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# YATES TYPE ESTIMATORS OF A COMMON MEAN

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## Abstract

Let $x$, $y$, $S$, $T$ and $W$ be independent random variables such that, $x \sim N(\mu, \alpha\sigma^2)$, $y \sim N(\mu, \beta\eta^2)$, $S/\sigma^2 \sim \chi^2(m)$, $T/\eta^2 \sim \chi^2(n)$ and $W/(\alpha\sigma^2+\beta\eta^2) \sim x^2(q)$, where $\mu$, $\sigma^2$, $\eta^2$ are unknown. For estimating $\mu$, consider the estimator $\hat{\mu}=x+(y-x)aS/[S+cT+d\{(y-x)^2+W\}]$, $a, c, d>0$. Note that the performance of $\hat{\mu}$ depends on $\tau=\beta\eta^2/\alpha\sigma^2$, which is unknown. Assume $q+n \geqq 2$ and let $a_0=(n+q-1)/(m+2)$, $c^*=c\alpha/\beta$, $d^*=d\alpha$. Two main results are:

i) for all $\tau>0$, $\hat{\mu}$ has a variance smaller than that of $x$ if $a \leqq 2 \min(1, c^*a_0, d^*a_0)$;

ii) for all $\tau \geqq \tau_0$, where $\tau_0>0$ is arbitrary, $\hat{\mu}$ has a variance smaller than that of $x$ if $a \leqq 2a_0 \min[c^*\tau_0/(1+\tau_0), d^*]$.

We also obtain some necessary conditions for $\hat{\mu}$ to have a variance smaller than that of $x$. It can be seen that with the exception of linked block designs for any design belonging to the class called $D_1$-class by Shah [16], Yates-Rao estimator for recovery of interblock information has the same form as that of $\hat{\mu}$. Hence, for such designs the above results can be used to examine if Yates estimator is good i.e., better than the intra-block estimator. Shah [16] resolved this question for linked block designs, which include the symmetrical BIBD's. Here, we consider asymmetrical BIBD's and show that Yates' estimator is good for all such designs listed in Fisher and Yates' table [5], with two exceptions. For one of these two designs, we show that Yates' estimator is not uniformly better than the intra-block estimator.


## 1. Introduction

The problem of estimating the common mean of two normal distributions and the related problem of recovery of inter-block information has been studied quite intensively in recent years. The reader can find a comprehensive list of references in the recent works by Brown and Cohen [2] and Khatri and Shah [9].

In the analysis of any incomplete block design under an Eisenhart model III (Eisenhart [3]), certain treatment contrasts admit two independent unbiased estimates commonly known as intra-block and inter-block estimators respectively. The problem commonly known as recovery of inter-block information seeks to combine these two estimators and was first considered by Yates [20], [21]. The procedure, suggested by Yates for special designs was later extended by Rao [11], [12] to all incomplete block designs. It may be noted that the motivation behind the recovery of inter-block information is not just to use the inter-block information but to use it for improving upon the customary intra-block estimator. It is therefore, desirable that the combined estimator should be unbiased and have a variance which does not exceed that of the intra-block estimator for any possible values of the unknown variances of the individual estimators. From the work of Graybill and Weeks [8], Graybill and Seshadri [7] and Shah [16], Yates-Rao estimator is known to be unbiased but very little is known about its variance.

While the properties of the Yates-Rao estimator remain largely unexplored, the desire to construct estimators which would be uniformly better than the intra-block estimator has lead to several modifications of the Yates-Rao estimator. Of these, the earlier works of Graybill and Deal [6], Seshadri [14], [15], Shah [16] and Stein [19] all ignored some between block comparisons and moreover, their results did not cover all incomplete block designs. The recent works by Brown and Cohen [2] and Khatri and Shah [9] make use of all between block comparisons and while the scope of the former is limited to BIBD's only, the later applies to all incomplete block designs. The Yates-Rao procedure, which is still the most widely used, does utilize all between block comparisons and up till now, the only known design for which it fails to give uniform improvement over the intra-block estimator is the symmetric BIBD with four treatments and three replications (see Shah [16]). Simulation studies by El-shaarawi, Prentice and Shah [4] as well as the numerical comparisons by Khatri and Shah [10] indicate that for large designs as used in these studies, Yates-Rao estimator compares favorably with that of Khatri and Shah.

In an earlier paper (Bhattacharya [1]), we considered a class of estimators which unified the two classes proposed in Brown and Cohen [2] and Khatri and Shah [9]. In this paper, we consider a further generalization which enables us to deal with the Yates-Rao estimator for all $D_1$-class design (Shah [16]), with the exception of linked block designs. Note that for linked block designs, the estimator proposed by Shah [16] covers the Yates-Rao estimator and as shown there the Yates-Rao estimator offers uniform improvement over the ordinary intra-block estimator iff the number of blocks is greater than five. Our main results are presented in Section 2 where $x$, $y$, $S$, $T$ and $W$ are independent random variables such that $x\sim N(\mu, \alpha\sigma^2)$, $y\sim N(\mu, \beta\eta^2)$, $S/\sigma^2\sim\chi^2(m)$, $T/\eta^2\sim\chi^2(n)$, $W/(\alpha\sigma^2+\beta\eta^2)\sim\chi^2(q)$ and $\mu$, $\sigma^2$, $\eta^2$ are unknown. For estimating $\mu$, we consider the estimator $\hat{\mu}=x+(y-x)aS/[S+cT+d\{(y-x)^2+W\}]$, $a, c, d>0$ and show that if $q+n\geqq 2$, $\tau=\beta\eta^2/\alpha\sigma^2$ and $a_0=(n+q-1)/(m+2)$, then (i) for all $\tau\geqq 0$, $\hat{\mu}$ has a variance smaller than that of $x$ if $a\leqq 2\min(1, c\alpha a_0/\beta, d\alpha a_0/\beta)$ and (ii) for all $\tau\geqq\tau_0$ where $\tau_0>0$ is arbitrary, $\hat{\mu}$ has a variance smaller than that of $x$ if

$$a\leqq 2a_0\min(c\tau_0/(1+\tau_0), d) .$$

We also obtain some necessary conditions for $\hat{\mu}$ to have a variance smaller than that of $x$. In Section 3, we apply the above results to asymmetrical BIBD's and show that Yates estimator offers uniform improvement over the intra-block estimator for all such designs listed in Fisher and Yates tables, [5] with two exceptions. For one of these two designs, we show that Yates estimator does not have the desired property. The other design appears to be the only one for which this property remains in doubt.

## 2. Main results

Let $x$, $y$, $S$, $T$, $W$ be independent random variables where $x\sim N(\mu, \alpha\sigma^2)$, $y\sim N(\mu, \beta\eta^2)$, $S/\sigma^2\sim\chi^2(m)$, $T/\eta^2\sim\chi^2(n)$ and $W/(\alpha\sigma^2+\beta\eta^2)\sim\chi^2(q)$. Let

$$\hat{\mu}=x+\phi(y-x) \tag{2.1}$$

with

$$\phi=aS/[S+cT+d\{(y-x)^2+W\}] \tag{2.2}$$

where $a, c, d>0$ are constants to be suitably chosen.

In Section 3 we shall see that the estimator proposed by Yates [20], [21] and generalized by Rao [11], [12] has the above form for all $D_1$-class designs (Shah [16]) with the exception of linked block designs.

Let $W_0$ be independent of $S$, $T$ and $W$ such that $W_0/(\alpha\sigma^2+\beta\eta^2)\sim\chi^2_3$. Let $\phi^*$ be the expression obtained by replacing $(y-x)^2$ in $\phi$ by $W_0$ and let $\tau=\beta\eta^2/\alpha\sigma^2$. It was shown in Brown and Cohen [2] and in Khatri and Shah [9] that $v(\hat{\mu})\leqq v(x)$ for all values of $(\sigma^2, \eta^2)$ iff

$$(1+\tau)\,\mathrm{E}\,\phi^{*2}\leqq 2\,\mathrm{E}\,\phi^* \tag{2.3}$$

for every $\tau>0$. Let $r=(1+\tau)\phi^*/a$. It is easy to verify that (2.3) is equivalent to $a\leqq 2\,\mathrm{E}\,r/\mathrm{E}\,r^2$. Thus we have

THEOREM 2.1. *$\hat{\mu}$ is uniformly better than $x$ iff $a\leqq 2\delta$ where $\delta=$*

$\inf_{\tau} \mathrm{E}\, r / \mathrm{E}\, r^2$.

Since it is not easy to evaluate $\delta$ in all cases, we shall try to obtain some non-trivial bounds for $\delta$. Let $z_1 = S/\sigma^2$, $z_2 \sim T/\eta^2 + (W_0+W)/(\alpha\sigma^2 + \beta\eta^2)$ and $u = (W_0+W)/z_2(\alpha\sigma^2+\beta\eta^2)$. It is easy to see that $z_1 \sim \chi^2_m$, $z_2 \sim \chi^2_{n+q+3}$, $u \sim \beta((q+3)/2, n/2)$ and that $z_1$, $z_2$, $u$ are all independently distributed. We shall write $c^* = c\alpha/\beta$, $d^* = d\alpha$, $\gamma = \tau/(1+\tau)$ and shall rewrite $r$ in the form

$$r = z_1/[(1-\gamma)z_1 + z_2 h(u, \gamma)] \tag{2.4}$$

where $h(u, \gamma) = d^*u + c^*(1-u)\gamma$. Primes will denote derivations w.r.t. $\gamma$, we shall attempt to obtain $\delta^* = \inf_{u,\gamma} f(u, \gamma)$ where $f(u, \gamma) = \mathrm{E}\,(r\,|\,u)/\mathrm{E}\,(r^2\,|\,u)$. Clearly $\delta^* \leqq \delta$ and hence it would be sufficient to have $a \leqq 2\delta^*$ to ensure $v(\hat{\mu}) \leqq v(x)$. Direct computations from (2.4) show that

$$r' = h(u, 1)[r^2 - c^*(1-u)r/h(u, 1)]/h(u, \gamma)\ . \tag{2.5}$$

We note that $f' = -g(u)h(u, 1)/h(u, \gamma)\, \mathrm{E}^2\,(r^2\,|\,u)$ where

$$g(u) = 2\,\mathrm{E}\,(r\,|\,u)\,\mathrm{E}\,(r^3\,|\,u) - c^*(1-u)\,\mathrm{E}\,(r\,|\,u)\,\mathrm{E}\,(r^2\,|\,u)/h(u, 1) - \mathrm{E}^2\,(r^2\,|\,u)\ .$$

We shall now prove the following lemma:

LEMMA 2.1. *If* $\mathrm{E}\,(r'\,|\,u) \geqq 0$, *then* $f' \leqq 0$.

PROOF. If $\mathrm{E}\,(r'\,|\,u) \geqq 0$, then $\mathrm{E}\,(r^2\,|\,u) \geqq c^*(1-u)\,\mathrm{E}\,(r\,|\,u)/h(u, 1)$ and hence $g(u) \geqq 2\,\mathrm{E}\,(r\,|\,u)\,\mathrm{E}\,(r^3\,|\,u) - 2\,\mathrm{E}^2\,(r^2\,|\,u) \geqq 0$ and hence $f' \leqq 0$.

We recall that $u$ is a beta variable which lies between 0 and 1. Since $r'' \geqq 0$ for $u > 0$, it follows that $r'$ is a non-decreasing function of $\gamma$ and hence either (i) $\mathrm{E}\,(r'\,|\,u) \geqq 0$ for $\gamma \in [0, 1)$ or (ii) $\mathrm{E}\,(r'\,|\,u) < 0$ for $\gamma \in [0, 1]$ or (iii) there exists $\lambda(u) \in (0, 1)$ such that $\mathrm{E}\,(r'\,|\,u) < 0$ for $0 \leqq \gamma < \lambda(u)$ and $\mathrm{E}\,(r'\,|\,u) \geqq 0$ for $\lambda(u) \leqq \gamma < 1$.

Thus, if $\mathrm{E}\,(r'\,|\,u) \geqq 0$, by virtue of Lemma 2.1, we have $\inf_{\gamma} f(u, \gamma) = \operatorname{limit}_{\gamma \to 1} f(u, \gamma) = a_0 h(u, 1)$, where $a_0 = (n+q-1)/(m+2)$ on the other hand if $\mathrm{E}\,(r'\,|\,u) < 0$, (2.5) gives $f(u, \gamma) > h(u, 1)/c^*(1-u) > 1$.

Thus, we have $\delta^* = \inf_{u,\gamma} f(u, \gamma) \geqq \min\,(1, ea_0)$, where

$$e = \inf_{u} h(u, 1) = \min\,(c^*, d^*)\ .$$

In view of Theorem 2.1 we have thus proved

THEOREM 2.2. *If* $a \leqq 2 \min\,(1, ea_0)$, *then* $\hat{\mu}$ *is uniformly better than* $x$.

In applications of the estimator $\hat{\mu}$ to the problem of recovery of inter-block information one may have prior knowledge that $\tau \geqq \tau_0$ where $\tau_0 > 0$ is a quantity which depends on the design parameters. Let $\gamma_0 = \tau_0/(1+\tau_0)$. It is easy to show that $r \leqq 1/e_0 z = r_0$, say, where $z = z_2/z_1$ and $e_0 = \min(c^*\gamma_0, d^*)$. Then, $\mathrm{E}\, r/\mathrm{E}\, r^2 \geqq \mathrm{E}\, r/\mathrm{E}\,(r_0 r)$. Using the same arguments as employed by Brown and Cohen [2] in the proof of their Theorem 2.1, we see that $f_0(u, \gamma) = \mathrm{E}\,(r|u)/\mathrm{E}\,(r_0 r|u)$ is non-increasing in $\gamma$ and hence $\inf_{\gamma} f_0(u, \gamma) = \operatorname{limit}_{\gamma \to 1} f_0(u, \gamma) = e_0 a_0$. Clearly $\delta = \inf_{\gamma} \mathrm{E}\, r/\mathrm{E}\, r^2 \geqq \inf_{u,\gamma} f_0(u, \gamma) = e_0 a_0$. Hence recalling Theorem 2.1, we have

THEOREM 2.3. *If* $a \leqq 2e_0 a_0$, *then* $\hat{\mu}$ *is better than* $x$ *for all* $\tau > \tau_0$.

We now present some necessary conditions for $v(\hat{\mu})$ to be less than $v(x)$. It can be seen that $\operatorname{limit}_{\gamma \to 1} \mathrm{E}\, r/\mathrm{E}\, r^2 = e_* a_0$, where $e_* = \mathrm{E}\,(1/h(u, 1))/\mathrm{E}\,(1/h^2(u, 1))$. Hence, $\delta \leqq e_* a_0$ and Theorem 2.1 gives

THEOREM 2.4. *If* $\hat{\mu}$ *has a variance uniformly smaller than that of* $x$, *then we must have* $a \leqq 2e_* a_0$.

Note that $e_* \leqq e^*$, where $e^* = \mathrm{E}\, h(u, 1) = d^*(q+3)/(n+q+3) + c^* n/(n+q+3)$. Hence we have

COROLLARY 2.1. *If* $\hat{\mu}$ *has a variance uniformly smaller than that of* $x$, *then we must have* $a \leqq 2e^* a_0$.

## 3. Applications

Consider a connected binary equireplicate incomplete block design. Let $b$ = number of blocks, $k$ = number of plots per block, $r$ = number of replications. Let $N$ $(v \times b)$ denote the incidence matrix of the design. Let $\phi_i$, $i = 1, \cdots, v-1$ denote the characteristic roots of $NN'/k$ other than $r$ and $p_i$ denote the corresponding characteristic vector. Let rank $N = t$, so that $NN'/k$ has exactly $t-1$ positive characteristic roots $\phi_1, \phi_2, \cdots, \phi_{t-1}$, all smaller than $r$. As shown in Roy and Shah [13], the problem of estimating an arbitrary treatment contrast can be reduced to that of estimating the canonical contrasts $\xi_i = \boldsymbol{p}_i'\theta$, where $\boldsymbol{\theta}$ $(v \times 1)$ stands for the vector of treatment effects in the linear model (Eisenhart model III). A set of minimal sufficient statistics for this problem is given by $\boldsymbol{x}$ $((v-1) \times 1)$, $\boldsymbol{y}$ $((t-1) \times 1)$, $S_1$, $S_2$ where $\boldsymbol{x} = (x_i)$, $x_i$ = intra-block estimate of $\xi_i$, $\boldsymbol{y} = (y_i)$, $y_i$ = inter-block estimate of $\xi_i$, $S_1$ = intra-block error $SS$, and $S_2$ = inter-block error $SS$ per plot. We also have: $x_1, x_2, \cdots, x_{v-1}$, $y_1, y_2, \cdots, y_{t-1}$, $S_1$, $S_2$ all independently distributed and $x_i \sim N(\xi_i, a_i\sigma_1^2)$, $y_i \sim N(\xi_i, b_i\sigma_2^2)$, $S_1/\sigma_1^2 \sim \chi^2(e_1)$, $S_2/\sigma_2^2 \sim \chi^2(e_2)$ where, $\sigma_1^2$ = intra-block error variance, $\sigma_2^2$ = inter-block error variance per plot, $e_1$ = d.f. for intra-block error, $e_2$ = d.f. for inter-block error $= b-t$, $a_i = 1/(r-\phi_i)$

and $b_i = 1/\psi_i$.

In the following we shall consider only $D_1$-class designs (Shah [16]). Note that for such designs $\psi_1, \psi_2, \cdots, \psi_{t-1}$ are all equal and let $\psi$ denote their common value. Then $a_1, a_2, \cdots, a_{t-1}$ are all equal to $1/(r-\psi)$ and $b_1, b_2, \cdots, b_{t-1}$ are all equal to $1/\psi$. Now, the Yates-Rao estimator of $\xi_i$ can be written as

$$\hat{\xi}_i = x_i + (y_i - x_i) S_1 / (c_1 S_1 + c_2 S_2 + c_3 S_3) ,$$

where

$$S_3 = \sum_{i=1}^{t-1} (y_i - x_i)^2 , \qquad c_1 = 1 - (v-k)(r-\psi)/\psi v(r-1) ,$$

$$c_2 = e_1 k(r-\psi)/\psi v(r-1) \quad \text{and} \quad c_3 = e_1 k(r-\psi)^2 / vr(r-1) .$$

We shall exclude the linked block designs so that we shall have $c_1, c_2, c_3$ all positive. Without loss of generality let us take $i=1$ and make the following match ups with the terms used in Section 2: $x_1 \sim x$, $y_1 \sim y$, $S_1 \sim S$, $S_2 \sim T$, $S_3 \sim W$, $\xi_1 \sim \mu$, $\sigma_1 \sim \sigma$, $\sigma_2 \sim \eta$, $a_1 \sim \alpha$, $b_1 \sim \beta$, $e_1 \sim m$, $e_2 \sim n$, $t-2 \sim q$.

Then $\hat{\xi}_1$ matches up with $\hat{\mu}$ with $a = 1/c_1$, $c = c_2/c_1$, $d = c_3/c_1$. Note that $\eta^2/\sigma^2 \geqq 1$ and hence $\tau \geqq b_1/a_1 = (r-\psi)/\psi = \tau_0$, say. Then $\gamma_0 = \tau_0/(1+\tau_0) = 1 - \psi/r$. Note also that $d^* = e_1 k(r-\psi)/vr(r-1)c_1 = c^* \gamma_0$ and $a_0 = (b-3)/(e_1+2)$. From Theorem 2.3 we then have

THEOREM 3.1. *$\hat{\xi}_1$ is uniformly better than $x_1$ if*

$$\nu = e_1 k(r-\psi)(b-3)/vr(r-1)(e_1+2) \geqq 1/2 .$$

Let $u_* = 1 - \psi u/r$, where $u$ has a beta distribution with parameters $(t+1)/2$ and $(b-t)/2$. Let $e_{**} = \mathrm{E}(1/u_*)/\mathrm{E}(1/u_*^2)$ and $c_* = c^* c_1$. Note that $h(u, 1) = c_* u_*$. Then Theorem 2.4 gives

THEOREM 3.2. *If $\hat{\xi}_1$ is uniformly better than $x_1$, then we must have*

$$\nu_* = c_* e_{**} (b-3)/(e_1+2) \geqq 1/2 .$$

Let $e^{**} = \mathrm{E}\, u_* = 1 - \psi(t+1)/r(b+1)$. Then, in the same way as Corollary 2.1 was deduced from Theorem 2.4, we obtain from Theorem 3.2,

COROLLARY 3.1. *If $\hat{\xi}_1$ is uniformly better than $x_1$, then we must have*

$$\nu^* = c_* e^{**} (b-3)/(e_1+2) \geqq 1/2 .$$

For BIBD's $\psi = (r-\lambda)/k$ where $\lambda$ = number of blocks in which any given pair of treatments occur together. Then $r - \psi = \lambda v/k$ and we have $\nu = e_1 \lambda (b-3)/r(r-1)(e_1+2)$. Values of $\nu$ were calculated for all asymmetrical BIBD's listed in Fisher-Yates tables [5] and turned out to be greater than 1/2 in all cases with two exceptions. The exceptional designs are (1) $v=4$, $b=6$, $k=2$ and (2) $v=5$, $b=10$, $k=2$. For the first one $\nu^*$ was calculated and found to be less than 1/2. For the second design, the value of $\nu^*$ turned out to be greater than 1/2 and hence $\nu_*$ was computed by numerical methods but this also turned out to be greater than 1/2.

Hence we have the result:

For all asymmetrical BIBD's listed in Fisher-Yates tables [5] with the two exceptions mentioned earlier, Yates estimator is uniformly better than the intra-block estimator. For the first of the two exceptional designs, Yates estimator is not uniformly better then the intra-block estimater. For the other exceptional design, which fails to satisfy the sufficient condition in Theorem 3.1 but satisfies the necessary condition in Theorem 3.2, no conclusion could be reached.

It should be remarked that in the above analysis we have considered the untruncated form of the Yates estimator. From the work of Shah [18] it is known that the truncated estimator is uniformly better than the untruncated estimator. It can also be verified that the necessary condition in Theorem 3.2 is valid also for the truncated estimator. Hence our conclusions above hold also for the truncated form of the Yates estimator.

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