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ON THE UNIFORM COMPLETE CONVERGENCE OF DENSITY FUNCTION ESTIMATES

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Abstract

Let f be a uniformly continuous density function. Let W be a non-negative weight function which is continuous on its compact support $[a, b]$ and $\int_a^b W(x)dx=1$. The complete convergence of

$$\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n} W\left(\frac{s-X_k}{b(n)}\right)-f(s)\right|$$

to zero is obtained under varying conditions on the bandwidths $b(n)$, support or moments of f, and smoothness of W. For example, one choice of the weight function for these results is Epanechnikov's optimal function and $nb^2(n)>n^\delta$ for some $\delta>0$. The uniform complete convergence of the mode estimate is also considered.

1. Introduction and preliminaries

The construction of a family of estimates of a density function $f(x)$ and of the mode has been studied by several people. Rosenblatt [9] considered a general class of density estimates:

$$f_n(x)=\frac{1}{nb(n)}\sum_{i=1}^{n} W\left(\frac{x-X_i}{b(n)}\right), \tag{1.1}$$

where $X_1,\cdots,X_n$ are i.i.d. random variables with continuous density function $f(x)$, $W(x)$ is a bounded integrable weight function such that

$$\int_{-\infty}^{\infty} W(x)dx=1$$

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and $b(n)$ is a bandwidth that tends to zero as $n\to\infty$. Thus, the question arises as to suitable choices of $W(x)$ and $b(n)$ so that the estimate function $f_n(x)$ is optimal (in some sense). The local properties of the estimate function in (1.1) have been studied extensively (see Rosenblatt [10] for a general survey), and a global measure of deviation of the curve $f_n(x)$ from $f(x)$ by

$$\|f_n-f\|_\infty=\sup_{x\in R}|f_n(x)-f(x)| \tag{1.2}$$

has been considered. Parzen [8] showed that if the (true) underlying density function $f(x)$ is uniformly continuous then $\|f_n-f\|_\infty$ converges in probability to zero under the following conditions:

(P1) $\phi_W(t)=\int_{-\infty}^{\infty}e^{itx}W(x)dx$ is absolutely integrable,

(P2) $\sqrt{n}\,b(n)\to\infty$ as $n\to\infty$.

The results of Nadaraya [7], Woodroofe [15], Deheuvels [4], Devroye and Wagner [5], and Silverman [12] on uniform consistency in the strong sense are discussed in the last section for comparison with the results of this paper.

The major results of the paper give a new class of "good" weight functions (which includes the optimal function) under mild conditions on the bandwidth sequence $b(n)$ where uniform consistency of the estimate $f_n(x)$ is obtained by the complete convergence (see Stout [13]) of $\|f_n-f\|_\infty$ to zero (which implies convergence with probability one). The main tools used in obtaining these results will be the smoothness of the weight function and sub-Gaussian techniques.

Throughout this paper, attention will be restricted to a density function which is uniformly continuous (or is continuous on its compact support $[a, b]$) (see Schuster [11] for a discussion of necessity of uniform continuity) and weight functions $W(x)$ which satisfy

(i) $\int_a^b W(x)dx=1$ and

(ii) $W(x)$ is nonnegative and continuous on $[a, b]$ and vanish outside $[a, b]$.

Let U_n be a polygonal approximating function on the space of continuous functions with domain $[a, b]$, $C[a, b]$. That is,

$$g\left(a+\frac{(b-a)i}{n}\right)=[U_n(g)]\left(a+\frac{(b-a)i}{n}\right)$$

for $i=0, 1, \cdots, n$ and $g\in C[a, b]$, and U_n is linear between the points $a+(b-a)i/n$ and $a+(b-a)(i+1)/n$. Recall that the modulus of continuity, $\omega_g(\delta)$, is defined by Billingsley [2] by

$$\omega_g(\gamma)=\sup_{|t-s|\leq\gamma}|g(t)-g(s)| \tag{1.3}$$

for $\gamma>0$, $s, t \in [a, b]$, and $g \in C[a, b]$.

DEFINITION (Chow [3]). A random variable X is said to be *sub-Gaussian* if there exists $\alpha \geqq 0$ such that

$$\mathrm{E}\left[\exp(tX)\right] \leqq \exp\left(\frac{\alpha^2 t^2}{2}\right) \qquad \text{for all } t \in R . \tag{1.4}$$

If X is sub-Gaussian, then let

$$\tau(X) = \inf\{\alpha \geqq 0 : \text{Inequality (1.4) holds}\} .$$

Some basic properties on sub-Gaussian random variables include:

1. If $\mathrm{P}[|X| \leqq K] = 1$ and $\mathrm{E}\, X = 0$, then

$$\mathrm{E}\left[\exp(tX)\right] \leqq \exp(K^2 t^2) . \tag{1.5}$$

2. If $\tau(X) = \alpha$, then

$$\mathrm{P}[|X| \geqq \lambda] \leqq 2 \exp(-\lambda^2/2\alpha^2) . \tag{1.6}$$

3. The sum of two independent sub-Gaussian random variables is sub-Gaussian.

Finally, a sequence of random variables $\{X_n\}$ is said to *converge completely* to a random variable X if

$$\sum_{n=1}^{\infty} \mathrm{P}[|X_n - X| > \varepsilon] < \infty \tag{1.7}$$

for each $\varepsilon > 0$. Thus, complete convergence implies convergence with probability one by Boole's inequality.

2. Main results

In this section the complete convergence of $\|f_n - f\|_\infty$ to zero is obtained under conditions on the modulus of continuity of the weight function $W(x)$ and the rate of convergence to zero by the bandwidth $b(n)$. Also, the uniform consistency of the mode estimate is obtained in this setting. The uniform consistency of the estimate $f_n(x)$ (in the complete sense) is accomplished by two lemmas.

LEMMA 1. (i) *If* $nb^2(n) > n^\delta$ *for some* $\delta > 0$ *and* (ii) $\int |x|^p f(x) dx < \infty$ *for some* $p > 0$, *and* (iii) $\omega_W((2b - 2a)/n^r b(n)) = o(b(n))$ *for some integer* $r > 1/p$, *then*

$$\sup_{-\infty < s < \infty} \left| f_n(s) - \frac{1}{b(n)} \mathrm{E}\, W\left(\frac{s - X_1}{b(n)}\right) \right| \to 0 \tag{2.1}$$

completely as $n\to\infty$ *where* $f_n(s)$ *is defined in* (1.1).

PROOF. From (ii) it follows that

$$\sum_{n=1}^{\infty} \mathrm{P}\,[|X_1|^p>n]<\infty\ . \tag{2.2}$$

Thus, define

$$Y_n=X_n I_{[|X_n|\leqq n^{1/p}]}$$

for each n. By (2.2) $\mathrm{P}\,[X_n\neq Y_n \text{ i.o.}]=0$ since

$$\sum_{n=1}^{\infty} \mathrm{P}\,[Y_n\neq X_n]=\sum_{n=1}^{\infty} \mathrm{P}\,[|X_n|>n^{1/p}]<\infty\ . \tag{2.3}$$

Next,

$$\begin{aligned} &\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n} W\left(\frac{s-X_k}{b(n)}\right)-\frac{1}{b(n)}\,\mathrm{E}\,W\left(\frac{s-X_1}{b(n)}\right)\right| \\ &\leqq \sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n}\left[W\left(\frac{s-X_k}{b(n)}\right)-W\left(\frac{s-Y_k}{b(n)}\right)\right]\right| \\ &\quad+\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n} W\left(\frac{s-Y_k}{b(n)}\right)-\frac{1}{nb(n)}\sum_{k=1}^{n}\mathrm{E}\,W\left(\frac{s-Y_k}{b(n)}\right)\right| \\ &\quad+\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n}\mathrm{E}\,W\left(\frac{s-Y_k}{b(n)}\right)-\frac{1}{b(n)}\,\mathrm{E}\,W\left(\frac{s-X_1}{b(n)}\right)\right|\ . \end{aligned} \tag{2.4}$$

The first and third terms of the right-hand side of Inequality (2.4) converge to 0 completely by (2.3) and the boundedness of W.

Using the compact support $[a, b]$ for W,

$$\frac{1}{nb(n)}\sum_{k=1}^{n} W\left(\frac{s-Y_k}{b(n)}\right)=0 \qquad \text{unless } a\leqq\frac{s-Y_k}{b(n)}\leqq b$$

for some $1\leqq k\leqq n$ or unless $-n^{1/p}+ab(n)\leqq s\leqq n^{1/p}+bb(n)$. Since $b(n)\to 0$, the $\sup\limits_{-\infty<s<\infty}$ in the second term of (2.4) need only be taken over $[-n^{1/p}+a, n^{1/p}+b]$. Let $\delta^n=2(b-a)/n^r b(n)$ and let $t_i=-n^{1/p}+a+(i(2n+(b-a))/n^{2r})$ for $1\leqq i\leqq n^{2r}$. Hence, $t_i-t_{i-1}=(2n^{1/p}+(b-a))/n^{2r}\leqq 2(b-a)/n^r$ for n large enough. Let $\tilde{W}_k(s)=W(s-Y_k/b(n))-\mathrm{E}\,W(s-Y_k/b(n))$ for each $k=1,\cdots,n$. Thus, $\mathrm{E}\,\tilde{W}_k(s)=0$ for each $s\in[a, b]$ and each k. Furthermore,

$$\begin{aligned} \omega_{\tilde{W}_k}(\delta_n)&=\sup_{|t-s|\leqq\delta_n}|\tilde{W}_k(s)-\tilde{W}_k(t)| \\ &\leqq \sup_{|t-s|\leqq\delta_n}\left|W\left(s-\frac{Y_k}{b(n)}\right)-W\left(t-\frac{Y_k}{b(n)}\right)\right| \\ &\quad+\sup_{|t-s|\leqq\delta_n}\left|\mathrm{E}\,W\left(s-\frac{Y_k}{b(n)}\right)-\mathrm{E}\,W\left(t-\frac{Y_k}{b(n)}\right)\right| \\ &\leqq 2\omega_W(\delta_n)\ . \end{aligned} \tag{2.5}$$

Hence, $\omega_{\tilde{W}_k}(\delta_n) \leqq 2\omega_W(\delta_n) = o(b(n))$ for each k from condition (iii). For $\varepsilon > 0$ let

$$A_n = \left[\sup_{-n^{1/p}+a \leqq s \leqq n^{1/p}+b} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k\left(\frac{s}{b(n)}\right) \right| > \varepsilon \right] = \left[\max_{1 \leqq i \leqq n^r} \sup_{s \in I_i} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k\left(\frac{s}{b(n)}\right) \right| > \varepsilon \right] \tag{2.6}$$

where $I_i = [t_{i-1}, t_i]$. Hence,

$$A_n \subset \left[\max_{1 \leqq i \leqq n^{2r}} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k\left(\frac{t_i}{b(n)}\right) \right| + \max_{1 \leqq i \leqq n^{2r}} \sup_{s \in I_i} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} \left[\tilde{W}_k\left(\frac{s}{b(n)}\right) - \tilde{W}_k\left(\frac{t_i}{b(n)}\right) \right] \right| > \varepsilon \right]. \tag{2.7}$$

However,

$$\max_{1 \leqq i \leqq n^{2r}} \sup_{s \in I_i} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} \left[\tilde{W}_k\left(\frac{s}{b(n)}\right) - \tilde{W}_k\left(\frac{t_i}{b(n)}\right) \right] \right| \leqq \frac{2}{b(n)} \omega_W(\delta_n). \tag{2.8}$$

Since $\omega_W(\delta_n) = o(b(n))$ by condition (iii), there exists $N(r)$ such that

$$A_n \subset \left[\max_{1 \leqq i \leqq n^{2r}} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k\left(\frac{t_i}{b(n)}\right) \right| > \frac{\varepsilon}{2} \right]$$

for all $n \geqq N(r)$. Using the basic properties of sub-Gaussian random variables ($\{\tilde{W}_k(t_i/b(n)) : k = 1, 2, \cdots\}$ for each i), for each $n \geqq N(r)$

$$\begin{aligned} P(A_n) &\leqq P\left[\max_{1 \leqq i \leqq n^{2r}} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k\left(\frac{t_i}{b(n)}\right) \right| > \frac{\varepsilon}{2} \right] \\ &\leqq \sum_{i=1}^{n^{2r}} P\left[\left| \frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k\left(\frac{t_i}{b(n)}\right) \right| > \frac{\varepsilon}{2} \right] \\ &\leqq n^{2r} 2 \exp\left[-\varepsilon^2 / 64 \|W\|_\infty^2 B_n \right] \end{aligned} \tag{2.9}$$

where $\|W\|_\infty = \sup_s |W(s)|$ and $B_n = \sum_{k=1}^{n} (1/nb(n))^2 = 1/nb^2(n)$. To obtain the complete convergence in (2.1), consider

$$\begin{aligned} \sum_{n=1}^{\infty} P(A_n) &= \sum_{n=1}^{N(r)} P(A_n) + \sum_{n=N(r)+1}^{\infty} P(A_n) \\ &\leqq N(r) + \sum_{n=N(r)+1}^{\infty} 2n^{2r} \exp\left(\frac{-\varepsilon^2 nb^2(n)}{64 \|W\|_\infty} \right) \\ &\leqq N(r) + \sum_{n=N(r)+1}^{\infty} 2n^{2r} \exp(-cn^{\delta}) \end{aligned} \tag{2.10}$$

where $c = \varepsilon^2 / 64 \|W\|_\infty$. Thus, the series in (2.10) converges by the integral test.

LEMMA 2. *If the underlying density, f, is uniformly continuous, then*

$$\sup_{-\infty<s<\infty}\left|\frac{1}{b(n)}\,\mathrm{E}\,W\left(\frac{s-X_1}{b(n)}\right)-f(s)\right|\to 0 \qquad as\ n\to\infty\,. \tag{2.11}$$

PROOF. Since f is uniformly continuous given $\varepsilon>0$ there exists $\delta>0$ such $\overset{x\to\infty}{\text{that}}$ $|f(x)-f(x')|\overset{x\to-\infty}{<}\varepsilon$ whenever $|x-x'|<\delta$. Let N be sufficiently large so that $|b(n)y|<\delta$ for all $n\geqq N$ and $y\in[a, b]$. Since $W(y)=0$ for $y\notin[a, b]$,

$$\begin{aligned}\left|\frac{1}{b(n)}\,\mathrm{E}\,W\left(\frac{s-X_1}{b(n)}\right)-f(s)\right| &= \left|\frac{1}{b(n)}\int_{-\infty}^{\infty} W\left(\frac{s-x}{b(n)}\right)f(x)dx-f(s)\right| \\ &= \left|\int_{-\infty}^{\infty} W(y)f(s-b(n)y)dy-f(s)\right| \\ &= \left|\int_a^b W(y)[f(s-b(n)y)-f(s)]dy\right| \\ &< \varepsilon\int_a^b W(y)dy=\varepsilon\end{aligned} \tag{2.12}$$

uniformly in s for all $n\geqq N$. Hence,

$$\sup_{-\infty<s<\infty}\left|\frac{1}{b(n)}\,\mathrm{E}\,W\left(\frac{s-X_1}{b(n)}\right)-f(s)\right|\to 0 \qquad \text{as } n\to\infty\,.$$

Thus, the proof of Theorem 1 is immediate from Lemmas 1 and 2 since for each $\varepsilon>0$

$$\begin{aligned}&\mathrm{P}\left[\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n} W\left(\frac{s-X_k}{b(n)}\right)-f(s)\right|>\varepsilon\right] \\ &\leqq \mathrm{P}\left[\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n}\left(W\left(\frac{s-X_k}{b(n)}\right)-\mathrm{E}\,W\left(\frac{s-X_1}{b(n)}\right)\right)\right|>\frac{\varepsilon}{2}\right] \\ &\quad +\mathrm{P}\left[\sup_{-\infty<s<\infty}\left|\frac{1}{b(n)}\,\mathrm{E}\,W\left(\frac{s-X_1}{b(n)}\right)-f(s)\right|>\frac{\varepsilon}{2}\right]\end{aligned} \tag{2.13}$$

and each of the terms in (2.13) is a convergent series in n. All of the conditions will be stated in Theorem 1 for easy reference.

THEOREM 1. *Let $\{X_n\}$ be independent random variables with the same density function $f(s)$ which is uniformly continuous. Let $W(x)$ be a nonnegative weight function which is continuous on its compact support and integrates to 1. If*

(a) $nb^2(n)>n^\delta$ *for some* $\delta>0$,

(b) $\int|x|^p f(x)dx<\infty$ *for some* $p>0$, *then*

(c) $\omega_W\left(\frac{2b-2a}{n^r b(n)}\right)=o(b(n))$ *for some integer* $r>\frac{1}{p}$, *then* $\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{i=1}^{n} W\left(\frac{s-X_i}{b(n)}\right)-f(s)\right|\to 0$
completely as $n\to\infty$.

The case of the density function having compact support is discussed in Taylor and Cheng [14] and being discontinuous off of its support is not entirely excluded in Theorem 1. The following steps indicate modifications which allows the theory to include a large class of density functions (for example, the uniform densities).

Step 1. For an unknown density function which is continuous only on $[a, b]$ and vanishes outside $[a, b]$, there is no change in Lemma 1.

Step 2. In Lemma 2 it is easy to verify that

$$\sup_{a+b(n)C\leq s\leq b-b(n)C}\left|\mathrm{E}\frac{1}{b(n)}W\left(\frac{s-X_1}{b(n)}\right)-f(s)\right|\to 0$$

as $n\to\infty$ where $C=\max\{|a|, |b|\}$.

Step 3. Combining steps 1 and 2, for each $\varepsilon>0$

$$\sup_{a+b(n)C\leq s\leq b-b(n)C}\left|\frac{1}{nb(n)}\sum_{k=1}^{n} W\left(\frac{s-X_k}{b(n)}\right)-f(s)\right|\to 0$$

completely as $n\to\infty$.

Hence, the complete convergence of the maximal deviation of the density estimate holds on arbitrary closed intervals inside of $[a, b]$. Similar consideration was also given in Woodroofe [15].

In Lemma 1 the modulus of continuity was used only to replace $f_n(s)$ by a polygonal approximation. Thus, the following corollary can be obtained with basically the same proof.

COROLLARY 1. *Let the density function* $f(s)$ *be as stated in Theorem* 1. *Let* $W(x)$ *be a nonnegative weight function which has compact support and integrates to* 1. *If*
(a) $nb^2(n)>n^\delta$ *for some* $\delta>0$, *and*
(b′) $\sup_{-n^{1/p}+a\leq s\leq n^{1/p}+b}|W(s)-U_{n^{2r}}(W)(s)|=o(b(n))$, *then*
$\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n} W\left(\frac{s-X_k}{b(n)}\right)-f(s)\right|\to 0$
completely as $n\to\infty$.

The condition $nb^2(n)>n^\delta$ need not hold for all n but only eventually. Also, the condition can be stated as (a′) $\int_d^\infty x^r \exp(-cxb^2(x))dx < \infty$

for some $d>0$ where $b(x)$ is a function which generates the bandwidths $b(1), b(2), \cdots$ and c is a constant.

In considering mode estimates, assume that the continuous density function $f(s)$ has a unique mode θ, that is,

$$f(\theta) = \max_{-\infty<s<\infty} f(s) .$$

The sample mode θ_n is also assumed to uniquely satisfy

$$f_n(\theta_n) = \max_{-\infty<s<\infty} f_n(s) \qquad \text{for each } n .$$

THEOREM 2. *If the regularity conditions of Theorem* 1 *or Corollary* 1 (*or condition* (b′)) *are satisfied, then*

$$|\theta_n - \theta| \to 0$$

completely as $n\to\infty$.

PROOF. Since $f(s)$ is uniformly continuous and has a unique mode θ, for $\varepsilon>0$ there exists $\eta>0$ such that $|x-\theta|\geqq\varepsilon$ implies that $|f(\theta)-f(x)| \geqq \eta$. Thus, it suffices to show that $f(\theta_n)\to f(\theta)$ completely. But,

$$(2.14) \quad \begin{aligned} |f(\theta_n)-f(\theta)| &\leqq |f(\theta_n)-f_n(\theta_n)|+|f_n(\theta_n)-f(\theta)| \\ &\leqq \sup_{-\infty<s<\infty} |f(s)-f_n(s)| + |\max_{-\infty<s<\infty} f_n(s) - \max_{-\infty<s<\infty} f(s)| \\ &\leqq 2 \sup_{-\infty<s<\infty} |f_n(s)-f(s)| \end{aligned}$$

pointwise for each n. From (2.14) and the complete convergence of $\|f_n - f\|_\infty$, it follows that $|\theta_n-\theta|\to 0$ completely.

3. Comparisons and useful weight functions

Brief comments and comparisons of these results with existing results will be listed in this section. Also, some useful weight functions will be considered.

Nadaraya [7] had the weaker bandwidth condition, $\sum_{n=1}^{\infty} \exp(-rnb^2(n)) < \infty$ for each $r>0$, but required W to be of bounded variation. Woodroofe [15] also considered weight functions with compact support. In addition, his conditions included: $W\in \mathrm{LIP}(\beta)$, $0<\beta\leqq 1$, and $b(n)^{-r}=o(n)$ with $n=o(b(n)^{-\delta})$, $1<r<\delta$. It will be shown that $W\in \mathrm{LIP}(\beta)$, $\beta>0$ or $\beta<-1$, is sufficient for the smoothness condition of Theorem 1(c).

Deheuvels [4] also used the pth moment condition of Theorem 1(b) in relating necessary and sufficient conditions on the bandwidth sequence (namely, $1/b(n)=o(n/\log n)$) and the almost sure convergence of $\sup_s |f_n(s) - f(s)|$ for a Riemann integrable weight function $W(x, y)$. In the nearest neighbor method, Devroye and Wagner [5] required only the condition $nb(n)/\log n \to \infty$ and $b(n)/n \to 0$ in obtaining almost sure convergence for bounded weight functions having compact support. Finally, Silverman [12] required the bandwidth condition of $b(n)\to 0$ and $\log n/nb(n) \to 0$ in obtaining almost sure convergence when W is uniformly continuous and of bounded variation.

For the results of this paper, the weight function $W(x)$ needed to be continuous on its compact support and satisfy a smoothness condition ((b) or (b′)). However, it need not be of bounded variation even if it satisfies a Lipschitz condition (which yields (b) or (b′)). Some useful weight functions for these results will now be listed. First, Epanechnikov's [6] optimal weight function

$$W(x)=\begin{cases} \dfrac{3}{4\sqrt{5}}\left(1-\dfrac{x^2}{5}\right) & \text{if } |x|\leqq\sqrt{5} \\ 0 & \text{otherwise} \end{cases}$$

can be used. In this case, let $a=-5$ and $b=5$. Then $|W(x)-W(y)|\leqq C|x-y|$ and $\omega_W((2b-2a)/b(n)n^r)\leqq C'(1/b(n)n^r)$ for constants C and C'. Thus, condition (c) is easily satisfied.

If the weight function W satisfies a Lipschitz condition of order α, then $|W(x)-W(y)|<M|x-y|^\alpha$ and

$$\omega_W\left(\frac{1}{b(n)n^r}\right)\leqq M\left(\frac{1}{b(n)n^r}\right)^\alpha$$

for some $M>0$. Thus, the bandwidth sequence $b(n)$ must be chosen so that

$$b^{\alpha+1}(n)n^{\alpha r}\to\infty \qquad \text{as } n\to\infty \text{ for some } r>0\,. \tag{3.1}$$

Case 1. $-1\leqq\alpha\leqq 0$. No bandwidth exists for (3.1).

Case 2. $\alpha>0$. If $b(n)=n^{-p}$ for some $p>0$, then r is an integer $\geqq 2p(1+\alpha)/\alpha$. Then, $b^{1+\alpha}(n)n^{\alpha r}\geqq n^{p(1+\alpha)}\to\infty$, and (3.1) is satisfied.

Case 3. $\alpha<-1$. Again, if $b(n)=n^{-p}$ for some $p>0$, then r is an integer $\geqq -p(\alpha+1)/2\alpha$. Then $b^{1+\alpha}(n)n^{\alpha r}\geqq n^{-p(1+\alpha)/2}\to\infty$, and (3.1) is satisfied.

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