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BETWEENNESS FOR REAL VECTORS AND LINES, II
RELATEDNESS OF BETWEENNESSES

EDWARD W. BARANKIN[1)] AND KOITI TAKAHASI[2)]

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1. Introduction

There is a substantial machinery of relatedness connecting betweenness relations with other betweenness relations, and it is constantly advantageous to draw upon it. We saw evidence of this in Section 6 of article I, the first article in this series ([2] in the References). In the present, second article we are going to develop more of this machinery and apply it to characterize a new betweenness in $\mathcal{K}^p$.

We start off in Section 2 by proving, in Theorem 2.1, the result that was stated without proof as part (2) of Theorem I.6.1. Further utilization of the procedures of this proof yield another result, Theorem 2.2, that is subsequently applied in the last section. In Section 3 we introduce and study the notion of a t-invariant betweenness relation on the domain $\mathcal{X}$ of a function t carrying $\mathcal{X}$ into a space $\mathcal{Y}$. The notion is alternatively formulated in terms of spread functions or span functions, and there is made explicit the one-to-one correspondence between t-invariant spread functions in $\mathcal{X}$ and general spread functions in $\mathcal{Y}$ (Theorem 3.2); similarly for span functions (Theorem 3.3). These facts, in their detail, in particular afford a spread function in $\mathcal{Y}$ corresponding to any given t-invariant spread function in $\mathcal{X}$. And this is precisely the subject that is studied more closely in Section 4, the last section, in the special case of $\mathcal{X}=\mathcal{O}_{\mathcal{K}}$, $\mathcal{Y}=\mathcal{K}^p$, and $t=t_0$, the function which maps an $x\in\mathcal{O}_{\mathcal{K}}$ into the linear manifold in $\mathcal{K}$ spanned by x. Theorem 4.1 presents a very general method of constructing t_0-invariant spread functions from spread functions of a less restricted type. Then it is shown that $\mathcal{V}_0^v$-betweenness and $\mathcal{V}_v^v$-betweenness are of this latter type and so they give rise, via Theorem 4.1, to t_0-invariant spread functions in $\mathcal{O}_{\mathcal{K}}$; and consequently, by the results of Section 3, also to spread functions in $\mathcal{K}^p$. The spread function in $\mathcal{K}^p$ thus derived from $\mathcal{V}_0^v$-betweenness is seen to be just that of $\mathcal{V}_0$-betweenness. The spread

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1) Tokyo, 1975.
2) Visiting Scholar, University of California, Berkeley, 1977.

function in $\mathcal{K}^p$ derived from $\mathcal{V}_\nu^\nu$-betweenness is new; its corresponding betweenness relation we label $\mathcal{V}_\nu$-*betweenness*, and we have worked out its characterization and stated it in Definition 4.3.

The notation in this present paper follows that of the first paper in this series. In referring here to formal statements and displays and sections in that first paper, we will use the obvious device (as already done here above) of inserting the Roman numeral I; thus, for example, Theorem I.6.1, or (I.6.8), or Section I.2.

2. Two theorems on $\mathcal{V}_\nu^\nu$-betweenness in $\mathcal{O}_{\mathcal{K}}$

In Theorem I.6.1 we stated a result without proof, namely, the statement (2) of that theorem. We begin this present paper by supplying that proof. And we will then go on, in this section, to prove a second result that will have application later on.

THEOREM 2.1. *Let* $\dim \mathcal{K} \geqq 3$, *and let* x *and* y *be elements of* $\mathcal{O}_{\mathcal{K}}$, *with* ζ *denoting the inner product* (x, y). *Let* $\nu > 0$ *and let the following condition hold:*

$$\nu \leqq \sqrt{\frac{1+\zeta}{2}}\,. \tag{2.1}$$

Finally we define

$$\rho \overset{\text{def.}}{=} \sqrt{1-\frac{2\nu^2}{1+\zeta}}\,. \tag{2.2}$$

Then, $z \in \mathcal{O}_{\mathcal{K}}$ *is* $\mathcal{V}_\nu^\nu$-*between* x *and* y *if and only if*

$$\left(z, \frac{\nu}{1+\zeta}(x+y)+w\right) \geqq \nu \tag{2.3}$$

for every $w \in \mathcal{K}$ *satisfying the conditions*

$$w \perp x, y\,, \qquad \|w\| = \rho\,. \tag{2.4}$$

To prove this theorem we consider first the case of $y=x$. In this case we know from our discussion in the first article that the only admissible z is x. (We see that this follows, prior to Theorem I.6.1, from the argument that was used in [2] to establish the third assertion in (I.6.8).) We proceed to show that this is what (2.3) implies. First of all, if $\rho=0$ then (2.3) reduces to $(z, x) \geqq 1$, and this is clearly satisfied only for $z=x$. If $\rho>0$, and we suppose $z \neq x$, then on writing $z=ax+w'$, with $w' \perp x$, (2.3) becomes

$$\nu a + (w', w) \geqq \nu\,; \tag{2.5}$$

and we have also

$$a^2+\|w'\|^2=1 \,. \tag{2.6}$$

If $w'=\theta$, these conditions immediately imply $a=+1$, and we have again $z=x$. If $w'\neq\theta$, then (2.5) will—according to (2.4)—hold for $w=-(\rho/\|w'\|)w'$. Putting in this w, we get

$$\nu a-\rho\|w'\|\geqq\nu \,. \tag{2.7}$$

This inequality cannot hold, since $a\leqq 1$ according to (2.6). Therefore w' must be θ, and so, as before, $a=1$ and $z=x$.

We now go on to consider the case of $y\neq x$. Let us set

$$A=\{\omega\in\mathcal{O}_{\mathcal{K}}\,|\,(x,\omega)\geqq\nu,\ (y,\omega)\geqq\nu\} \tag{2.8}$$

and

$$B=\{\omega\in\mathcal{O}_{\mathcal{K}}\,|\,(x,\omega)=(y,\omega)=\nu\}\,; \tag{2.9}$$

and then define also

$$A'=\{z\in\mathcal{O}_{\mathcal{K}}\,|\,(z,\omega)\geqq\nu \textit{ for all } \omega\in A\} \tag{2.10}$$

and

$$B'=\{z\in\mathcal{O}_{\mathcal{K}}\,|\,(z,\omega)\geqq\nu \textit{ for all } \omega\in B\}\,. \tag{2.11}$$

From the definition of $\mho^{v}_{\nu}$-betweenness we know that A' is precisely the set of all $z\in\mathcal{O}_{\mathcal{K}}$ which are $\mho^{v}_{\nu}$-between x and y. We see also that $B\subseteq A$ and therefore $A'\subseteq B'$. The course of our proof is going to be to first obtain an explicit characterization of B, and then to establish the reverse inclusion $B'\subseteq A'$. This will show that likewise B' is the set of all $z\in\mathcal{O}_{\mathcal{K}}$ which are $\mho^{v}_{\nu}$-between x and y, and the characterization of B will then have given us the assertion of our present theorem.

If $\omega\in B$ we have $(x-y,\omega)=0$; that is, $\omega\perp x-y$. Since x and y have equal norms, the orthocomplement of $\{x-y\}$ is spanned by $x+y$ and the orthocomplement of the set $\{x,y\}$. Therefore we can write $\omega=a(x+y)+w$, with $w\perp x,y$. Applying the fact that $(x,\omega)=\nu$ we obtain the evaluation $a=\nu/(1+\text{と})$. The fact that $\|\omega\|=1$ then gives us $\|w\|=\rho$. Thus, any $\omega\in B$ is of the form

$$\omega=\frac{\nu}{1+\text{と}}(x+y)+w \qquad \textit{with } w\perp x,y \textit{ and } \|w\|=\rho\,. \tag{2.12}$$

Conversely, we see easily that every ω of the form (2.12) is an element of B. It follows that the conditions (2.3) and (2.4) of the theorem describe precisely the set B'. And therefore, for the proof of the theorem, it remains only to show that $B'=A'$, a fact that will be established when we prove that $B' \subseteq A'$.

Let $z \in B'$ and let us write it in the form

$$z = r(x+y) + s(x-y) + u\ , \qquad u \perp x, y\ . \tag{2.13}$$

Since $\|z\| = 1$, we have

$$2(1+\text{と})r^2 + 2(1-\text{と})s^2 + \|u\|^2 = 1\ . \tag{2.14}$$

Expressing the fact that $z \in B'$—using (2.12)—gives us

$$2\nu r + (u, w) \geqq \nu \qquad \textit{for every } w \perp x, y, \textit{ with } \|w\| = \rho\ . \tag{2.15}$$

If $u = \theta$, (2.15) gives us $r \geqq 1/2$. Applying this to (2.14), we get $s^2 \leqq 1/4$. Summing up, we have

$$|s| \leqq \frac{1}{2} \leqq r\ , \qquad \textit{if } u = \theta\ . \tag{2.16}$$

Now suppose $u \neq \theta$. Setting $w = -(\rho/\|u\|)u$ in (2.15), we obtain

$$2\nu r - \rho\|u\| \geqq \nu\ , \tag{2.17}$$

or

$$r \geqq \frac{1}{2}\left(1 + \frac{\rho\|u\|}{\nu}\right)\ . \tag{2.18}$$

Applying this result to (2.14), we have

$$2(1-\text{と})s^2 \leqq 1 - \frac{1+\text{と}}{2}\left(1 + \frac{\rho\|u\|}{\nu}\right)^2 - \|u\|^2\ . \tag{2.19}$$

Now notice that, for any value of $\|u\|$,

$$\begin{aligned} 1 - \|u\|^2 &\leqq \left(1 + \frac{\rho\|u\|}{\nu}\right)^2 \\ &= \frac{1+\text{と}}{2}\left(1 + \frac{\rho\|u\|}{\nu}\right)^2 + \frac{1-\text{と}}{2}\left(1 + \frac{\rho\|u\|}{\nu}\right)^2\ . \end{aligned} \tag{2.20}$$

Bringing this inequality to bear on (2.19), we get the result

$$s^2 \leqq \frac{1}{4}\left(1 + \frac{\rho\|u\|}{\nu}\right)^2\ . \tag{2.21}$$

Combining this with (2.18), and noticing that the result reduces to (2.16) in the case of $\|u\| = 0$, we can now finally state: *for all values of* $\|u\|$,

$$|s| \leqq \frac{1}{2}\left(1 + \frac{\rho\|u\|}{\nu}\right) \leqq r\ . \tag{2.22}$$

We now want to show that $z \in A'$. Let ω_1 be any element of A, and let us write

$$(2.23) \qquad \omega_1 = \alpha(x+y)+\beta(x-y)+v\ , \qquad v \perp x, y\ .$$

From the inequalities $(x, \omega_1) \geqq \nu$ and $(y, \omega_1) \geqq \nu$, we readily find

$$(2.24) \qquad (1+\xi)\alpha-\nu \geqq 0$$

and

$$(2.25) \qquad (1-\xi)|\beta| \leqq (1+\xi)\alpha-\nu\ .$$

From the fact that ω_1 is of norm 1, combined with the inequality (2.24), we obtain

$$(2.26) \qquad \|v\| \leqq \rho\ .$$

Consider now

$$\begin{aligned}(2.27) \quad (z, \omega_1) &= (z, \alpha(x+y)+\beta(x-y)+v) \\ &= \left(z, \frac{\nu}{1+\xi}(x+y)+\left[\left(\alpha-\frac{\nu}{1+\xi}\right)(x+y)+\beta(x-y)+v\right]\right) \\ &= \left(z, \left[\frac{\nu}{1+\xi}(x+y)+w\right]+\left[\left(\alpha-\frac{\nu}{1+\xi}\right)(x+y)\right.\right. \\ &\qquad \left.\left. +\beta(x-y)+(v-w)\right]\right)\ .\end{aligned}$$

The vector w that has been introduced here is defined as follows:

$$(2.28) \qquad w = \begin{cases} +\dfrac{\rho}{\|v\|}v & \textit{if } (u, v) < 0\ , \\ -\dfrac{\rho}{\|v\|}v & \textit{if } (u, v) \geqq 0,\ v \neq \theta\ , \\ \textit{any particular } w_0 \perp x, y \textit{ with } \|w_0\| = \rho \textit{ and such that} \\ (u, w_0) \leqq 0 \textit{ if } v = \theta. \end{cases}$$

If we now continue our evaluation of (z, ω_1), putting in the expression (2.13) for z, and applying the fact that $(z, \omega) \geqq \nu$ for any ω of the form (2.12), we get

$$(2.29) \qquad (z, \omega_1) \geqq \nu + 2(1+\xi)r\left(\alpha-\frac{\nu}{1+\xi}\right)+2(1-\xi)s\beta+(u, v-w)\ .$$

Let us examine the term $(u, v-w)$. If $(u, v) < 0$, then

$$(2.30) \qquad (u, v-w) = \left(1-\frac{\rho}{\|v\|}\right)(u, v) \geqq 0 \qquad \textit{by } (2.26)\ .$$

If $(u, v) \geqq 0$ and $v \neq \theta$, then

$$(u, v-w) = \left(1 + \frac{\rho}{\|v\|}\right)(u, v) \geqq 0 \,. \tag{2.31}$$

And finally, if $v = \theta$,

$$(u, v-w) = -(v, w_0) \geqq 0 \qquad by\ (2.28)\,. \tag{2.32}$$

Thus, $(u, v-w) \geqq 0$ in all cases, and therefore (2.29) gives us

$$\begin{aligned} (z, \omega_1) &\geqq \nu + 2(1+\text{と})r\left(\alpha - \frac{\nu}{1+\text{と}}\right) + 2(1-\text{と})s\beta \\ &\geqq \nu + 2r[(1+\text{と})\alpha - \nu] - 2|s|[(1-\text{と})|\beta|] \\ &\geqq \nu + 2r[(1+\text{と})\alpha - \nu] - 2|s|[(1+\text{と})\alpha - \nu] \qquad by\ (2.25) \\ &= \nu + 2[(1+\text{と})\alpha - \nu](r - |s|) \\ &\geqq \nu \qquad by\ (2.22)\ and\ (2.24)\,. \end{aligned} \tag{2.33}$$

This completes the proof that $z \in A'$, hence that $B' \subseteq A'$, hence that $B' = A'$, and so the proof of Theorem 2.1 is complete.

We next prove the following theorem:

THEOREM 2.2. *Let x and y be elements of $\mathcal{O}_{\mathcal{K}}$, with* $\text{と} \overset{\text{def.}}{=} (x, y)$, $\nu > 0$, *condition* (2.1) *holding, and ρ defined by* (2.2). *Then, if z' and z'' are any elements of $\mathcal{O}_{\mathcal{K}}$ that are* ひ^{v}_{ν}*-between x and y, we have* $(z', z'') \geqq \text{と}$.

This result is quite obvious in the cases of $\dim \mathcal{K} = 1$ and 2, and also when $y = x$ in $\mathcal{K}$ of dimension $\geqq 3$. We shall therefore confine ourselves, for the detailed proof, to the case of $\dim \mathcal{K} \geqq 3$ and $y \neq x$.

Let us write

$$\begin{aligned} z' &= r'(x+y) + s'(x-y) + u'\,, \qquad u' \perp x, y\,, \\ z'' &= r''(x+y) + s''(x-y) + u''\,, \qquad u'' \perp x, y\,. \end{aligned} \tag{2.34}$$

By (2.18) we have

$$r' \geqq \frac{1}{2}\left(1 + \frac{\rho\|u'\|}{\nu}\right), \qquad r'' \geqq \frac{1}{2}\left(1 + \frac{\rho\|u''\|}{\nu}\right), \tag{2.35}$$

and by (2.19)

$$\begin{aligned} 2(1-\text{と})s'^2 &\leqq 1 - \frac{1+\text{と}}{2}\left(1 + \frac{\rho\|u'\|}{\nu}\right)^2 - \|u'\|^2\,, \\ 2(1-\text{と})s''^2 &\leqq 1 - \frac{1+\text{と}}{2}\left(1 + \frac{\rho\|u''\|}{\nu}\right)^2 - \|u''\|^2\,. \end{aligned} \tag{2.36}$$

With these inequalities we now proceed to examine

$$
\begin{aligned}
(2.37)\quad (z', z'') &= 2(1+\text{と})r'r''+2(1-\text{と})s's''+(u', u'') \\
&\geqq 2(1+\text{と})r'r''-2(1-\text{と})|s'|\cdot|s''|+(u', u'') \\
&\geqq 2(1+\text{と})r'r''-(1-\text{と})(s'^2+s''^2)+(u', u'') \\
&\geqq \frac{1}{2}(1+\text{と})\left(1+\frac{\rho\|u'\|}{\nu}\right)\left(1+\frac{\rho\|u''\|}{\nu}\right) \\
&\quad -\frac{1}{2}\left[1-\frac{1+\text{と}}{2}\left(1+\frac{\rho\|u'\|}{\nu}\right)^2-\|u'\|^2\right. \\
&\quad \left.+1-\frac{1+\text{と}}{2}\left(1+\frac{\rho\|u''\|}{\nu}\right)^2-\|u''\|^2\right]+(u', u'') \\
&= \text{と}+(1+\text{と})\frac{\rho}{\nu}(\|u'\|+\|u''\|)+\frac{1+\text{と}}{2}\left(\frac{\rho}{\nu}\right)^2\|u'\|\cdot\|u''\| \\
&\quad +\frac{1+\text{と}}{4}\left(\frac{\rho}{\nu}\right)^2(\|u'\|^2+\|u''\|^2) \\
&\quad +\left[\frac{1}{2}(\|u'\|^2+\|u''\|^2)+(u', u'')\right].
\end{aligned}
$$

Since $|(u, u'')|\leqq\|u'\|\cdot\|u''\|\leqq(1/2)(\|u'\|^2+\|u''\|^2)$, we see that this last expression is $\geqq$ と, and this is the result we were after. The proof of the theorem is complete.

3. Invariant betweenness relations

In Lemma I.6.2 we saw that a betweenness relation is straightforwardly induced in the domain of a function when a betweenness relation is given in the range-space of the function. The fact is that the functional relation also permits us, in certain circumstances, to go in the other direction from a given betweenness relation to an induced betweenness relation. To understand this, let us begin by making a definition. (It will be noted that we are now switching around somewhat the notation that was used in Lemma I.6.2.)

DEFINITION 3.1. Let t be a function on the space $\mathcal{X}$ to the space $\mathcal{Y}$. A betweenness relation B in $\mathcal{X}$ will be said to be *t-invariant* if the following condition holds: if $B(x, y; z)$ then also $B(x_1, y_1; z_1)$ for any x_1, y_1, z_1 such that $t(x_1)=t(x)$, $t(y_1)=t(y)$ and $t(z_1)=t(z)$.

If the betweenness relation B in $\mathcal{X}$ is the relation induced, in the manner of Lemma I.6.2, from a betweenness relation B' in $\mathcal{Y}$ under a function t on $\mathcal{X}$ to $\mathcal{Y}$, then we see without any difficulty that B is t-invariant. Now, conversely suppose that B is t-invariant. Then for any two triplets, $\langle x, y, z\rangle$ and $\langle x_1, y_1, z_1\rangle$, of points in $\mathcal{X}$, such that

$t(x_1)=t(x)$, $t(y_1)=t(y)$ and $t(z_1)=t(z)$, either both $B(x, y; z)$ and $B(x_1, y_1; z_1)$, or both $B^c(x, y; z)$ and $B^c(x_1, y_1; z_1)$. (Recall that $B^c(x, y; z)$ denotes that z is not between x and y.) Consequently, a relation, $B^{[t]}$, is defined on the range (in $\mathcal{Y}$) of t by the equation

$$B^{[t]}(x', y'; z') \equiv B(x, y; z) \qquad \textit{for any } x, y, z \in \mathcal{X} \textit{ such that } t(x)=x',\ t(y)=y' \textit{ and } t(z)=z' . \tag{3.1}$$

And we find readily that $B^{[t]}$ fulfills the conditions of Definition I.2.1; that is, $B^{[t]}$ is a betweenness relation. It is furthermore easy to prove that the procedures of Lemma I.6.2 and the present (3.1) are inverses of each other. All of these results can now be summed up in the following theorem:

THEOREM 3.1. *Let t be a function on the space $\mathcal{X}$ onto the space $\mathcal{Y}$. There is a 1-1 correspondence between the class of all betweenness relations in $\mathcal{Y}$ and the class of all t-invariant betweenness relations in $\mathcal{X}$. Under this correspondence, if B' is a betweenness relation in $\mathcal{Y}$, then the corresponding betweenness relation in $\mathcal{X}$ is B'_t, defined by*

$$B'_t(x, y; z) \equiv B'(t(x), t(y); t(z)) . \tag{3.2}$$

And if B is a t-invariant betweenness relation in $\mathcal{X}$, then the corresponding betweenness relation in $\mathcal{Y}$ is $B^{[t]}$, given by the equivalence (3.1).

It is useful to characterize this correspondence in terms of spread functions and span functions. Let B be a t-invariant betweenness relation in $\mathcal{X}$, t being a function on $\mathcal{X}$ onto $\mathcal{Y}$. Let $B^{[t]}$ be the betweenness relation in $\mathcal{Y}$ corresponding to B, according to the above theorem. Let τ be one of the possibly two (see Section I.2) spread functions belonging to B, and $\tau^{[t]}$ be one of those corresponding to $B^{[t]}$. If now $\{x, y\}$ is any set in $\tilde{\mathcal{A}}^{\cdot}$ (=the collection of all non-empty, at most 2-point subsets of $\mathcal{X}$), then we may calculate as follows:

$$\begin{aligned} \tau(\{x, y\}) &= \{z \in \mathcal{X} \mid B(x, y; z)\} \\ &= \{z \in \mathcal{X} \mid B^{[t]}(t(x), t(y); t(z))\} \\ &= \{z \in \mathcal{X} \mid t(z) \in \tau^{[t]}(\{t(x), t(y)\})\} \\ &= t^{-1}\tau^{[t]}(\{t(x), t(y)\}) ; \end{aligned} \tag{3.3}$$

and this result may be written

$$\tau(A) = t^{-1}\tau^{[t]}(tA) , \qquad A \in \tilde{\mathcal{A}}^{\cdot} . \tag{3.4}$$

Corresponding to $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^{\cdot}$ in $\mathcal{X}$, let $\mathcal{B}$ be the class of all subsets of $\mathcal{Y}$, $\tilde{\mathcal{B}}$ be the class of all at most 2-point subsets of $\mathcal{Y}$, and $\tilde{\mathcal{B}}^{\cdot}$ be the class of all non-empty, at most 2-point subsets of $\mathcal{Y}$. We see that as A runs over all sets in $\tilde{\mathcal{A}}^{\cdot}$, tA runs over all sets in $\tilde{\mathcal{B}}^{\cdot}$. Therefore, (3.4) gives

$$\bigcap_{A \in \tilde{\mathcal{A}}^{\cdot}} \tau(A) = t^{-1} \bigcap_{A' \in \tilde{\mathcal{B}}^{\cdot}} \tau^{[t]}(A') . \tag{3.5}$$

It follows that the two intersections in this equation are either both empty or both non-empty. Consequently, according to (3.5), the equation (3.4) will hold also for $A=\emptyset$ (the empty set) provided the proper matching is made between the choice of τ and the choice of $\tau^{[t]}$. That is, if there is chosen the τ with $\tau(\emptyset)=\emptyset$, then the $\tau^{[t]}$ with $\tau^{[t]}(\emptyset)=\emptyset$ is to be chosen. And if there is chosen (if it exists) the τ with $\tau(\emptyset)\neq\emptyset$, then there will be the alternative $\tau^{[t]}$ with $\tau^{[t]}(\emptyset)\neq\emptyset$ (—and conversely—), and it is to be chosen. With the understanding that this is done, we may replace (3.4) with the statement

$$\tau(A) = t^{-1}\tau^{[t]}(tA) , \qquad A \in \tilde{\mathcal{A}} . \tag{3.6}$$

(The reader will recognize that this discussion is based on the findings in Section I.2.)

The inverse of the formula (3.6) can be derived as follows. From (3.6) it follows that $\tau^{[t]}(tA)=t\tau(A)$, and therefore we have:

for $A' \in \tilde{\mathcal{B}}$,

$$\tau^{[t]}(A') = t\tau(A) \tag{3.7}$$

where A *is any set in* $\tilde{\mathcal{A}}$ *such that* $tA=A'$.

Our discussion has shown that, in spite of the ambiguity of spread functions corresponding to a given betweenness relation, there is actually a 1-1 correspondence between the spread functions of t-invariant betweenness relations in $\mathcal{X}$ and spread functions of betweenness relations in $\mathcal{Y}$; and this correspondence is fully expressed by (3.6) and (3.7). Before we state this result in the form of a theorem, let us obtain another result, namely, a characterization of the spread functions of t-invariant betweenness relations in $\mathcal{X}$. We begin by making the following definitions:

DEFINITION 3.2. Let t be a function on the space $\mathcal{X}$ onto the space $\mathcal{Y}$. A set $A\subseteq\mathcal{X}$ will be said to be a *t-invariant* subset of $\mathcal{X}$ if $t^{-1}tA = A$. A function, f, on any class $\mathcal{D}$ of subsets of $\mathcal{X}$, with values in $\mathcal{A}$, will be called a *t-invariant function* if $f(A)$ is a t-invariant set for every $A \in \mathcal{D}$.

By (3.6) we see that a spread function of a t-invariant betweenness relation is t-invariant. We want to show now that the converse of this is also true, namely, that any t-invariant spread function determines a t-invariant betweenness relation. Let us recall (see Section I.2) that a spread function is characterized by the two properties

$$A \subseteq \tau(A) , \tag{3.8}$$

$$A \subseteq \tau(B) \Rightarrow \tau(A) \subseteq \tau(B) \tag{3.9}$$

for all sets A and B in $\tilde{\mathcal{A}}$. And let us note explicitly that the betweenness relation B determined by a given spread function τ is defined by the equivalence

$$B(x, y; z) \equiv [z \in \tau(\{x, y\})] . \tag{3.10}$$

Let the spread function τ be t-invariant. Let the points x, y and z be such that $B(x, y; z)$. To prove that B is t-invariant we must show that if x_1, y_1, z_1 are any particular three points with

$$t(x_1) = t(x) , \qquad t(y_1) = t(y) , \qquad t(z_1) = t(z) , \tag{3.11}$$

then also $B(x_1, y_1; z_1)$. That is, we must show that $z \in \tau(\{x, y\})$ implies $z_1 \in \tau(\{x_1, y_1\})$. By (3.8) we have $\{x, y\} \subseteq \tau(\{x, y\})$, and therefore $\{x_1, y_1\} \subseteq t^{-1}t\{x, y\}$ (by (3.11)) $\subseteq t^{-1}t\tau(\{x, y\}) = \tau(\{x, y\})$ (by t-invariance). It follows then by (3.9) that $\tau(\{x_1, y_1\}) \subseteq \tau(\{x, y\})$. By repeating this argument with the pairs x, y and x_1, y_1 interchanged, we come up with the reverse inclusion, and hence we have

$$\tau(\{x_1, y_1\}) = \tau(\{x, y\}) . \tag{3.12}$$

Now, by (3.11), $z_1 \in t^{-1}t\{z\}$, and therefore the statement $z \in \tau(\{x, y\})$ implies $z_1 \in t^{-1}t\tau(\{x, y\}) = \tau(\{x, y\}) = \tau(\{x_1, y_1\})$ (by (3.12)), which is what was to be shown. Our result is the following:

LEMMA 3.1. *The class of all spread functions of t-invariant betweenness relations in $\mathcal{X}$ is identical with the class of all t-invariant spread functions in $\mathcal{X}$.*

With this lemma we are now able to state our previous result as follows:

THEOREM 3.2. *The class of all t-invariant spread functions in $\mathcal{X}$ is in a* 1-1 *correspondence with the class of all spread functions in $\mathcal{Y}$. Under this correspondence the spread function, τ, in $\mathcal{X}$ corresponding to a specified spread function, $\tau^{[t]}$, in $\mathcal{Y}$ is given by* (3.6); *and conversely, the $\tau^{[t]}$ in $\mathcal{Y}$ corresponding to a specified τ in $\mathcal{X}$ is given by* (3.7). *This correspondence is identical with that of Lemma* 3.1 *under identification of spread functions with their betweenness relations.*

Continuing to employ the notation of Section I.2, we see that from (3.6) we obtain

$$\tau^{(1)}(A)=t^{-1}\tau^{[t](1)}(tA)\ , \qquad A\in\mathcal{A}\ . \tag{3.13}$$

And from this in turn it follows that the same relation holds between the core extensions of τ and $\tau^{[t]}$; that is,

$$g_\tau(A)=t^{-1}g_{\tau^{[t]}}(tA)\ , \qquad A\in\mathcal{A}\ . \tag{3.14}$$

Notice that from this we have $g_\tau(t^{-1}tA)=t^{-1}g_{\tau^{[t]}}(tt^{-1}tA)=t^{-1}g_{\tau^{[t]}}(tA)$; that is,

$$g_\tau(A)=g_\tau(t^{-1}tA)\ , \qquad A\in\mathcal{A}\ . \tag{3.15}$$

According to (3.14), $g_{\tau^{[t]}}(tA)=tg_\tau(A)$, and therefore, by (3.15), $g_{\tau^{[t]}}(tA)=tg_\tau(t^{-1}tA)$. If we set $tA=A'$, we thus obtain the inverse formula to (3.14):

$$g_{\tau^{[t]}}(A')=tg_\tau(t^{-1}A')\ , \qquad A'\in\mathcal{B}\ . \tag{3.16}$$

We want to go on now to show that the same reciprocal relations (3.14) and (3.16) hold between span functions in general, and give us a 1-1 correspondence between the class of all t-invariant span functions in $\mathcal{X}$ and the class of all span functions in $\mathcal{Y}$.

First of all, let us establish the following fact:

LEMMA 3.2. *The class of all t-invariant span functions in $\mathcal{X}$ is identical with the class of all span function extensions of t-invariant spread functions in $\mathcal{X}$.*

It is of course obvious that the spread function restriction of a t-invariant span function is t-invariant. Conversely, suppose f is a span function extension of a t-invariant spread function τ. Let g_τ be the core extension of τ. Then, by Lemma I.3.3 and (3.14), we have, for any $A\in\mathcal{A}$,

$$f(A)=g_\tau(f(A))=t^{-1}g_{\tau^{[t]}}(tf(A))\ . \tag{3.17}$$

This shows that f is a t-invariant function. The lemma is therefore proved.

Now let f be any particular t-invariant span function in $\mathcal{X}$. Let τ be the restriction of f to $\tilde{\mathcal{A}}$, and let g_τ be the core extension of τ. By Lemma I.3.3 and (3.15) we have

$$f(A)=f(g_\tau(A))=f(g_\tau(t^{-1}tA))\ . \tag{3.18}$$

that is,

$$f(A)=f(t^{-1}tA)\ , \qquad A\in\tilde{\mathcal{A}}\ . \tag{3.19}$$

We define a function $f^{[t]}$ on $\mathcal{B}$ to $\mathcal{B}$ as follows:

$$f^{[t]}(A')=tf(t^{-1}A')\ , \qquad A'\in\mathcal{B}\ . \tag{3.20}$$

This formula can be inverted. Indeed, since f is t-invariant it follows immediately that $f(t^{-1}A')=t^{-1}f^{[t]}(A')$. From this we have that, for any $A\in\mathcal{A}$, $f(t^{-1}tA)=t^{-1}f^{[t]}(tA)$. And so, by (3.19), we conclude that

$$f(A)=t^{-1}f^{[t]}(tA)\ , \qquad A\in\mathcal{A}\ . \tag{3.21}$$

Let us proceed to demonstrate now that $f^{[t]}$ is a span function in $\mathcal{Y}$. We must show that $f^{[t]}$ has the two characteristic properties (I.3.25) and (I.3.26) of Definition I.3.1. These properties stated for our present span function f are as follows: for every $A\in\mathcal{A}$ and every $B\in\mathcal{A}$,

$$A\subseteq f(A)\ , \tag{3.22}$$

$$A\subseteq f(B)\Rightarrow f(A)\subseteq f(B)\ . \tag{3.23}$$

Accordingly, we must show that, for all A' and B' in $\mathcal{B}$,

$$A'\subseteq f^{[t]}(A')\ , \tag{3.24}$$

and

$$A'\subseteq f^{[t]}(B')\Rightarrow f^{[t]}(A')\subseteq f^{[t]}(B')\ . \tag{3.25}$$

By (3.22) we have $t^{-1}A'\subseteq f(t^{-1}A')$, and therefore $A'=tt^{-1}A'\subseteq tf(t^{-1}A')$; thus, by (3.20), we have exactly (3.24). Now suppose that $A'\subseteq f^{[t]}(B')$. Then $t^{-1}A'\subseteq t^{-1}f^{[t]}(B')$, and therefore, by the isotone property of span functions (see Lemma I.3.1), $f(t^{-1}A')\subseteq f(t^{-1}f^{[t]}(B'))$. It follows, by (3.20), that $f^{[t]}(A')\subseteq tf(t^{-1}f^{[t]}(B'))$. But this last expression is equal to $tf(t^{-1}\cdot f^{[t]}(tt^{-1}B'))=tf(f(t^{-1}B'))=tf(t^{-1}B')$ (by the idempotency of f—see Lemma I.3.1) $=f^{[t]}(B')$ (by (3.20)). Thus, we have established (3.25), and therewith it is proved that $f^{[t]}$ is a span function.

Let A' be, in particular, a set in $\tilde{\mathcal{B}}$ and let $A\in\tilde{\mathcal{A}}$ be such that $tA=A'$. Then $f^{[t]}(A')=tf(t^{-1}A')=tf(t^{-1}tA)=tf(A)$ (by (3.19)) $=tg_\tau(A)=tg_\tau(t^{-1}tA)$ (by (3.15)) $=tg_\tau(t^{-1}A')=g_{\tau^{[t]}}(A')$ (by 3.16)) $=\tau^{[t]}(A')$. Thus, the span function $f^{[t]}$ is an extension of the spread function $\tau^{[t]}$ which corresponds, according to Theorem 3.2, to the spread function τ which is the restriction of f.

Now, similar arguments show that if we take any span function $f^{[t]}$ in $\mathcal{Y}$ and define f by (3.21), then f is a t-invariant span function in $\mathcal{X}$ and is an extension of the spread function in $\mathcal{X}$ that corresponds, under Theorem 3.2, to the spread function of $f^{[t]}$. We thereby complete the establishment of the result we were after, and we now state this result in the following theorem.

THEOREM 3.3. *There is a* 1-1 *correspondence between the class of all t-invariant span functions in* $\mathcal{X}$ *and the class of all span functions in* $\mathcal{Y}$. *This correspondence, for* f *in* $\mathcal{X}$ *and* $f^{[t]}$ *in* $\mathcal{Y}$, *is given by the reciprocal formulas* (3.20) *and* (3.21). *Under this correspondence, corresponding span functions have spread function restrictions that correspond under Theorem* 3.2.

The results of this section give us the means of constructing a betweenness relation in $\mathcal{Y}$ from a given t-invariant betweenness relation in $\mathcal{X}$, or vice-versa. We are going to be interested in the case of $\mathcal{X}$=the unit sphere, $\mathcal{O}_{\mathcal{K}}$, in our unitary space $\mathcal{K}$, and $\mathcal{Y}=\mathcal{K}^p$, the space of all 1-dimensional linear manifolds in $\mathcal{K}$. And our interest is especially directed to the construction of betweenness relations in $\mathcal{K}^p$ from given betweenness relations in $\mathcal{O}_{\mathcal{K}}$. We deal with this in the next section.

4. Derivation of line betweennesses from vector betweennesses

We shall employ the notation $[\![A]\!]$ to denote the closed linear manifold spanned by the set A of vectors in $\mathcal{K}$. We define the function t_0 on $\mathcal{O}_{\mathcal{K}}$ to $\mathcal{K}^p$ as follows:

$$t_0(x) \overset{\text{def.}}{=} [\![\{x\}]\!] , \qquad x \in \mathcal{O}_{\mathcal{K}} . \tag{4.1}$$

The results of the preceding section can now be brought to bear on this situation between the spaces $\mathcal{O}_{\mathcal{K}}$ and $\mathcal{K}^p$, and in particular—what interests us especially—we can obtain betweenness relations in $\mathcal{K}^p$ from t_0-invariant betweenness relations in $\mathcal{O}_{\mathcal{K}}$. Toward this goal, our first task will be to produce some t_0-invariant betweenness notions in $\mathcal{O}_{\mathcal{K}}$. We shall, in fact, do this by establishing a general principle for the construction of a t_0-invariant betweenness from a certain other kind of betweenness in $\mathcal{O}_{\mathcal{K}}$. And this principle—see Theorem 4.1 below—will be found readily applicable to provide us with specific cases of t_0-invariant betweenness.

We start with certain definitions and preliminary results. If A is any set of vectors in $\mathcal{K}$, we denote by A^- the reflection of A through the origin; that is,

$$A^- \overset{\text{def.}}{=} \{x \in \mathcal{K} \mid -x \in A\} . \tag{4.2}$$

Clearly, if $A \subseteq \mathcal{O}_{\mathcal{K}}$, then also $A^- \subseteq \mathcal{O}_{\mathcal{K}}$. With this we now make the following definitions:

DEFINITION 4.1. Let C denote the class of all subsets of $\mathcal{K}$, and let C' be a subclass of C that is closed under the reflection operation

$(\ )^-$. Let f be a function on C' to C. We say that the function f is *symmetric* if

$$f(A^-)=f(A)^- . \tag{4.3}$$

And we say that f is *reflective* if

$$f(A)^-=f(A) . \tag{4.4}$$

We can prove immediately

LEMMA 4.1. *If a reflective function f has, in addition, the two properties*

$$A\subseteq f(A) , \qquad A\subseteq f(B)\Rightarrow f(A)\subseteq f(B) , \tag{4.5}$$

then it is necessarily symmetric. Thus, reflective spread functions and reflective span functions are symmetric.

If the domain, C', of f is closed under unions then reflectivity and (4.5) *imply furthermore that*

$$f(A\cup A^-)=f(A) . \tag{4.6}$$

In particular, therefore, every reflective span function has this property.

To prove this lemma, observe that the first relation in (4.5) and reflectivity imply, for any $A\in C'$, that $A^-\subseteq f(A)^-=f(A)$. Hence, by an application of the second relation in (4.5), we have $f(A^-)\subseteq f(A)$. Replacing A by A^- we get the reverse inclusion as well; and therefore there results the equality $f(A^-)=f(A)$. Again applying the reflectivity property, we have our assertion of symmetry.

To prove (4.6), we note that the first relation of (4.5) together with the already-established inclusion, $A^-\subseteq f(A)$, implies that $A\cup A^-\subseteq f(A)$. Therefore, the second relation of (4.5) gives the inclusion $f(A\cup A^-)\subseteq f(A)$. On the other hand, we have $A\subseteq A\cup A^-\subseteq f(A\cup A^-)$, and therefore $f(A)\subseteq f(A\cup A^-)$. Hence, (4.6) results.

The following fact will be needed later:

LEMMA 4.2. *The core extension of a symmetric spread function is symmetric.*

This statement is true whatever be the subset of $\mathcal{K}$ (closed under reflection) in which the spread function is defined. However, we are interested in $\mathcal{O}_{\mathcal{K}}$ in particular, and we shall exhibit the steps of the proof in that case. Thus, let τ be a symmetric spread function in $\mathcal{O}_{\mathcal{K}}$, and let g_τ be its core extension. Let $C_{\mathcal{O}}$ denote the class of all subsets of $\mathcal{O}_{\mathcal{K}}$, and $\tilde{C}_{\mathcal{O}}$ denote the class of all at-most-two-point subsets of $\mathcal{O}_{\mathcal{K}}$. Then, for any $A\in C_{\mathcal{O}}$, we see that

$$(4.7)\qquad \tau^{(1)}(A^-)=\bigcup_{\substack{C\in\tilde{\mathcal{C}}_{\mathcal{O}}\\ C\subseteq A^-}}\tau(C)=\bigcup_{\substack{C\in\tilde{\mathcal{C}}_{\mathcal{O}}\\ C\subseteq A}}\tau(C^-)=\bigcup_{\substack{C\in\tilde{\mathcal{C}}_{\mathcal{O}}\\ C\subseteq A}}(\tau(C))^-$$
$$=(\bigcup_{\substack{C\in\tilde{\mathcal{C}}_{\mathcal{O}}\\ C\subseteq A}}\tau(C))^-=(\tau^{(1)}(A))^- .$$

From this result it then follows straightforwardly that $g_\tau(A^-)=(g_\tau(A))^-$ for any $A\in\mathcal{C}_{\mathcal{O}}$.

The notion of reflectivity is actually an alternative formulation of t_0-invariance:

LEMMA 4.3. *Let f be any function defined on a suitable subclass of $\mathcal{C}_{\mathcal{O}}$ to $\mathcal{C}_{\mathcal{O}}$. Then f is t_0-invariant if and only if it is reflective. In particular, a spread function or a span function in $\mathcal{O}_{\mathcal{K}}$ is t_0-invariant if and only if it is reflective.*

The proof of this lemma requires only showing that for any $B\subseteq\mathcal{O}_{\mathcal{K}}$ the condition $t_0^{-1}t_0B=B$ is equivalent to the condition $B^-=B$. But the first of these conditions is readily seen to be the condition $B\cup B^-=B$, and this is easily seen to be equivalent to $B^-=B$.

We now introduce the following definition:

DEFINITION 4.2. Let $\mathcal{C}$ be as above and $\mathcal{C}'$ be any subclass of $\mathcal{C}$; and let f be a function on $\mathcal{C}'$ to $\mathcal{C}$. Let us say that a non-empty, non-$\{\theta\}$ set $A\subseteq\mathcal{K}$ is an *acute-angle set* if

$$(4.8)\qquad \inf_{\substack{x,\,y\in A\\ x,\,y\neq\theta}}\frac{(x,y)}{\|x\|\cdot\|y\|}\geqq 0\ ;$$

and let us say that A is a *strictly acute-angle set* if the inequality definitely holds in (4.8). More generally, let us say that A is a *κ-limited acute-angle set* if ($\kappa\in[0,1]$ and)

$$(4.9)\qquad \inf_{\substack{x,\,y\in A\\ x,\,y\neq\theta}}\frac{(x,y)}{\|x\|\cdot\|y\|}\geqq \kappa\ ;$$

and say that A is a *strictly κ-limited acute-angle set* if the inequality definitely holds in (4.9).

Then, f will be called a (*strictly*) *acute-angle function* resp. a (*strictly*) *κ-limited acute-angle function* if it preserves (strict) acuteness resp. (strict) κ-limited acuteness.

We are now in a position to formulate the theorem we announced earlier, which will provide a source of t_0-invariant betweennesses.

THEOREM 4.1. *Let τ be a symmetric spread function in $\mathcal{O}_{\mathcal{K}}$, and let it furthermore be either a strictly acute-angle function or, for some $\kappa>0$, a (strictly) κ-limited acute-angle function. Let g_τ denote the core extension of τ. For two elements, x and y, in $\mathcal{O}_{\mathcal{K}}$ we set*

$$y_x = \begin{cases} y, & \text{if } (x, y) \geqq 0, \\ -y, & \text{if } (x, y) < 0. \end{cases} \tag{4.10}$$

Let J denote the interval $[\kappa, 1]$ or $(\kappa, 1]$ according as τ is a κ-limited or a strictly κ-limited acute-angle function. (If $\kappa=0$, J is necessarily $(0, 1]$.) Then, the function Δ, on $\tilde{C}_{\mathcal{O}}$ to $C_{\mathcal{O}}$, defined by

$$\Delta(\emptyset) \overset{\text{def.}}{=} \tau(\emptyset) = \emptyset \tag{4.11}$$

and

$$\Delta(\{x, y\}) \overset{\text{def.}}{=} \begin{cases} \tau(\{x, y_x\}) \cup \tau(\{-x, -y_x\}), & \text{if } (x, y_x) \in J, \\ g_\tau(\{x, -x, y, -y\}), & \text{if } (x, y_x) \notin J \end{cases} \tag{4.12}$$

is a t_0-invariant spread function in $\mathcal{O}_{\mathcal{K}}$.

Let us notice, first of all, that the pair of sets, $\{x, y_x\}$ and $\{-x, -y_x\}$, is identical with the pair of sets, $\{x_y, y\}$ and $\{-x_y, -y\}$, when $(x, y) \neq 0$; and therefore that the definition (4.12) truly gives a function of the *set* $\{x, y\}$.

Let us observe, next, that there are no alternatives to (4.11). That is, by virtue of the assumed properties of τ, a non-$\emptyset$ possibility for the value of $\tau(\emptyset)$ does not exist. And because of those properties of τ, the same is true for $\Delta(\emptyset)$. Let us see that these statements are true. For any particular $x \in \mathcal{O}_{\mathcal{K}}$, $\{x\}$ is a (strictly) κ-limited acute-angle set (whatever be κ), and therefore $\tau(\{x\})$ is a (strictly) κ-limited acute-angle set. The same is true of $\tau(\{-x\}) = (\tau(\{x\}))^-$. Since these two subsets of $\mathcal{O}_{\mathcal{K}}$ are reflections of each other and are strictly acute-angle sets, they are therefore necessarily disjoint: $\tau(\{x\}) \cap \tau(\{-x\}) = \emptyset$. It follows that the intersection of the sets $\tau(A)$ for all $A \in \tilde{C}_{\mathcal{O}}^{\cdot}$ (=class of all 1- and 2-point subsets of $\mathcal{O}_{\mathcal{K}}$) is empty, and thus $\emptyset$ is the only possible value for $\tau(\emptyset)$. To see that the same is true for Δ, let us suppose the contrary: suppose there is a $z \in \mathcal{O}_{\mathcal{K}}$ that belongs to every $\Delta(\{x, y\})$. We can then choose in particular an $x \perp z$, and so, on taking $y=x$, we have $z \in \Delta(\{x\})$. Therefore $z \in \tau(\{x\})$ or $z \in \tau(\{-x\})$. If the first of these is the case then since also $x \in \tau(\{x\})$ we have $\{x, z\} \subseteq \tau(\{x\})$; in the second case we have $\{-x, z\} \subseteq \tau(\{-x\})$. But in either of these two cases we have the contradictory assertion of two mutually orthogonal elements being contained in a strictly acute-angle set. Hence the assumption of the non-emptiness of $\bigcap_{A \in \tilde{C}_{\mathcal{O}}^{\cdot}} \Delta(A)$ is false. It follows, therefore—if indeed Δ is a spread function over $\tilde{C}^{\cdot}$—that there is only one possible value for $\Delta(\emptyset)$, namely $\emptyset$.

We now proceed to show that Δ is a spread function over $\tilde{C}^{\cdot}$. The fact that τ is a spread function means that we have, for all points x, y, x_1, y_1, $\in \mathcal{O}_{\mathcal{K}}$,

$$\{x, y\} \subseteq \tau(\{x, y\}) \tag{4.13}$$

and

$$\{x_1, y_1\} \subseteq \tau(\{x, y\}) \Rightarrow \tau(\{x_1, y_1\}) \subseteq \tau(\{x, y\}) \ . \tag{4.14}$$

We must exhibit the same properties of Δ, that is,

$$\{x, y\} \subseteq \Delta(\{x, y\}) \ , \tag{4.15}$$

$$\{x_1, y_1\} \subseteq \Delta(\{x, y\}) \Rightarrow \Delta(\{x_1, y_1\}) \subseteq \Delta(\{x, y\}) \ . \tag{4.16}$$

To establish (4.15), we note first that $x \in \tau(\{x, y_x\})$ and that either $y \in \tau(\{x, y_x\})$ or $y \in \tau(\{-x, -y_x\})$. Therefore we have

$$\{x, y\} \subseteq \tau(\{x, y_x\}) \cup \tau(\{-x, -y_x\}) \ . \tag{4.17}$$

If $(x, y_x) \in J$ then (4.17) is exactly the statement (4.15). If $(x, y_x) \notin J$ then we apply the further fact that

$$\tau(\{x, y_x\}) \cup \tau(\{-x, -y_x\}) \subseteq g_\tau(\{x, -x, y, -y\}) \ , \tag{4.18}$$

and this together with (4.17) again gives us (4.15).

We now prove (4.16). We first treat the case of $(x, y_x) \in J$. In this case the left side of the implication (4.16) is

$$\{x_1, y_1\} \subseteq \tau(\{x, y_x\}) \cup \tau(\{-x, -y_x\}) \ . \tag{4.19}$$

The two sets $\tau(\{x, y_x\})$ and $\tau(\{-x, -y_x\})$ are reflections of each other and are, in the present case of $(x, y_x) \in J$, (strictly) κ-limited acute-angle sets. It is a consequence of these facts that also $-x_1$ and $-y_1$ belong to the right-hand side of (4.19), and that the two particular elements x_1 and $(y_1)_{x_1}$ (which have a non-negative inner product) cannot be distributed one in each of the two sets $\tau(\{x, y_x\})$ and $\tau(\{-x, -y_x\})$: both elements belong to the first of these two sets, or both belong to the second of the two. If they both belong to the first set, then we have

$$\{x_1, (y_1)_{x_1}\} \subseteq \tau(\{x, y_x\}) \ , \tag{4.20}$$

and therefore also, by symmetry,

$$\{-x_1, -(y_1)_{x_1}\} \subseteq \tau(\{-x, -y_x\}) . \tag{4.21}$$

By (4.14) we get, from these two inclusions, the inclusions

$$\begin{aligned} &\tau(\{x_1, (y_1)_{x_1}\}) \subseteq \tau(\{x, y_x\}) , \\ &\tau(\{-x_1, -(y_1)_{x_1}\}) \subseteq \tau(\{-x, -y_x\}) , \end{aligned} \tag{4.22}$$

and from these we get

$$\tau(\{x_1, (y_1)_{x_1}\}) \cup \tau(\{-x_1, -(y_1)_{x_1}\}) \subseteq \Delta(\{x, y\}) . \tag{4.23}$$

Now, we have already seen, above, that x_1 and $(y_1)_{x_1}$ are both elements of a (strictly) κ-limited acute-angle set; that is, $(x_1, (y_1)_{x_1}) \in J$. Therefore the left-hand side of (4.23) is exactly $\Delta(\{x_1, y_1\})$; and thus (4.23) is the conclusion we had wished to reach, namely, the right side of the implication (4.16). This implication is now established for the case of $(x, y_x) \in J$.

Now suppose $(x, y_x) \notin J$. Then the left side of (4.16) is

$$\{x_1, y_1\} \subseteq g_\tau(\{x, -x, y, -y\}) . \tag{4.24}$$

Since g_τ is symmetric (see Lemma 4.2) we have

$$\begin{aligned} (g_\tau(\{x, -x, y, -y\}))^- &= g_\tau(\{x, -x, y, -y\}^-) \\ &= g_\tau(\{x, -x, y, -y\}) , \end{aligned} \tag{4.25}$$

and therefore we get from (4.24) that

$$\{x_1, -x_1, y_1, -y_1\} \subseteq g_\tau(\{x, -x, y, -y\}) . \tag{4.26}$$

From this it follows that

$$\tau(\{x_1, (y_1)_{x_1}\}) \equiv g_\tau(\{x_1, (y_1)_{x_1}\}) \subseteq g_\tau(\{x, -x, y, -y\}) \tag{4.27}$$

and

$$\tau(\{-x_1, -(y_1)_{x_1}\}) \equiv g_\tau(\{-x_1, -(y_1)_{x_1}\}) \subseteq g_\tau(\{x, -x, y, -y\}) , \tag{4.28}$$

and also that

$$g_\tau(\{x_1, -x_1, y_1, -y_1\}) \subseteq g_\tau(\{x, -x, y, -y\}) . \tag{4.29}$$

In the case of $(x_1, (y_1)_{x_1}) \in J$, the inclusions (4.27) and (4.28) combine to give us the statement

$$\Delta(\{x_1, y_1\}) \subseteq \Delta(\{x, y\}) . \tag{4.30}$$

In the case of $(x_1, (y_1)_{x_1}) \notin J$, (4.29) is the statement (4.30). Thus, the implication (4.16) is now fully established also in the case of $(x, y_x) \notin J$.

We have now proved that Δ is a spread function. It remains only to prove that it is t_0-invariant. But it is clear from (4.11) and (4.12)—

with the help of (4.25)—that Δ is a reflective function. Therefore, by Lemma 4.3, it is t_0-invariant. This completes the proof of Theorem 4.1.

We have said that Theorem 4.1 would be a ready means of providing us with t_0-invariant betweenness relations, and hence with betweenness relations in $\mathcal{K}^p$. We may see now that this is so. Let us look at $\mathfrak{V}_0^v$-betweenness in $\mathcal{O}_{\mathcal{K}}$ (see Definition I.5.2). It is intuitively clear that its spread function is a symmetric and strictly acute-angle function and therefore the construction of Theorem 4.1, with $J=(0, 1]$, can be applied to it. It is not difficult to see what the t_0-invariant sets $\Delta(\{x, y\})$ look like in this case, and it is then not difficult either to see that the betweenness relation in $\mathcal{K}^p$ corresponding to that of Δ in $\mathcal{O}_{\mathcal{K}}$, according to Theorem 3.1, is nothing other than our old familiar $\mathfrak{V}_0$-betweenness (see Definition I.5.3). Thus, we have come upon another way of generating $\mathfrak{V}_0$-betweenness in the real case.

But now let us look at another example, which will give us a new particular notion of betweenness in $\mathcal{K}^p$. Let $\nu>0$ and let us consider $\mathfrak{V}_\nu^v$-betweenness in $\mathcal{O}_{\mathcal{K}}$. We may appeal to Theorem I.6.1 to see that τ_ν, the spread function of $\mathfrak{V}_\nu^v$, is a symmetric function. And Theorem 2.2 above tells us further that τ_ν is a $(2\nu^2-1)$-limited acute-angle function. We can therefore apply Theorem 4.1, with $J=[2\nu^2-1, 1]$. Let us confine ourselves, for the following deliberations, to the more interesting case of $\dim \mathcal{K} \geqq 3$. Then—letting Δ_ν denote the t_0-invariant spread function derived from τ_ν according to Theorem 4.1—we find from Theorem I.6.1 that

$$\text{(4.31)} \qquad \Delta_\nu(\{x, y\})=\mathcal{O}_{\mathcal{K}} \qquad \textit{if } |(x, y)|<2\nu^2-1\,,$$

(—observe that $|(x, y)|\equiv(x, y_x)$—) and that for $|(x, y)|\geqq 2\nu^2-1$ the set $\Delta_\nu(\{x, y\})$ consists of all those $z \in \mathcal{O}_{\mathcal{K}}$ such that either

$$\text{(4.32)} \qquad \left(z, \frac{\nu}{1+|\mathfrak{t}|}(x+y_x)+w\right)\geqq \nu \qquad \textit{for all } w\perp x, y \textit{ with } \|w\|=\rho_+$$

or

$$\text{(4.33)} \qquad \left(z, \frac{\nu}{1+|\mathfrak{t}|}(-x-y_x)+w\right)\geqq \nu \qquad \textit{for all } w\perp x, y \textit{ with } \|w\|=\rho_+ \,.$$

We are continuing to employ, for conciseness of expression, symbols of the type previously introduced; specifically,

$$\text{(4.34)} \qquad \mathfrak{t} \overset{\text{def.}}{=} (x, y)\,, \qquad \rho_+ \overset{\text{def.}}{=} \sqrt{1-\frac{2\nu^2}{1+|\mathfrak{t}|}}\,.$$

Let us examine (4.32) more closely. By Theorem 2.2, if z satisfies (4.32) then $(z, x) \geqq |\xi|$ and $(y_x, z) \geqq |\xi|$. It follows then that (4.32) is equivalent to this statement:

$$\frac{\nu}{1+|\xi|}(|(z, x)|+|(y_x, z)|)+(z, w) \geqq \nu \tag{4.35}$$

$$\text{for all } w \perp x, y \text{ with } \|w\| = \rho_+, \quad \text{and } (z, x) \geqq |\xi|,\ (y_x, z) \geqq |\xi| .$$

Notice next that

$$[(y_x, z) \geqq |\xi|] \Longleftrightarrow [\xi \cdot (y, z) \geqq 0,\ |(y, z)| \geqq |\xi|] \tag{4.36}$$

and that $|(y_x, z)| \equiv |(y, z)|$. Therefore (4.35) can be restated still further as

$$\frac{\nu}{1+|\xi|}(|(z, x)|+|(y, z)|)+(z, w) \geqq \nu \tag{4.37}$$

$$\text{for all } w \perp x, y \text{ with } \|w\| = \rho_+, \text{ and: } (z, x) \geqq |\xi|,\ |(y, z)| \geqq |\xi|,\ \xi \cdot (y, z) \geqq 0 .$$

The statement (4.33) obtains from (4.32) by replacing x and y by $-x$ and $-y$, respectively. Therefore we get a statement equivalent to (4.33) by making these replacements in (4.37). The result of this is that only the first and third of the last three inequalities change their form; they become

$$(z, x) \leqq -|\xi| , \qquad \xi \cdot (y, z) \leqq 0 . \tag{4.38}$$

Now, the disjunction of the mentioned pair of inequalities in (4.37) with (4.38) is readily seen to be

$$|(z, x)| \geqq |\xi| , \qquad (z, x) \cdot \xi \cdot (y, z) \geqq 0 . \tag{4.39}$$

And so we can state: For $|(x, y)| \geqq 2\nu^2 - 1$, the set $\Delta_\nu(\{x, y\})$ consists of all those $z \in \mathcal{O}_{\mathcal{K}}$ such that the following conditions are satisfied:

$$\text{(4.40)} \quad \begin{cases} \text{i)} \ \dfrac{\nu}{1+|\xi|}(|(z, x)|+|(y, z)|)+(z, w) \geqq \nu \\ \qquad\qquad \text{for all } w \perp x, y \text{ with } \|w\| = \rho_+ , \\ \text{ii)} \ |(z, x)| \geqq |(x, y)| \\ \text{iii)} \ |(y, z)| \geqq |(x, y)| \\ \text{iv)} \ (z, x)(x, y)(y, z) \geqq 0 . \end{cases}$$

We now have Δ_ν completely determined. Let Δ_ν^p be the spread function in $\mathcal{K}^p$ corresponding to Δ_ν according to Theorem 3.2. An examination of the correspondence presented by this theorem shows that—in particular in the case at hand—the spread function Δ_ν^p can be enunciated as follows: Let L, M and N be elements of $\mathcal{K}^p$, and let x, y and z be any particular unit vectors in L, M and N, respectively; then, $N \in \Delta_\nu^p(\{L, M\})$ if and only if $z \in \Delta_\nu(\{x, y\})$. By this means, then, we can formally present this new notion of betweenness in $\mathcal{K}^p$. We do so in the following definition, giving it the name $\mathcal{U}_\nu$-betweenness, corresponding to the $\mathcal{U}_0$-betweenness of Definition I.5.3 which, as we have seen above, derives from $\mathcal{U}_0^v$-betweenness in the same way that our present new notion derives from $\mathcal{U}_\nu^v$-betweenness. Before presenting the definition, however, let us make an observation. The full list of four conditions in (4.40) has the advantage of displaying the close parallel that exists between $\mathcal{U}_\nu$-betweenness and $\mathcal{U}_0$-betweenness, as seen by Definition I.5.3 or, still better, by Lemma 2.2 in [1]. But in the present case of $\mathcal{U}_\nu$-betweenness a special situation obtains; namely, ii) and iii) of (4.40) are consequences of i). We can see this by first replacing i) by the following condition which is clearly equivalent to it:

$$|(z, x)|+|(y, z)| \geqq (1+|(x, y)|)\left(1+\frac{1}{\nu}|(z, w)|\right) \quad \textit{for all } w \perp x, y \textit{ with } \|w\| \leqq \rho_+ . \tag{4.41}$$

If we put $w=\theta$ in this inequality we get

$$|(z, x)|+|(y, z)| \geqq 1+|(x, y)| , \tag{4.42}$$

and from this it follows that neither of the terms on the left can be $<|(x, y)|$. Taking this result into account, our definition is then as follows:

DEFINITION 4.3. Let $\nu \in (0, 1]$. Let L, M and N be elements of $\mathcal{K}^p$, and let x, y and z be any particular unit vectors in L, M and N, respectively.

If $|(x, y)|<2\nu^2-1$, we say of every N that it is *$\mathcal{U}_\nu$-between* L and M.

If $|(x, y)| \geqq 2\nu^2-1$, we say that N is $\mathcal{U}_\nu$-between L and M if the following conditions are satisfied:

$$\begin{cases} \text{i)} \quad |(z, x)|+|(y, z)| \geqq (1+|(x, y)|)\left(1+\frac{1}{\nu}|(z, w)|\right) \\ \qquad\qquad \textit{for all } w \perp x, y \textit{ with } \|w\| \leqq \rho_+ , \\ \text{ii)} \quad (z, x)(x, y)(y, z) \geqq 0 . \end{cases} \tag{4.43}$$

We have not stipulated in this definition that $\dim \mathcal{K} \geqq 3$, for the reason that the definition applies, in fact, to $\dim \mathcal{K}<3$ as well. It is not difficult to see that when $\dim \mathcal{K}=1$ or 2 the conditions (4.43) reduce to the conditions (I.5.18), thus asserting that in these dimensionality cases $\mathcal{V}_\nu$-betweenness reduced to $\mathcal{V}_0$-betweenness. And this assertion is correct, as we see by the statement (1) in Theorem I.6.1.

UNIVERSITY OF CALIFORNIA
THE INSTITUTE OF STATISTICAL MATHEMATICS

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