Ann. Inst. Statist. Math.
30 (1978), Part A, 435-442

A FINELY TUNED CONTINUITY CORRECTION

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(Received May 15, 1978)

Abstract

The role of the continuity correction of 1/2, when approximating discrete binomial probabilities with normal probabilities, is examined. It is shown that a substantial improvement is available, one that involves very little more computational effort (it can easily be performed on a pocket calculator), and gives big gains in accuracy.

1. Introduction

The problem is almost a classical one; probabilities from the discrete binomial distribution are to be approximated with the continuous normal distribution. For example, the tail probability $\sum_{m=k}^{N}\binom{N}{m}p^m(1-p)^{N-m}$ is very simply approximated by $1-\Phi\{(k-1/2-Np)/\sqrt{Np(1-p)}\}$. The role of the value 1/2 in the argument of Φ, is that of a correction term to allow a function defined only on the integers, to be approximated by a continuous function (Yates [9]). It is in improving this *continuity correction* of 1/2, that is the aim of this paper.

In order to finely tune the continuity correction, the error term is expanded in powers of $N^{-1/2}$. The correction which forces the coefficient of $N^{-1/2}$ to be zero, is essentially the Gram-Charlier approximation in disguise. In its continuity correction form however, it performs extremely poorly when approximating tail probabilities. The reason may be found in Feller [3], where the same objection is made for the classical continuity correction of 1/2. But, going one step further, and also forcing the coefficient of N^{-1} to be zero gives a correction which in a large region of the (N, p) plane, $N\in Z^+$, $p\in(0, 1)$, outperforms the extremely accurate Camp-Paulson approximation (Camp [1]; Johnson and Kotz [5], Chapter 3, Section 8).

A similar approximation is clearly possible for the distribution function of any integer-valued random variable, provided that conditions exist under which the individual probabilities can be simultaneously replaced by ordinates of a normal density function. Cox [2] discusses the continuity correction, but he attacks a different problem than is being considered here.

Define,

$$B(N, k, p) \equiv \sum_{m=0}^{k} \binom{N}{m} p^m q^{N-m}, \qquad \text{where } q=1-p,$$

$$T(N, k, p) \equiv \sum_{m=k}^{N} \binom{N}{m} p^m q^{N-m},$$

$$\phi(x) \equiv (2\pi)^{-1/2} e^{-x^2/2},$$

$$\Phi(x) \equiv \int_{-\infty}^{x} \phi(y)dy,$$

$$F(x) \equiv \int_{x}^{\infty} \phi(y)dy.$$

The central limit theorem states that if a random variable X is binomially distributed with parameters N and p, then $\mathrm{P}\{Z \geqq (k-Np)(Npq)^{-1/2}\} \simeq \mathrm{P}\{X \geqq k\}$, where Z is a standard normal; i.e. $F((k-Np)(Npq)^{-1/2}) \simeq T(N, k, p)$. Unfortunately one could equally claim that $\mathrm{P}\{Z > (k-1-Np)(Npq)^{-1/2}\} \simeq \mathrm{P}\{X > k-1\}$; i.e. $F((k-1-Np)(Npq)^{-1/2}) \simeq T(N, k, p)$. One way out of this dilemma is to use a *continuity correction* c in the following way

$$F((k-Np-c)(Npq)^{-1/2}) \simeq T(N, k, p). \tag{1.1}$$

The value $c=1/2$ is the most common correction chosen, but why? By way of illustration of the superiority of the finely tuned correction of (2.13), (2.14), consider the example in Feller [3] of $N=500$, $p=.1$. Now $\sum_{m=50}^{55} \binom{500}{m} (.1)^m (.9)^{500-m} = .317573$; the normal approximation with continuity correction 1/2 is $\Phi(.8199) - \Phi(-.0745) \simeq .3238$ (obtained by rounding arguments to three decimal places, and interpolating in the normal tables); while the finely tuned correction gives $\Phi(.8266) - \Phi(-.0547) \simeq .3177$, showing an error of only .00013.

Section 2 gives the mathematical reasoning behind finding a good continuity correction, culminating in the finely tuned approximations of (2.13) and (2.14). Section 3 computes tables of errors, in particular showing the regions where our continuity correction approximation is superior. Finally, conclusions and recommendations are made in Section 4.

2. The best continuity correction

Quite simply, the problem may be posed in the following way:

We wish to approximate the upper [lower] tail of the binomial probability distribution, $T(N, k, p)$ $[B(N, k, p)]$, by $F((k-Np-c)(Npq)^{-1/2})$ $[\Phi((k-Np+d)(Npq)^{-1/2})]$, for some judicious choice of c $[d]$. When this problem is solved we are able to approximate any binomial probability. Typically the value of c $[d]$ chosen is $c=1/2$ $[d=1/2]$. It will become clear that a better choice for c $[d]$ is available; one that involves a little more computation, *but* is correct to a higher order of magnitude.

Let us now expand the individual binomial probability, $\binom{N}{m}p^m q^{N-m}$, using Stirling's formula,

$$\log n! = \frac{1}{2}\log(2\pi n) + n\log n - n + \frac{1}{12n} + O(n^{-3})\ . \tag{2.1}$$

Taking logarithms, and putting

$$\delta_b \equiv (b-Np)/(Npq)^{1/2}\ , \qquad b \in \boldsymbol{R}\ , \tag{2.2}$$

we obtain,

$$\begin{aligned}
&\log N! - \log m! - \log(N-m)! + m\log p + (N-m)\log q \\
&= -\frac{1}{2}\log 2\pi - \frac{1}{2}\log p - \frac{1}{2}\log q - \frac{1}{2}\log N \\
&\quad -(Np+\delta_m\sqrt{Npq}+1/2)\log(1+\delta_m\sqrt{q/Np}) \\
&\quad -(Nq-\delta_m\sqrt{Npq}+1/2)\log(1-\delta_m\sqrt{p/Nq}) + R_N\ ,
\end{aligned}$$

where $R_N \equiv (1/12N)\{1-(p+\delta_m\sqrt{pq/N})^{-1}-(q-\delta_m\sqrt{pq/N})^{-1}\}+O(N^{-3})$. Use of the Taylor series expansion, $\log(1+x)=x-x^2/2+x^3/3+\cdots$, gives

$$\begin{aligned}
-\log(\sqrt{2\pi Npq}) - \frac{\delta_m^2}{2}\{1+S_N\} + \log(1+\delta_m\sqrt{q/Np})^{-1/2} \\
+\log(1-\delta_m\sqrt{p/Nq})^{-1/2} + R_N\ ,
\end{aligned}$$

where $S_N \equiv -(\delta_m/3)((q^2-p^2)/\sqrt{Npq}) + (\delta_m^2/6)((q^3+p^3)/Npq) + O(N^{-3/2})$. Hence

$$\begin{aligned}
\binom{N}{m}p^m q^{N-m} = \frac{1}{\sqrt{2\pi Npq}}\{\exp(-\delta_m^2/2)\}^{1+S_N}(1+\delta_m\sqrt{q/Np})^{-1/2} \\
\cdot (1-\delta_m\sqrt{p/Nq})^{-1/2}e^{R_N}\ .
\end{aligned}$$

Now $a^{1+\varepsilon} = a + \varepsilon a\log a + (\varepsilon^2/2)a\log^2 a + (\varepsilon^3/6)a\log^3 a + \cdots$, and so the above equals,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi Npq}}\{&\exp(-\delta_m^2/2) + S_N(\exp(-\delta_m^2/2))(-\delta_m^2/2) \\
&+ S_N^2(\exp(-\delta_m^2/2))(\delta_m^4/8) + O(N^{-3/2})\}
\end{aligned} \tag{2.3}$$

$$\cdot\left\{1+\frac{\delta_m}{2}\frac{(p-q)}{\sqrt{Npq}}+\frac{\delta_m^2}{8}\frac{(3p^2-2pq+3q^2)}{Npq}+\frac{(pq-1)}{12Npq}+O(N^{-3/2})\right\} .$$

If we keep only terms of $O(N^{-1/2})$ and lower, then

$$\binom{N}{m}p^m q^{N-m}=\frac{\phi(\delta_m)}{\sqrt{Npq}}\left\{1+\frac{\delta_m(q-p)(\delta_m^2-3)}{6\sqrt{Npq}}+O(N^{-1})\right\} . \tag{2.4}$$

We will use (2.4) to derive an initial continuity correction and (2.3) to fine-tune our result into its final form.

THEOREM 1. *The choice of* $c=1/2+(q-p)(\delta_{k-1/2}^2-1)/6$ *in* (1.1), *gives an error of approximation of* $O(N^{-1})$.

PROOF. Write $T(N, k, p)-F((k-Np-c)(Npq)^{-1/2})$ as,

$$\sum_{m=k}^{N}\left[\binom{N}{m}p^m q^{N-m}-\{\Phi(\delta_{m+1/2})-\Phi(\delta_{m-1/2})\}\right]$$
$$-\{\Phi(\delta_{k-1/2})-\Phi(\delta_{k-c})\}-\{1-\Phi(\delta_{N+1/2})\} ,$$

where δ_b, for all real b, is given by (2.2). Now $\Phi(\delta_{m+1/2})-\Phi(\delta_{m-1/2})$ is, by integrating the Taylor series expansion of $\phi(x)$,

$$\frac{\phi(\delta_m)}{\sqrt{Npq}}+\frac{\phi''(\delta_m)}{24(\sqrt{Npq})^3}+O(N^{-5/2})=\frac{\phi(\delta_m)}{\sqrt{Npq}}\left\{1+\frac{(\delta_m^2-1)}{24Npq}+O(N^{-2})\right\} . \tag{2.5}$$

Hence,

$$\begin{aligned} &T(N, k, p)-F(\delta_{k-c}) \\ &=\sum_{m=k}^{N}\frac{\phi(\delta_m)}{\sqrt{Npq}}\left\{\frac{\delta_m(q-p)(\delta_m^2-3)}{6\sqrt{Npq}}+O(N^{-1})\right\} \\ &-\left\{\frac{\phi(\delta_k)}{\sqrt{Npq}}(c-1/2)-\frac{\phi'(\delta_k)}{2Npq}(c^2-1/4)\right\}+O(N^{-3/2}) . \end{aligned} \tag{2.6}$$

Approximating the sum by an integral with respect to δ, it becomes,

$$(q-p)\int_{\delta_{k-1/2}}^{\delta_{N+1/2}}\phi(\delta)\delta(\delta^2-3)d\delta/6\sqrt{Npq}+O(N^{-3/2})$$
$$=(q-p)\phi(\delta_{k-1/2})(\delta_{k-1/2}^2-1)/6\sqrt{Npq}+O(N^{-3/2}) .$$

Therefore, finally,

$$\begin{aligned} &T(N, k, p)-F(\delta_{k-c}) \\ &=\frac{\phi(\delta_{k-1/2})}{\sqrt{Npq}}\{(q-p)(\delta_{k-1/2}^2-1)/6\}-\frac{\phi(\delta_k)}{\sqrt{Npq}}\{c-1/2\}+O(N^{-1}) \\ &=\frac{\phi(\delta_{k-1/2})}{\sqrt{Npq}}\{(q-p)(\delta_{k-1/2}^2-1)/6-(c-1/2)\}+O(N^{-1}) . \end{aligned} \tag{2.7}$$

Requiring the error term to be $O(N^{-1})$ forces us to then choose

$$c=1/2+(q-p)(\delta_{k-1/2}^2-1)/6 . \tag{2.8}$$

Thus we use $F(\delta_{k-c})=F(\delta_{k-1/2}-(q-p)(\delta_{k-1/2}^2-1)/6\sqrt{Npq})$ to approximate $T(N, k, p)$. A similar analysis with the lower tail leads us to

$$d=1/2-(q-p)(\delta_{k+1/2}^2-1)/6 , \tag{2.9}$$

and hence to use $\Phi(\delta_{k+d})=\Phi(\delta_{k+1/2}-(q-p)(\delta_{k+1/2}^2-1)/6\sqrt{Npq})$ to approximate $B(N, k, p)$.

The continuity correction approximation,

$$F(\delta_{k-1/2}-(q-p)(\delta_{k-1/2}^2-1)/6\sqrt{Npq}) , \tag{2.10}$$

possesses obvious similarities to the normal Gram-Charlier approximation,

$$G(N, k, p)\equiv 1-\{\Phi(\delta_{k-1/2})-(q-p)(\delta_{k-1/2}^2-1)\phi(\delta_{k-1/2})/6\sqrt{Npq}) . \tag{2.11}$$

They are equivalent up to $O(N^{-1/2})$, and hence the error in using (2.11) is also $O(N^{-1})$. A study by Raff [8] has shown that although (2.11) is generally more accurate than $F(\delta_{k-1/2})$, there are other approximations available which require just as much computation, but are much more accurate, such as the Poisson Gram-Charlier for small p, and the Camp-Paulson for larger p.

We will now fine tune our continuity correction in order to compete with the Camp-Paulson approximation.

THEOREM 2. *The choice of*

$$c=1/2+(q-p)(\delta_{k-1/2}^2-1)/6+\{\delta_{k-1/2}^3(-5/72+7pq/36) + \delta_{k-1/2}(1/36-pq/36)\}/\sqrt{Npq}$$

in (1.1), *gives an error of approximation of* $O(N^{-3/2})$.

PROOF. If we keep all terms of $O(N^{-1})$ and lower, then from (2.3), (2.5) and (2.6), we find that upon using (2.8),

$$\begin{aligned} T(N, k, p)-F(\delta_{k-c}) &= \sum_{m=k}^{N} \frac{\phi(\delta_m)}{\sqrt{Npq}}\left\{\delta_m^6\left(\frac{1}{72}-\frac{pq}{18}\right)+\delta_m^4\left(-\frac{1}{6}+\frac{7pq}{12}\right)+\delta_m^3\left(\frac{1}{3}-pq\right)\right. \\ &\quad \left.-\frac{1}{24}+\frac{pq}{12}\right\}\Big/(Npq)-\frac{\phi'(\delta_k)}{2Npq}(c-1/2)+\frac{\phi'(\delta_k)}{2Npq}(c^2-1/4) \\ &= \int_{\delta_{k-1/2}}^{\delta_{N+1/2}} \phi(\delta)\left\{\delta^6\left(\frac{1}{72}-\frac{pq}{18}\right)+\delta^4\left(-\frac{1}{6}+\frac{7pq}{12}\right)+\delta^2\left(\frac{1}{3}-pq\right)\right. \\ &\quad \left.-\frac{1}{24}+\frac{pq}{12}\right\}d\delta/(Npq)-\frac{\delta_k\phi(\delta_k)}{Npq}\left(\frac{1}{72}-\frac{pq}{18}\right) \end{aligned} \tag{2.12}$$

$$\cdot \{\delta_{k-1/2}^4 - 2\delta_{k-1/2}^2 + 1\} + O(N^{-3/2}) .$$

Put $G_k(y) = \int_y^\infty x^k \phi(x) dx$. Then by recursion,

$$G_6(y) = \phi(y)y^5 + 5\phi(y)y^3 + 15\phi(y)y + 15\Phi(y)$$

$$G_4(y) = \phi(y)y^3 + 3\phi(y)y + 3\Phi(y)$$

$$G_2(y) = \phi(y)y + \Phi(y) .$$

If we write $c = 1/2 + (q-p)(\delta_{k-1/2}^2 - 1)/6 + e/\sqrt{Npq}$, then $T(N, k, p) - F(\delta_{k-c}) = \phi(\delta_{k-1/2})\{\delta_{k-1/2}^3(-5/72 + 7pq/36) + \delta_{k-1/2}(1/36 - pq/36) - e\}/Npq + O(N^{-3/2})$. Solving for e to make the term of $O(N^{-1})$ zero, gives the result.

It remains then to write Theorem 2 in a way that can *easily be used by practitioners*:

$$B(N, k, p) \simeq \Phi(a + b\delta_{k+1/2} + c\delta_{k+1/2}^2 + d\delta_{k+1/2}^3) , \tag{2.13}$$

where

$$a = (q-p)/6\sqrt{Npq} ,$$

$$b = 1 - (1/36 - pq/36)/Npq ,$$

$$c = -a = -(q-p)/6\sqrt{Npq} ,$$

$$d = (5/72 - 7pq/36)/Npq ,$$

$$\delta_{k+1/2} \equiv (k - Np + 1/2)(Npq)^{-1/2} ;$$

and

$$T(N, k, p) \simeq F(a + b\delta_{k-1/2} + c\delta_{k-1/2}^2 + d\delta_{k-1/2}^3) \equiv FT(N, k, p) . \tag{2.14}$$

3. Comparisons

In what follows, we will be approximating $T(N, k, p)$ by the classical approximation, $H(N, k, p) \equiv F(\delta_{k-1/2})$; the Gram-Charlier approximation $GC(N, k, p)$, given by (2.11); the finely tuned approximation $FT(N, k, p)$, given by (2.14), and finally the Camp-Paulson approximation (Camp [1]),

$$CP(N, k, p) \equiv \Phi(y/(3z^{1/2})) , \tag{3.1}$$

where

$$y = \{(N-k+1)p/(kq)\}^{1/3} \cdot \{9 - 1/(N-k+1)\} + (1/k) - 9$$

$$z = \{(N-k+1)p/(kq)\}^{2/3} \cdot \{1/(N-k+1)\} + 1/k ; \qquad 0 < k \leq N .$$

Note that there are typographical errors in both Raff [8], p. 299, and Johnson and Kotz [5], p. 64, in giving the Camp-Paulson approximation.

The above were chosen for their simplicity and accuracy. A very much more complicated approximation has been given by Peizer and Pratt [7], but it involves use of their specially computed table; it is not easy to use but was found to be more accurate than Camp-Paulson. In summarizing their findings, both Gebhardt [4] and Molenaar [6] recommend the Camp-Paulson approximation as a superior approximation for p away from zero and one. It is here that we will demonstrate the simpler approximation $FT(N, k, p)$ to be comparable. As our criterion for accuracy we will use

$$\sup_{0<k<N} |A(N, k, p)-T(N, k, p)| , \tag{3.2}$$

where A denotes the particular approximation used. This is slightly different from the criterion of Raff [8] and Gebhardt [4], but well suited to tail probabilities. Table 1 shows this error multiplied by 10,000.

Table 1. Error of approximation*

N	p	H	GC	CP	FT
10	0.05	987.4	207.5	49.8	214.8
	0.1	496.0	143.5	21.4	111.4
	0.2	294.9	48.8	16.3	42.6
	0.3	177.5	58.1	7.4	26.3
	0.5	26.9	26.9	3.0	4.0
50	0.05	405.3	87.9	18.4	49.3
	0.1	243.7	46.2	8.0	22.0
	0.2	139.0	16.9	3.2	6.2
	0.3	81.5	9.3	1.7	2.4
	0.5	5.4	5.4	0.9	0.2
100	0.05	267.1	51.3	9.1	25.2
	0.1	174.7	20.3	4.0	7.8
	0.2	99.0	8.0	1.6	2.1
	0.3	57.8	4.4	0.9	0.8
	0.5	2.7	2.7	0.5	0.0
150	0.05	227.7	31.4	6.0	14.1
	0.1	143.4	13.0	2.8	4.2
	0.2	81.0	5.2	1.1	1.1
	0.3	47.3	2.9	0.6	0.4
	0.5	1.8	1.8	0.3	0.0
200	0.05	191.4	22.4	4.4	8.7
	0.1	124.5	9.7	2.0	2.7
	0.2	70.3	3.8	0.8	0.7
	0.3	41.0	2.2	0.4	0.3
	0.5	1.2	1.2	0.2	0.0

* All entries are multiplied by 10,000.

4. Conclusions

From Table 1 and other tables computed but not shown here, the continuity correction (2.14) clearly outperforms the one-half continuity correction, and the more accurate Gram-Charlier approximation; and for (roughly) $Np>20$ it does better than the extremely accurate Camp-Paulson approximation. Apart from small N, (2.14) is indeed comparable to Camp-Paulson, yet has the simplicity of the one-half correction. For a given problem where N and p are fixed, the coefficients a, b, c and d of (2.14) need be calculated only *once*; then after the standardized deviate $\delta_{k-1/2}$ is found, the answer is immediate from one table look-up. When using Camp-Paulson, (3.1) has to be completely recalculated each time. Also, if one is forced to do the computation by hand, (2.14) only needs knowledge of $(Npq)^{-1/2}$ (essential anyway if one is using $1-\Phi(\delta_{k-1/2})$); all the terms are then obtained by simple multiplication and division. However (3.1) requires a calculating machine with square root and cube root facilities. We conclude therefore, that not only do the finely tuned corrections of (2.13), (2.14) have theoretical interest, but practical interest also.

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