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RECORD VALUES AND THE EXPONENTIAL DISTRIBUTION

M. AHSANULLAH*

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Abstract

A sequence $\{X_n,\ n\geqq 1\}$ of independent and identically distributed random variables with continuous cumulative distribution function $F(x)$ is considered. X_j is a record value of this sequence if $X_j>\max(X_1, \cdots, X_{j-1})$. Let $\{X_{L(n)},\ n\geqq 0\}$ be the sequence of such record values. Some properties of $X_{L(n)}$ and $X_{L(n)}-X_{L(n-1)}$ are studied when $\{X_n,\ n\geqq 1\}$ has the exponential distribution. Characterizations of the exponential distribution are given in terms of the sequence $\{X_{L(n)},\ n\geqq 0\}$.

1. Introduction and notations

Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables (rv) with cumulative distribution function (CDF) $F(x)$ and probability density function (pdf) $f(x)$. Set $Y_n=\max\{X_1, \cdots, X_n\}$, for $n\geqq 1$. We say X_j is a record value of $\{X_n\}$, if $Y_j>Y_{j-1}$. By definition X_1 is a record value. The indices at which record values occur are given by the record value times $\{L(n),\ n\geqq 0\}$, where $L(0)=1$ and $L(n)=\min\{j\,|\,j>L(n-1),\ X_j>X_{L(n-1)}\}$. We will denote $R(x)=-\log_e \bar{F}(x)$, $\bar{F}(x)=1-F(x)$ and $r(x)=(d/dx)R(x)$. If F is the distribution function of a nonnegative rv, we will call F is "new better than used" (NBU), if $\bar{F}(x+y)\leqq\bar{F}(x)\bar{F}(y)$, $x, y\geqq 0$, and F is "new worse than used" (NWU), if $\bar{F}(x+y)\geqq\bar{F}(x)\bar{F}(y)$, $x, y\geqq 0$. We will say that F belongs to the class C_1, if F is either NBU or NWU. If $F(x)$ has the density $f(x)$, the ratio $r(x)=f(x)/\bar{F}(x)$, for $\bar{F}(x)>0$, is called the hazard rate. We will say F belongs to the class C_2, if $r(x)$ is either monotone increasing or monotone decreasing. We will call the rv $X\in E(x, \sigma)$, if the pdf $f(x)$ of X is of the form

$$(1.1)\qquad f(x)=\begin{cases} \sigma^{-1}\exp(-x/\sigma)\ , & \text{for } x>0,\ \sigma>0\ , \\ 0\ , & \text{otherwise}\ ; \end{cases}$$

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* The work was partly completed when the author was at the Department of Statistics, University of Brasilia, Brazil.

and $X \in G_n(x, \sigma)$, if the pdf $f(x)$ of X is of the type

$$(1.2) \qquad f(x)=\begin{cases} (\Gamma(n))^{-1}\sigma^{-1}x^{n-1}\exp(-x/\sigma), & \text{for } x>0,\ \sigma>0, \\ 0, & \text{otherwise}. \end{cases}$$

Tata [2] gave a characterization of exponential distribution by the independence of the statistics $X_{L(0)}$ and $X_{L(1)}-X_{L(0)}$. In this paper we will give a generalization of Tata's [2] result and will also study some properties of $X_{L(n)}$ and $Z_n=X_{L(n)}-X_{L(n-1)}$, when the rv $X \in E(x, \sigma)$.

2. Main results

LEMMA 2.1. *If $X \in E(x, \sigma)$, then $X_{L(n)} \in G_{n+1}(x, \sigma)$ and $Z_n \in E(x, \sigma)$.*

PROOF. The distribution of $X_{L(n)}$ is known (see Karlin [1], p. 268) as

$$(2.1) \qquad \mathrm{P}\{X_{L(n)} \leqq x\}=\int_{-\infty}^{x} (R^n(y)/n!)dF(y).$$

Substituting $R(y)=-\log_e \bar{F}(y)=y\sigma^{-1}$, for $y>0$ and $R(y)=0$ for $y\leqq 0$, it follows that $X_{L(n)} \in G_{n+1}(x, \sigma)$. The joint pdf f_1 of $X_{L(0)}, X_{L(1)}, \cdots, X_{L(n)}$ is known (see Resnick [3], p. 69) as

$$(2.2) \qquad f_1(x_0, x_1, \cdots, x_n) =\begin{cases} r(x_0)r(x_1)\cdots r(x_{n-1})f(x_n), & 0<x_0<x_1<\cdots<x_n<\infty, \\ 0, & \text{otherwise}. \end{cases}$$

Integrating out $x_0, x_1, \cdots, x_{n-2}$, we get the joint pdf f_2 of $X_{L(n-1)}$ and $X_{L(n)}$ as

$$(2.3) \qquad f_2(x_{n-1}, x_n)=\begin{cases} (R(x_{n-1}))^{n-1}(\Gamma(n))^{-1}r(x_{n-1})f(x_n), & 0<x_{n-1}<x_n<\infty, \\ 0, & \text{otherwise}. \end{cases}$$

Substituting $R(x)=x\sigma^{-1}$, $r(x)=\sigma^{-1}$, $f(x)=\sigma^{-1}\exp(-x/\sigma)$ and using the transformations $Z_n=X_{L(n)}-X_{L(n-1)}$, $U_n=X_{L(n-1)}$, we get the joint pdf f_3 of Z_n and U_n as

$$(2.4) \qquad f_3(z, u)=\begin{cases} u^{n-1}\sigma^{-n-1}(\Gamma(n))^{-1}\exp(-(z+u)/\sigma), & 0<u,\ z<\infty, \\ 0, & \text{otherwise}. \end{cases}$$

Integrating (2.4) w.r.t. u, we see that $Z_n \in E(x, \sigma)$.

THEOREM 2.1. *Let $\{X_n, n \geqq 1\}$ i.i.d. random variables with common distribution F which is absolutely continuous (with respect to Lebesgue measure) and $F(0)=0$. Then for $X_n \in E(x, \sigma)$, it is necessary and sufficient that $X_{L(n-1)}$ and Z_n are independent.*

PROOF. Suppose $X_n \in E(x, \sigma)$, then it follows from (2.4) that $X_{L(n-1)}$ and Z_n are independent. Suppose now that $X_{L(n-1)}$ and Z_n are independent. We have from (2.3), the joint pdf f_4 of Z_n and U_n as

$$f_4(z, u) = \begin{cases} (R(u))^{n-1}(\Gamma(n))^{-1} r(u) f(u+z) , & 0<u, z<\infty , \\ 0 , & \text{otherwise} . \end{cases} \tag{2.5}$$

But the pdf f_5 of U_n is

$$f_5(u) = \begin{cases} (R(u))^{n-1}(\Gamma(n))^{-1} f(u) , & 0<u<\infty , \\ 0 , & \text{otherwise} . \end{cases} \tag{2.6}$$

Since Z_n and U_n are independent, we get from (2.5) and (2.6),

$$f(u+z)/\bar{F}(u) = g(z) ,$$

where $g(z)$ is the pdf of Z_n. Integrating w.r.t. z from 0 to z_1, we get

$$\frac{\bar{F}(u) - \bar{F}(u+z_1)}{\bar{F}(u)} = G(z_1) , \tag{2.7}$$

where $G(z_1) = \int_0^{z_1} g(z) dz$. Now letting $u \to 0^+$ and using $F(0)=0$, we see that $G(z_1) = F(z_1)$. Hence we get from (2.7),

$$\bar{F}(u+z_1) = \bar{F}(u) \bar{F}(z_1) . \tag{2.8}$$

The only continuous solution of (2.8) with the boundary condition $F(0) = 0$, is $\bar{F}(x) = \exp(-x/\sigma)$.

THEOREM 2.2. *Let $\{X_n, n \geqq 1\}$ be a sequence of i.i.d. random variables which has everywhere continuous distribution function F with density f and $F(0)=0$. Further if X_k belongs to the class C_1 and Z_n and X_k $(k \geqq 1)$ are identically distributed, then $X_k \in E(x, \sigma)$.*

PROOF. From (2.5), the pdf f_6 of Z_n can be written as

$$f_6(z) = \begin{cases} \int_0^\infty (R(u))^{n-1}(\Gamma(n))^{-1} r(u) f(u+z) du , & 0<z<\infty , \\ 0 , & \text{otherwise} . \end{cases} \tag{2.9}$$

By the assumption of identical distribution of Z_n and X_k, we must have

$$\int_0^\infty (R(u))^{n-1}(\Gamma(n))^{-1} r(u) f(u+z) du = f(z) , \qquad \text{for all } z>0 . \tag{2.10}$$

Substituting

$$\int_0^\infty (R(u))^{n-1} f(u) du = \Gamma(n) , \tag{2.11}$$

we have

$$\int_0^\infty (R(u))^{n-1} r(u) f(u+z) du = f(z) \int_0^\infty (R(u))^{n-1} f(u) du , \quad \text{for all } z>0 , \tag{2.12}$$

i.e.

$$\int_0^\infty (R(u))^{n-1} f(u) [f(u+z)/\bar{F}(u) - f(z)] du = 0 , \quad \text{for all } z>0 . \tag{2.13}$$

Integrating with respect to z from z_1 to ∞, we get from (2.13),

$$\int_0^\infty (R(u))^{n-1} f(u) [\bar{F}(u+z_1)/\bar{F}(u) - \bar{F}(z_1)] du = 0 , \quad \text{for all } z_1>0 . \tag{2.14}$$

If $F(x)$ is NBU, then (2.14) is true if

$$\bar{F}(u+z_1)/\bar{F}(u) = \bar{F}(z_1) , \quad \text{for all } z_1>0 . \tag{2.15}$$

The continuous solution of (2.15) with the boundary condition $\bar{F}(0)=1$ is $\bar{F}(x)=\exp(-x/\sigma)$, where σ is arbitrary. Similarly if $F(x)$ is NWU, then (2.14) is satisfied if (2.15) is true and hence $X_k \in E(x, \sigma)$.

THEOREM 2.3. *If X_k, $k \geqq 1$ has an everywhere continuous distribution function F which has density f with $F(0)=0$. Further if X_k belongs to C_2 and Z_n and Z_{n+1}, $n \geqq 1$, are identically distributed, then $X_k \in E(x, \sigma)$.*

PROOF. From (2.5), it follows that

$$P(Z_n>z) = \begin{cases} \bar{F}_{Z_n}(z) = \int_0^\infty (R(u))^{n-1} (\Gamma(n))^{-1} r(u) \bar{F}(u+z) du , & \text{for all } z \geqq 0 , \\ 0 , & \text{otherwise} . \end{cases}$$

Since Z_n and Z_n are identically distributed, we get

$$\int_0^\infty (R(u))^{n} r(u) \bar{F}(u+z) du = n \int_0^\infty (R(u))^{n-1} r(u) \bar{F}(u+z) du , \quad \text{for all } z \geqq 0 . \tag{2.16}$$

But

$$n \int_0^\infty (R(u))^{n-1} r(u) \bar{F}(u+z) du = \int_0^\infty (R(u))^{n} f(u+z) du . \tag{2.17}$$

Substituting (2.17) in (2.16) we get on simplification,

$$\text{(2.18)} \quad \int_0^\infty (R(u))^{n-1} r(u) \bar{F}(u+z) \left[1 - \frac{r(u+z)}{r(u)}\right] du = 0 , \qquad \text{for all } z \geqq 0 .$$

Thus if $X_k \in C_2$, then (2.18) is true if

$$\text{(2.19)} \quad r(u+z) = r(u) \qquad \text{for almost all } u, \text{ and any fixed } z \geqq 0 .$$

Hence $X_k \in E(x, \sigma)$.

HEALTH AND WELFARE CANADA, OTTAWA

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