta and $dg$ is the Haar measure on $G$. The set of all the elements $A_{ij}^{(n)}$ forms a complete orthogonal system in $L_2(G)$. It follows that $X(g)$ can be written

$$X(g) = \sum_{n=1}^{\infty} \sum_{i,j=1}^{d_n} Z_{ji}^{(n)} A_{ij}^{(n)}(g) , \tag{2.4}$$

where

$$Z_{ji}^{(n)} = d_n \int_G X(g) A_{ij}^{(n)}(g) dg . \tag{2.5}$$

Both equations (2.4) and (2.5) should be understood is the q.m. sense. Since $\mathrm{E}[X(g)]=0$ it follows that $\mathrm{E}[Z_{ji}^{(n)}]=0$. The following theorem is known (Yaglom [11]).

THEOREM 2.1. *The stochastic process* (2.4) *is weakly homogeneous if and only if*

(2.6) $$\mathrm{E}\,[Z_{ji}^{(n)}\overline{Z_{lk}^{(m)}}]=\delta_{nm}\delta_{jl}\delta_{ik}f_n\ ,$$

*where* $f_n \geqq 0$ *and* $\sum nf_n < \infty$. *Moreover, the covariance function of* $X(g)$ *is given by*

(2.7) $$R(g)=\sum_n \chi^{(n)}(g)f_n$$

*where* $\chi^{(n)}(g)=\mathrm{tr}\,[A^{(n)}(g)]$ *are the characters of* $G$. *The* $f_n$*'s are the spectral parameters. We note that*

(2.8) $$\mathrm{E}\,\{|Z_{ji}|^2\}=\mathrm{Var}\,\{Z_{ji}^{(n)}\}=f_n\ .$$

We suppose now that $X(g)$ is Gaussian. Then the coefficients $Z_{ji}^{(n)}$ will have a complex normal joint distribution. We denote by $N_p^c(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ a multivariate complex normal distribution with mean $\boldsymbol{\mu}(p\times 1)$ and covariance matrix $\boldsymbol{\Sigma}(p\times p)$, Hermitian and non-negative definite. In the case $p=1$, if the random variable (r.v.) $Y$ is $N_1^c(\mu, \sigma^2)$, then $\mathrm{Re}\,Y$ and $\mathrm{Im}\,Y$ are independent r.v.'s $N_1(\mathrm{Re}\,\mu, \sigma^2/2)$ and $N_1(\mathrm{Im}\,\mu, \sigma^2/2)$, respectively. For further details on the complex normal distribution, see Brillinger [1] and Goodman [2].

The following theorem is immediate.

THEOREM 2.2. *If* $X(g)$ *is weakly homogeneous, real and Gaussian, with mean zero, then the r.v.'s* $Z_{ji}^{(n)}$ *are mutually independent, with a complex normal distribution* $N_1^c(0, f_n)$, *if* $f_n>0$ *and degenerate at the origin, if* $f_n=0$.

If we have a realization of the process $\{X(g),\ g\in G\}$ we obtain $Z_{ji}^{(n)}$ from (2.5) and using (2.8) we may estimate $f_n$ through

(2.9) $$\hat{f}_n=\frac{1}{d_n^2}\sum_{i,j=1}^{d_n}|Z_{ji}^{(n)}|^2\ .$$

Then we have the result below.

THEOREM 2.3. *The r.v.* $2d_n^2\hat{f}_n/f_n$ *is distributed as a chi-square r.v. with* $2d_n^2$ *degrees of freedom,* $\chi^2(2d_n^2)$, *if* $f_n>0$ *and it is degenerate at the origin if* $f_n=0$. *Moreover, the* $\hat{f}_n$*'s are mutually independent.*

PROOF. Since $Z_{ji}^{(n)}$ is a $N_1^c(0, f_n)$ r.v. then $\mathrm{Re}\,Z_{ji}^{(n)}$ and $\mathrm{Im}\,Z_{ji}^{(n)}$ are independent r.v.'s, each one with distribution $N_1(0, f_n/2)$. Since $2[\mathrm{Re}\,Z_{ji}^{(n)}]^2/f_n$ and $2[\mathrm{Im}\,Z_{ji}^{(n)}]^2/f_n$ are independent $\chi^2(1)$ it follows that $2\{[\mathrm{Re}\,Z_{ji}^{(n)}]^2+[\mathrm{Im}\,Z_{ji}^{(n)}]^2\}/f_n=2|Z_{ji}^{(n)}|^2/f_n$ is a $\chi^2(2)$, and by the independence of the $Z_{ji}^{(n)}$'s, the theorem follows.

Replacing $Z_{ji}^{(n)}$ given by (2.5) in (2.9) we obtain

$$\hat{f}_n=\int_G \int_G X(g)X(h) \sum_{i,j=1}^{d_n} A_{ij}^{(n)}(g^{-1}h)dgdh \ . \tag{2.10}$$

If we have $T$ independent realizations of $X(g)$, consider the estimator

$$f_n^{(T)}=T^{-1}\sum_{t=1}^{T}\hat{f}_{n,t} \ , \tag{2.11}$$

where $\hat{f}_{n,t}$ is the estimator (2.9) corresponding to the $t$th realization. It follows that $f_n^{(T)}$ is unbiased for $f_n$ and the theorem that follows is immediate.

THEOREM 2.4. *If $X(g)$ is Gaussian, mean zero and $f_n^{(T)}$ is given by* (2.11), *then the r.v.* $2Td_n^2 f_n^{(t)}/f_n$ *is a* $\chi^2(2Td_n^2)$, *if* $f_n>0$ *and degenerate at the origin if* $f_n=0$. *Moreover, the* $f_n^{(T)}$, $n\geqq 1$, *are mutually independent.*

Based on (2.7) we estimate $R(g)$ by

$$R^{(T)}(g)=\sum_{n=1}^{N_T}\gamma^{(n)}(g)f_n^{(T)} \ , \tag{2.12}$$

where $N_T$ is a positive integer, with $N_T\to\infty$ as $T\to\infty$. If $B(g)$ denotes the bias of $R^{(T)}(g)$, we have the next theorem.

THEOREM 2.5.

$$\text{(a)}\quad \mathrm{E}\,[R^{(T)}(g)]=O(N_T^{-1}) \ ; \tag{2.13}$$

$$\text{(b)}\quad \lim_{T\to\infty} T\,\mathrm{Cov}\,\{R^{(T)}(g_1),\, R^{(T)}(g_2)\}=d_n^{-2}\sum_{n=1}^{N_T}\gamma^{(n)}(g_1)\gamma^{(n)}(g_2)f_n^2 \ ; \tag{2.14}$$

$$\text{(c)}\quad B(g)=O(N_T^{-1}) \ . \tag{2.15}$$

PROOF. (2.13) follows immediately and

$$\begin{aligned}\mathrm{Cov}\,\{R^{(T)}(g_1),\, R^{(T)}(g_2)\}&=\mathrm{Cov}\left\{\sum_{n=1}^{N_T}\gamma^{(n)}(g_1)f_n^{(T)},\ \sum_{m=1}^{N_T}\gamma^{(m)}(g_2)f_m^{(T)}\right\}\\ &=T^{-1}d_n^{-2}\sum_{n=1}^{N_T}\gamma^{(n)}(g_1)\gamma^{(n)}(g_2)f_n^2 \ ,\end{aligned}$$

using the fact that the $f_n^{(T)}$'s are mutually independent and that $\mathrm{Var}\,[f_n^{(T)}]=f_n^2/Td_n^2$. The bias of $R^{(T)}(g)$ is given by

$$B(g)=\sum_{n>N_T}\gamma^{(n)}(g)f_n \ , \qquad \text{and} \qquad |B(g)|\leqq\sum_{n>N_T}\frac{n}{N_T}f_n\leqq\sum_n\frac{n}{N_T}f_n \ ,$$

and since $\sum_n nf_n<\infty$, the theorem is proved.

As an example, take the case of a stochastic process $\{X(P),\ P\in S_2\}$

on the unit sphere $S_2 \subset R$. The expressions (2.4) and (2.5) become

$$X(P)=\sum_{n=0}^{\infty}\sum_{k=-n}^{n} Z_{nk}Y_{nk}(P)\ , \qquad P\in S_2\ , \tag{2.16}$$

$$Z_{nk}=\int_{S_2} X(P)Y_{nk}(P)dP\ , \tag{2.17}$$

respectively, where $\{Y_{nk}(P),\ -n\leq k\leq n\}$ are the real spherical harmonics of order $n$ and $dP$ is the measure over $S_2$. In this case, the covariance function $R(P, Q)$ depends only on the angular distance $\theta$ between the points $P$ and $Q$ and (2.7) becomes

$$R(\theta)=(4\pi)^{-1}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)f_n\ , \tag{2.18}$$

where $P_n(\cdot)$ are the Legendre polynomials of degree $n$. The estimator $\hat{f}_n$ given by (2.9) and (2.10) reduces to

$$\hat{f}_n=(4\pi)^{-1}\int_{S_2}\int_{S_2} X(P)X(Q)P_n(\cos\theta)dPdQ\ . \tag{2.19}$$

For further details on stochastic processes on the sphere see Jones [4] and Roy [8], [9].

## 3. Time-dependent processes

We consider now a process $\{X(g, t)\colon g\in G,\ t\in Z\}$, where $G$ is a CAG and $Z=\{0, \pm 1, \pm 2, \cdots\}$. We assume that the process is real and for each $t\in Z$, $\{X(g, t),\ g\in G\}$ is continuous in q.m. Hence, for each $t\in Z$ we have (using the notation of last section)

$$X(g, t)=\sum_{n=1}^{\infty}\sum_{i,j=1}^{d_n} Z_{ji}^{(n)}(t)A_{ij}^{(n)}(g)\ , \qquad g\in G\ , \tag{3.1}$$

where

$$Z_{ji}^{(n)}(t)=d_n\int_G X(g, t)A_{ij}^{(n)}(g)dg\ , \tag{3.2}$$

both equations understood in the q.m. sense.

We say that $X(g, t)$ is *weakly homogeneous with respect to* $g$ and *weakly stationary with respect to* $t$ if for all $g_1, g_2$ of $G$ and $t, \tau$ of $Z$ we have

$$\begin{aligned} &\text{(a)}\quad \mathrm{E}\,[X(g_1, t)]=\text{constant}=\mu\ , \qquad \text{and we assume } \mu=0\ ,\\ &\text{(b)}\quad \mathrm{E}\,[X(g_1, t+\tau)X(g_2, t)]=R(g, \tau)\ , \qquad \text{with } g=g_1g_2^{-1}\ . \end{aligned} \tag{3.3}$$

The analogue of Theorem 2.1 is the following (see Hannan [3]).

THEOREM 3.1. *If $X(g, t)$ is weakly homogeneous and stationary, then*

$$\mathrm{E}\{Z_{ji}^{(n)}(t+\tau)Z_{lk}^{(m)}(t)\}=\delta_{nm}\delta_{jl}\delta_{ik}\int_{-\pi}^{\pi} e^{i\lambda\tau}dF_n(\lambda) \ , \tag{3.4}$$

*where $F_n(\lambda)$ is a sequence of real, non-decreasing functions, such that*

$$\sum_n n\int_{-\pi}^{\pi} dF_n(\lambda)<\infty \ . \tag{3.5}$$

*Moreover,*

$$R(g, \tau)=\sum \gamma^{(n)}(g)\int_{-\pi}^{\pi} e^{i\lambda\tau}dF_n(\lambda) \ . \tag{3.6}$$

We say that $X$ is *strictly homogeneous with respect to* $g$ and *strictly stationary with respect to* $t$ if the distribution of $X(g_1+g, t_1+t),\cdots, X(g_k+g, t_k+t)$ is independent of $g\in G$ and of $t\in Z$, for every finite collection $(g_1, t_1),\cdots, (g_k, t_k)$.

We suppose that the process $X(g, t)$ satisfies the

ASSUMPTION 3.1. For each $t\in Z$, $X(g, t)$ is continuous in q.m., strictly homogeneous and stationary and having bounded moments with respect to $g$, that is

$$|\mathrm{E}\{X(g_1, t_1)\cdots X(g_k, t_k)\}|\leqq M_k(t_1,\cdots, t_k) \ , \tag{3.7}$$

uniformly in $g_1,\cdots, g_k$, for all $t_1,\cdots, t_k$, $k=1, 2, 3,\cdots$.

The following theorems are proved along the same lines as in Roy [7], [8] and hence we omit the proofs. We denote by $\mathrm{Cum}\{Y_1,\cdots, Y_k\}$ the $k$th order cumulant of the r.v.'s $Y_1,\cdots, Y_k$.

THEOREM 3.2. *The cumulant*

$$\mathrm{Cum}\{Z_{j_1i_1}^{(n_1)}(u_1+t),\cdots, Z_{j_{k-1}i_{k-1}}^{(n_{k-1})}(u_{k-1}+t), Z_{j_ki_k}^{(n_k)}(t)\}$$

*is independent of $t$ and it will be denoted by*

$$c_{j_1i_1,\cdots,j_ki_k}^{(n_1,\cdots,n_k)}(u_1,\cdots, u_{k-1}) \ .$$

This result is necessary to prove the central limit theorem below, since eventhough $X(g, t)$ is homogeneous and stationary, the process $\{Z_{ji}^{(n)}(t),\ t\in Z\}$ given by (3.2) need not to be stationary.

To prove Theorem 3.2 the following lemmas are necessary.

LEMMA 3.1.

$$\mathrm{E}\{Z_{j_1i_1}^{(n_1)}(t_1)\cdots Z_{j_ki_k}^{(n_k)}(t_k)\}$$
$$=d_n^k\int_G\cdots\int_G \prod_{l=1}^{k}\overline{A_{i_lj_l}^{(n_l)}(g_l)}\ \mathrm{E}\left\{\prod_{l=1}^{k} X(g_l, t_l)\right\}dg_1\cdots dg_k \ ,$$

*for all* $t_l, g_l, j_l, i_l, l=1, 2, \cdots, k, k=1, 2, \cdots$.

LEMMA 3.2.

$$\begin{aligned}&\mathrm{Cum}\,\{Z^{(n_1)}_{j_1 i_1}(t_1), \cdots, Z^{(n_k)}_{j_k i_k}(t_k)\}\\&\quad = d_n^k \int_G \cdots \int_G \prod_{l=1}^{k} \overline{A^{(n_l)}_{i_l j_l}(g_l)}\, \mathrm{Cum}\,\{X(g_1, t_1), \cdots, X(g_k, t_k)\} dg_1 \cdots dg_k\,,\end{aligned}$$

*for all* $t_l, g_l, j_l, i_l, l=1, \cdots, k, k=1, 2, \cdots$.

We further assume that the following supposition is valid.

ASSUMPTION 3.2($l$). For given $l \geqq 0$

$$\begin{aligned}\sum_{u_1, \cdots, u_{k-1}=-\infty}^{\infty} \{1+|u_j|^l\}|\mathrm{Cum}\,\{&X(g_1, u_1+t), \cdots,\\ &X(g_{k-1}, u_{k-1}+t), X(g_k, t)\}| \leqq C_k < \infty\,,\end{aligned}$$

uniformly in $g_1, \cdots, g_k$, $j=1, \cdots, k-1$, $k=2, 3, \cdots$.

It is easy to see using Lemma 3.2 and the Monotone Convergence Theorem, that this assumption implies that

$$\sum_{u_1, \cdots, u_{k-1}=-\infty}^{\infty} \{1+|u_j|^l\}|c^{(n_1, \cdots, n_k)}_{j_1 i_1, \cdots, j_k i_k}(u_1, \cdots, u_{k-1})| < \infty\,. \tag{3.8}$$

For $l=0$, we can then define the *spectral cumulant of order* $k$ as

$$\begin{aligned}f^{(n_1, \cdots, n_k)}_{j_1 i_1, \cdots, j_k i_k}&(\lambda_1, \cdots, \lambda_{k-1})\\&= (2\pi)^{-k+1} \sum_{u_1, \cdots, u_{k-1}} c^{(u_1, \cdots, u_k)}_{j_1 i_1, \cdots, j_k i_k}(u_1, \cdots, u_{k-1}) \exp\left(-i\sum_{j=1}^{k-1} u_j \lambda_j\right)\,,\end{aligned} \tag{3.9}$$

for $-\infty < \lambda_j < +\infty$, $j=1, \cdots, k-1$, $k=2, 3, \cdots$.

In particular, for the second-order spectrum, we have from Theorem 3.1

$$f^{(n, m)}_{ji, lk}(\lambda) = \delta_{jl}\delta_{ik}\delta_{nm} f_n(\lambda)\,, \tag{3.10}$$

and the Assumption 3.2(0) tells us that $f_n(\lambda) = F_n'(\lambda)$. In order to estimate $f_n(\lambda)$ we consider the finite Fourier transform of the coefficients $Z^{(n)}_{ji}(\lambda)$, defined by

$$d^{(n, T)}_{ji}(\lambda) = (2\pi T)^{-1/2} \sum_{t=0}^{T-1} Z^{(n)}_{ji}(t) e^{-i\lambda t}\,, \tag{3.11}$$

$-\infty < \lambda < +\infty$, $i, j=1, \cdots, d_n$, given the values $X(g, t)$, $g \in G$ and $t=0, 1, \cdots, T-1$ of the process.

From Theorem 3.2 and Theorem 4.4.1 of Brillinger [1], we have the following result.

THEOREM 3.3. *Let $X(g, t)$ satisfying Assumption 3.1 and Assumption 3.2(1). Let $s_k(T)$ be an integer with $\lambda_k(T)=2\pi s_k(T)/T\rightarrow\lambda_k$, as $T\rightarrow\infty$, $k=1,\cdots,K$. Assume $2\lambda_k(T)$, $\lambda_k(T)\pm\lambda_l(T)\not\equiv 0 \pmod{2\pi}$, for $1\leqq k<l\leqq K$ and let $d_{ji}^{(n,T)}(\lambda)$ be defined by (3.11). Then $d_{ji}^{(n,T)}(\lambda_k(T))$, $k=1,\cdots,K$ are asymptotically independent $N_1^c(0, f_n(\lambda_k))$ random variables. If $\lambda\equiv 0 \pmod{\pi}$ then $d_{ji}^{(n,T)}(\lambda)\xrightarrow{\mathcal{D}}N_1(0, f_n(\lambda))$ independently of the previous variates.*

The theorem suggests that as an estimate of $f_n(\lambda)$ we may take

$$I_n^{(T)}(\lambda)=\frac{1}{d_n^2}\sum_{i,j=1}^{d_n}|d_{ji}^{(n,T)}(\lambda)|^2\,,\qquad n\geqq 0 \tag{3.12}$$

which is called the *periodogram of order n*.

COROLLARY. *Under the conditions of the Theorem 3.3, for $n\geqq 0$, $I_n^{(T)}(\lambda_k)$, $k=1,\cdots,K$, are asymptotically independent $f_n(\lambda_k)\chi^2(2d_n^2)/2d_n^2$ random variables. For $\lambda\equiv 0 \pmod{\pi}$, $I_n^{(T)}(\lambda)$ is $f_n(\lambda)\chi^2(d_n^2)/d_n^2$ independently of the previous variates and for $n\neq m$, $I_n^{(T)}(\lambda)$ and $I_m^{(T)}(\lambda)$ are asymptotically independent.*

Consistent spectral estimates for the case of the sphere are considered by Roy [9]. A class of estimates for the case of the circle are considered by Roy [7] and Roy and Dufour [10].

UNIVERSITY OF SÃO PAULO

## REFERENCES

[1] Brillinger, D. R. (1975). *Time Series—Data Analysis and Theory*, Holt, Rinehart and Winston, N.Y.

[2] Goodman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution (An introduction), *Ann. Math. Statist.*, **34**, 152-177.

[3] Hannan, E. J. (1971). *Multiple Time Series*, John Wiley and Sons, N.Y.

[4] Jones, R. H. (1963). Stochastic processes on a sphere, *Ann. Math. Statist.*, **34**, 213-215.

[5] Morettin, P. A. (1979). Homogeneous random processes on locally compact Abelian groups, to appear Annals of Braz. Academy of Sciences.

[6] Pontryagin, L. S. (1966). *Topological Groups*, 2nd edition, Gordon and Breach, N.Y.

[7] Roy, R. (1972). Spectral analysis for a random process on the circle, *J. Appl. Prob.*, **9**, 745-757.

[8] Roy, R. (1973). Estimation of the covariance function of a homogeneous processes on the sphere, *Ann. Statist.*, **1**, 780-785.

[9] Roy, R. (1976a). Spectral analysis for a random process on the sphere, *Ann. Inst. Statist. Math.*, **28**, 91-97.

[10] Roy, R. and Dufour, J. M. (1976b). On spectral estimation for a homogeneous random process on the circle, *Stoch. Processes Appl.*, **4**, 107-120.

[11] Yaglom, A. M. (1961). Second-order homogeneous random fields, *Proc. Fourth Berkeley Symp.*, 593-622.