ON THE UNIFORM COMPLETE CONVERGENCE OF DENSITY FUNCTION ESTIMATES

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Abstract

Let $f$ be a uniformly continuous density function. Let $W$ be a non-negative weight function which is continuous on its compact support $[a, b]$ and $\int_a^b W(x)dx = 1$. The complete convergence of

$$\sup_{-\infty < s < \infty} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} W\left( \frac{s - X_k}{b(n)} \right) - f(s) \right|$$

to zero is obtained under varying conditions on the bandwidths $b(n)$, support or moments of $f$, and smoothness of $W$. For example, one choice of the weight function for these results is Epanechnikov’s optimal function and $nb^2(n) > n^3$ for some $\delta > 0$. The uniform complete convergence of the mode estimate is also considered.

1. Introduction and preliminaries

The construction of a family of estimates of a density function $f(x)$ and of the mode has been studied by several people. Rosenblatt [9] considered a general class of density estimates:

$$f_n(x) = \frac{1}{nb(n)} \sum_{i=1}^{n} W\left( \frac{x - X_i}{b(n)} \right),$$

where $X_1, \ldots, X_n$ are i.i.d. random variables with continuous density function $f(x)$, $W(x)$ is a bounded integrable weight function such that

$$\int_{-\infty}^{\infty} W(x)dx = 1$$

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and \( b(n) \) is a bandwidth that tends to zero as \( n \to \infty \). Thus, the question arises as to suitable choices of \( W(x) \) and \( b(n) \) so that the estimate function \( f_n(x) \) is optimal (in some sense). The local properties of the estimate function in (1.1) have been studied extensively (see Rosenblatt [10] for a general survey), and a global measure of deviation of the curve \( f_n(x) \) from \( f(x) \) by

\[
\| f_n - f \|_\infty = \sup_{x \in \mathbb{R}} | f_n(x) - f(x) |
\]

has been considered. Parzen [8] showed that if the (true) underlying density function \( f(x) \) is uniformly continuous then \( \| f_n - f \|_\infty \) converges in probability to zero under the following conditions:

(P1) \( \phi_w(t) = \int_{-\infty}^{\infty} e^{itx} W(x) dx \) is absolutely integrable,

(P2) \( \sqrt{n} b(n) \to \infty \) as \( n \to \infty \).

The results of Nadaraya [7], Woodroofe [15], Deheuvels [4], Devroye and Wagner [5], and Silverman [12] on uniform consistency in the strong sense are discussed in the last section for comparison with the results of this paper.

The major results of the paper give a new class of "good" weight functions (which includes the optimal function) under mild conditions on the bandwidth sequence \( b(n) \) where uniform consistency of the estimate \( f_n(x) \) is obtained by the complete convergence (see Stout [13]) of \( \| f_n - f \|_\infty \) to zero (which implies convergence with probability one). The main tools used in obtaining these results will be the smoothness of the weight function and sub-Gaussian techniques.

Throughout this paper, attention will be restricted to a density function which is uniformly continuous (or is continuous on its compact support \([a, b]\)) (see Schuster [11] for a discussion of necessity of uniform continuity) and weight functions \( W(x) \) which satisfy

(i) \( \int_a^b W(x) dx = 1 \) and

(ii) \( W(x) \) is nonnegative and continuous on \([a, b]\) and vanish outside \([a, b]\).

Let \( U_n \) be a polygonal approximating function on the space of continuous functions with domain \([a, b]\), \( C[a, b] \). That is,

\[
g\left( a + \frac{b-a}{n} i \right) = [U_n(g)] \left( a + \frac{b-a}{n} i \right)
\]

for \( i = 0, 1, \ldots, n \) and \( g \in C[a, b] \), and \( U_n \) is linear between the points \( a + (b-a)i/n \) and \( a + (b-a)(i+1)/n \). Recall that the modulus of continuity, \( \omega_b(\delta) \), is defined by Billingsley [2] by

\[
\omega_b(\gamma) = \sup_{|t-s| \leq \gamma} | g(t) - g(s) |
\]
for $\gamma > 0$, $s, t \in [a, b]$, and $g \in C[a, b]$.

**Definition (Chow [3]).** A random variable $X$ is said to be sub-Gaussian if there exists $\alpha \geq 0$ such that

\begin{equation}
E[\exp(tX)] \leq \exp\left(\frac{\alpha^2 t^2}{2}\right) \quad \text{for all } t \in \mathbb{R}.
\end{equation}

If $X$ is sub-Gaussian, then let

$$\tau(X) = \inf \{ \alpha \geq 0 : \text{Inequality (1.4) holds} \}.$$

Some basic properties on sub-Gaussian random variables include:

1. If $P[|X| \leq K] = 1$ and $E X = 0$, then

\begin{equation}
E[\exp(tX)] \leq \exp(K^2 t^2).
\end{equation}

2. If $\tau(X) = \alpha$, then

\begin{equation}
P[|X| \geq \lambda] \leq 2 \exp\left(-\frac{\lambda^2}{2\alpha^2}\right).
\end{equation}

3. The sum of two independent sub-Gaussian random variables is sub-Gaussian.

Finally, a sequence of random variables $\{X_n\}$ is said to converge completely to a random variable $X$ if

\begin{equation}
\sum_{n=1}^{\infty} P[|X_n - X| > \varepsilon] < \infty
\end{equation}

for each $\varepsilon > 0$. Thus, complete convergence implies convergence with probability one by Boole’s inequality.

2. Main results

In this section the complete convergence of $\|f_n - f\|_\infty$ to zero is obtained under conditions on the modulus of continuity of the weight function $W(x)$ and the rate of convergence to zero by the bandwidth $b(n)$. Also, the uniform consistency of the mode estimate is obtained in this setting. The uniform consistency of the estimate $f_n(x)$ (in the complete sense) is accomplished by two lemmas.

**Lemma 1.** (i) If $nb^4(n) > n^5$ for some $\delta > 0$ and (ii) $\int |x|^\delta f(x)dx < \infty$ for some $p > 0$, and (iii) $\omega_n((2b^2 - 2a)/n^rb(n)) = o(b(n))$ for some integer $r > 1/p$, then

\begin{equation}
\sup_{-\infty < s < \infty} \left| f_n(s) - \frac{1}{b(n)} \mathbb{E} W\left(\frac{s - X_n}{b(n)}\right) \right| \to 0
\end{equation}
completely as \( n \to \infty \) where \( f_n(s) \) is defined in (1.1).

**Proof.** From (ii) it follows that

\[
\sum_{n=1}^{\infty} P \left[ |X_i|^p > n \right] < \infty.
\]

Thus, define

\[
Y_n = X_n I_{\{ |X_n| \leq n^{1/p} \}}
\]

for each \( n \). By (2.2) \( P \left[ X_n \neq Y_n \text{ i.o.} \right] = 0 \) since

\[
\sum_{n=1}^{\infty} P \left[ Y_n \neq X_n \right] = \sum_{n=1}^{\infty} P \left[ |X_n| > n^{1/p} \right] < \infty.
\]

Next,

\[
\sup_{-\infty < t < \infty} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} W\left( \frac{s-X_k}{b(n)} \right) - \frac{1}{b(n)} \sum_{k=1}^{n} E W\left( \frac{s-Y_k}{b(n)} \right) \right|
\]

\[
\leq \sup_{-\infty < t < \infty} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} W\left( \frac{s-X_k}{b(n)} \right) - W\left( \frac{s-Y_k}{b(n)} \right) \right|
\]

\[
+ \sup_{-\infty < t < \infty} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} W\left( \frac{s-Y_k}{b(n)} \right) - \frac{1}{b(n)} \sum_{k=1}^{n} E W\left( \frac{s-Y_k}{b(n)} \right) \right|
\]

\[
+ \sup_{-\infty < t < \infty} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} E W\left( \frac{s-Y_k}{b(n)} \right) - \frac{1}{b(n)} \sum_{k=1}^{n} E W\left( \frac{s-X_k}{b(n)} \right) \right|
\].

The first and third terms of the right-hand side of Inequality (2.4) converge to 0 completely by (2.3) and the boundedness of \( W \).

Using the compact support \([a, b]\) for \( W \),

\[
\frac{1}{nb(n)} \sum_{k=1}^{n} W\left( \frac{s-Y_k}{b(n)} \right) = 0 \quad \text{unless } a \leq \frac{s-Y_k}{b(n)} \leq b
\]

for some \( 1 \leq k \leq n \) or unless \(-n^{1/p} + ab(n) \leq s \leq n^{1/p} + bb(n)\). Since \( b(n) \to 0 \), the \( \sup \) in the second term of (2.4) need only be taken over \([-n^{1/p} + a, n^{1/p} + b]\). Let \( \delta = 2(b-a)/n^2 b(n) \) and let \( t_i = -n^{1/p} + a + (i(2n + (b-a))/n^2) \) for \( 1 \leq i \leq n^2 \). Hence, \( t_i - t_i-1 = (2n^{1/p} + (b-a))/n^2 \leq 2(b-a)/n^2 \) for \( n \) large enough. Let \( \tilde{W}_k(s) = W(s-Y_k/b(n)) - E W(s-Y_k/b(n)) \) for each \( k = 1, \ldots, n \). Thus, \( E \tilde{W}_k(s) = 0 \) for each \( s \in [a, b] \) and each \( k \). Furthermore,

\[
\omega_{\tilde{W}_k}(\delta_n) = \sup_{|s-t| \leq \delta_n} |\tilde{W}_k(s) - \tilde{W}_k(t)|
\]

\[
\leq \sup_{|s-t| \leq \delta_n} \left| W\left( \frac{s-Y_k}{b(n)} \right) - W\left( \frac{t-Y_k}{b(n)} \right) \right|
\]

\[
+ \sup_{|s-t| \leq \delta_n} \left| E W\left( \frac{s-Y_k}{b(n)} \right) - E W\left( \frac{t-Y_k}{b(n)} \right) \right|
\]

\[
\leq 2\omega_W(\delta_n).
\]
Hence, \( \omega_w(\delta_n) \leq 2\omega_w(\delta_n) = o(b(n)) \) for each \( k \) from condition (iii). For \( \varepsilon > 0 \) let

\[
A_n = \sup_{-n^{1/p} \leq s \leq n^{1/p} \alpha} \left| -\frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k \left( \frac{s}{b(n)} \right) \right| > \varepsilon
\]

(2.6)

\[
= \left[ \max_{1 \leq i \leq n^{2r}} \sup_{s \in I_i} \left| -\frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k \left( \frac{s}{b(n)} \right) \right| > \varepsilon \right]
\]

where \( I_i = [t_{i-1}, t_i] \). Hence,

\[
A_n \leq \max_{1 \leq i \leq n^{2r}} \left| -\frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k \left( \frac{s}{b(n)} \right) \right| + \max_{1 \leq i \leq n^{2r}} \sup_{s \in I_i} \left| -\frac{1}{nb(n)} \sum_{k=1}^{n} \left[ \tilde{W}_k \left( \frac{s}{b(n)} \right) - \tilde{W}_k \left( \frac{t_i}{b(n)} \right) \right] \right| > \varepsilon .
\]

(2.7)

However,

\[
\max_{1 \leq i \leq n^{2r}} \sup_{s \in I_i} \left| -\frac{1}{nb(n)} \sum_{k=1}^{n} \left[ \tilde{W}_k \left( \frac{s}{b(n)} \right) - \tilde{W}_k \left( \frac{t_i}{b(n)} \right) \right] \right| \leq \frac{2}{b(n)} \omega_w(\delta_n) .
\]

(2.8)

Since \( \omega_w(\delta_n) = o(b(n)) \) by condition (iii), there exists \( N(r) \) such that

\[
A_n \leq \max_{1 \leq i \leq n^{2r}} \left| -\frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k \left( \frac{t_i}{b(n)} \right) \right| > \varepsilon
\]

(2.9)

for all \( n \geq N(r) \). Using the basic properties of sub-Gaussian random variables \( \{ \tilde{W}_k(t_i/b(n)) : k = 1, 2, \ldots \} \) for each \( i \), for each \( n \geq N(r) \)

\[
P(A_n) \leq \sum_{i=1}^{n^{2r}} P \left[ \left| -\frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k \left( \frac{t_i}{b(n)} \right) \right| > \frac{\varepsilon}{2} \right]
\]

\[
\leq \sum_{i=1}^{n^{2r}} P \left( \frac{1}{nb(n)} \sum_{k=1}^{n} \tilde{W}_k \left( \frac{t_i}{b(n)} \right) \right) > \frac{\varepsilon}{2}
\]

\[
\leq n^{2r} 2 \exp \left\{ -\varepsilon^2 / 64 \| W \|_\infty^2 B_n \right\}
\]

where \( \| W \|_\infty = \sup_{s} |W(s)| \) and \( B_n = \sum_{k=1}^{n} (1/nb(n))^2 = 1/nb^2(n) \). To obtain the complete convergence in (2.1), consider

\[
\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{N(r)} P(A_n) + \sum_{n=N(r)+1}^{\infty} P(A_n)
\]

(2.10)

\[
\leq N(r) + \sum_{n=N(r)+1}^{\infty} 2n^{2r} \exp \left( -\frac{\varepsilon^2 nb(n)}{64 \| W \|_\infty^2} \right)
\]

\[
\leq N(r) + \sum_{n=N(r)+1}^{\infty} 2n^{2r} \exp \left( -cn^2 \right)
\]

where \( c = \varepsilon^2 / 64 \| W \|_\infty \). Thus, the series in (2.10) converges by the integral test.
Lemma 2. If the underlying density, \( f \), is uniformly continuous, then

\[
\sup_{-\infty < s < \infty} \left| \frac{1}{b(n)} \mathbb{E} \left( \frac{s - X_i}{b(n)} \right) - f(s) \right| \to 0 \quad \text{as } n \to \infty.
\]

Proof. Since \( f \) is uniformly continuous given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(x) - f(x')| < \varepsilon \) whenever \( |x - x'| < \delta \). Let \( N \) be sufficiently large so that \( |b(n)y| < \delta \) for all \( n \geq N \) and \( y \in [a, b] \). Since \( W(y) = 0 \) for \( y \notin [a, b] \),

\[
\mathbb{E} \left( \frac{s - X_i}{b(n)} \right) - f(s) = \left| \frac{1}{b(n)} \int_{-\infty}^{\infty} W \left( \frac{s - x}{b(n)} \right) f(x) \, dx - f(s) \right|
\]

\[
= \left| \int_{-\infty}^{\infty} W(y) f(s - b(n)y) \, dy - f(s) \right|
\]

\[
= \left| \int_{a}^{b} W(y) [f(s - b(n)y) - f(s)] \, dy \right|
\]

\[
< \varepsilon \int_{a}^{b} W(y) \, dy = \varepsilon
\]

uniformly in \( s \) for all \( n \geq N \). Hence,

\[
\sup_{-\infty < s < \infty} \left| \frac{1}{b(n)} \mathbb{E} \left( \frac{s - X_i}{b(n)} \right) - f(s) \right| \to 0 \quad \text{as } n \to \infty.
\]

Thus, the proof of Theorem 1 is immediate from Lemmas 1 and 2 since for each \( \varepsilon > 0 \),

\[
P \left[ \sup_{-\infty < s < \infty} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} W \left( \frac{s - X_k}{b(n)} \right) - f(s) \right| > \varepsilon \right]
\]

\[
\leq P \left[ \sup_{-\infty < s < \infty} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} \left( W \left( \frac{s - X_k}{b(n)} \right) - \mathbb{E} \mathbb{E} \left( \frac{s - X_k}{b(n)} \right) \right) \right| > \frac{\varepsilon}{2} \right]
\]

\[
+ P \left[ \sup_{-\infty < s < \infty} \left| \frac{1}{b(n)} \mathbb{E} \left( \frac{s - X_i}{b(n)} \right) - f(s) \right| > \frac{\varepsilon}{2} \right]
\]

and each of the terms in (2.13) is a convergent series in \( n \). All of the conditions will be stated in Theorem 1 for easy reference.

Theorem 1. Let \( \{X_n\} \) be independent random variables with the same density function \( f(s) \) which is uniformly continuous. Let \( W(x) \) be a nonnegative weight function which is continuous on its compact support and integrates to 1. If

(a) \( nb^2(n) > \nu \) for some \( \delta > 0 \),

(b) \( \int |x|^p f(x) \, dx < \infty \) for some \( p > 0 \), then
(c) \( \omega_w \left( \frac{2b-2a}{n^b(n)} \right) = o(b(n)) \) for some integer \( r > \frac{1}{p} \), then

\[
\sup_{-\infty < \epsilon < \infty} \left| \frac{1}{nb(n)} \sum_{i=1}^{n} W \left( \frac{s-X_i}{b(n)} \right) - f(s) \right| \to 0
\]

completely as \( n \to \infty \).

The case of the density function having compact support is discussed in Taylor and Cheng [14] and being discontinuous off of its support is not entirely excluded in Theorem 1. The following steps indicate modifications which allows the theory to include a large class of density functions (for example, the uniform densities).

**Step 1.** For an unknown density function which is continuous only on \([a, b]\) and vanishes outside \([a, b]\), there is no change in Lemma 1.

**Step 2.** In Lemma 2 it is easy to verify that

\[
\sup_{a + b(n) \in \mathbb{R}^+} \left| \frac{1}{b(n)} \sum_{i=1}^{n} W \left( \frac{s-X_i}{b(n)} \right) - f(s) \right| \to 0
\]

as \( n \to \infty \) where \( C = \max \{|a|, |b|\} \).

**Step 3.** Combining steps 1 and 2, for each \( \epsilon > 0 \)

\[
\sup_{a + b(n) \in \mathbb{R}^+} \left| \frac{1}{nb(n)} \sum_{i=1}^{n} W \left( \frac{s-X_i}{b(n)} \right) - f(s) \right| \to 0
\]

completely as \( n \to \infty \).

Hence, the complete convergence of the maximal deviation of the density estimate holds on arbitrary closed intervals inside of \([a, b]\). Similar consideration was also given in Woodrooffe [15].

In Lemma 1 the modulus of continuity was used only to replace \( f_n(s) \) by a polygonal approximation. Thus, the following corollary can be obtained with basically the same proof.

**Corollary 1.** Let the density function \( f(s) \) be as stated in Theorem 1. Let \( W(x) \) be a nonnegative weight function which has compact support and integrates to 1. If

(a) \( nb^3(n) \geq n^\delta \) for some \( \delta > 0 \), and

(b') \( \sup_{-n^{1/p} \leq s \leq n^{1/p} + b} |W(s) - U_{n^{1/p}}(W)(s)| = o(b(n)) \), then

\[
\sup_{-\infty < \epsilon < \infty} \left| \frac{1}{nb(n)} \sum_{i=1}^{n} W \left( \frac{s-X_k}{b(n)} \right) - f(s) \right| \to 0
\]

completely as \( n \to \infty \).

The condition \( nb^3(n) \geq n^\delta \) need not hold for all \( n \) but only eventually. Also, the condition can be stated as
(a') \[ \int_{d}^{\infty} x e^{-cx^b(x)} dx < \infty \]
for some \( d > 0 \) where \( b(x) \) is a function which generates the bandwidths \( b(1), b(2), \ldots \) and \( c \) is a constant.

In considering mode estimates, assume that the continuous density function \( f(s) \) has a unique mode \( \theta \), that is,
\[ f(\theta) = \max_{-\infty < s < \infty} f(s). \]
The sample mode \( \theta_n \) is also assumed to uniquely satisfy
\[ f_n(\theta_n) = \max_{-\infty < s < \infty} f_n(s) \quad \text{for each } n. \]

**Theorem 2.** If the regularity conditions of Theorem 1 or Corollary 1 (or condition (b')) are satisfied, then
\[ |\theta_n - \theta| \to 0 \]
completely as \( n \to \infty \).

**Proof.** Since \( f(s) \) is uniformly continuous and has a unique mode \( \theta \), for \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that \( |x - \theta| \geq \varepsilon \) implies that \( |f(\theta) - f(x)| \geq \eta \). Thus, it suffices to show that \( f(\theta_n) \to f(\theta) \) completely. But,
\begin{align*}
(2.14) \quad |f(\theta_n) - f(\theta)| & \leq |f(\theta_n) - f_n(\theta_n)| + |f_n(\theta_n) - f(\theta)| \\
& \leq \sup_{-\infty < s < \infty} |f(s) - f_n(s)| + |f_n(\theta_n) - f(\theta)| \\
& \leq 2 \sup_{-\infty < s < \infty} |f_n(s) - f(s)|
\end{align*}
pointwise for each \( n \). From (2.14) and the complete convergence of \( \|f_n - f\|_\infty \), it follows that \( |\theta_n - \theta| \to 0 \) completely.

3. Comparisons and useful weight functions

Brief comments and comparisons of these results with existing results will be listed in this section. Also, some useful weight functions will be considered.

Nadaraya [7] had the weaker bandwidth condition, \( \sum_{n=1}^{\infty} \exp(-r n b^2(n)) \leq C \) for each \( r > 0 \), but required \( W \) to be of bounded variation. Woodward [15] also considered weight functions with compact support. In addition, his conditions included: \( W \in \text{LIP}(\beta) \), \( 0 < \beta \leq 1 \), and \( b(n)^{-r} = o(n) \) with \( n = o(b(n)^{-r}) \), \( 1 < r < \delta \). It will be shown that \( W \in \text{LIP}(\beta), \beta > 0 \) or \( \beta < -1 \), is sufficient for the smoothness condition of Theorem 1(c).

Deheuvels [4] also used the \( p \)th moment condition of Theorem 1(b) in relating necessary and sufficient conditions on the bandwidth sequence.
(namely, $1/b(n) = o(n/\log n)$) and the almost sure convergence of $\sup_s |f_n(s) - f(s)|$ for a Riemann integrable weight function $W(x, y)$. In the nearest neighbor method, Deveugle and Wagner [5] required only the condition $nb(n)/\log n \to \infty$ and $b(n)/n \to 0$ in obtaining almost sure convergence for bounded weight functions having compact support. Finally, Silverman [12] required the bandwidth condition of $b(n) \to 0$ and $\log n/nb(n) \to 0$ in obtaining almost sure convergence when $W$ is uniformly continuous and of bounded variation.

For the results of this paper, the weight function $W(x)$ needed to be continuous on its compact support and satisfy a smoothness condition ((b) or (b')). However, it need not be of bounded variation even if it satisfies a Lipschitz condition (which yields (b) or (b')). Some useful weight functions for these results will now be listed. First, Epanechnikov's [6] optimal weight function

$$W(x) = \begin{cases} \frac{3}{4\sqrt{5}} \left(1 - \frac{x^2}{5}\right) & \text{if } |x| \leq \sqrt{5} \\ 0 & \text{otherwise} \end{cases}$$

can be used. In this case, let $a = -5$ and $b = 5$. Then $|W(x) - W(y)| \leq C|x-y|$ and $\omega_w((2b-2a)/b(n)n^r) \leq C'(1/b(n)n^r)$ for constants $C$ and $C'$. Thus, condition (c) is easily satisfied.

If the weight function $W$ satisfies a Lipschitz condition of order $\alpha$, then $|W(x) - W(y)| < M|x-y|^{\alpha}$ and

$$\omega_w\left(\frac{1}{b(n)n^r}\right) \leq M\left(\frac{1}{b(n)n^r}\right)^{\alpha}$$

for some $M > 0$. Thus, the bandwidth sequence $b(n)$ must be chosen so that

(3.1) \hspace{1cm} b^{\alpha+}(n)n^{\alpha r} \to \infty \quad \text{as } n \to \infty \text{ for some } r > 0.

**Case 1.** \hspace{1cm} $-1 \leq \alpha \leq 0$. No bandwidth exists for (3.1).

**Case 2.** \hspace{1cm} $\alpha > 0$. If $b(n) = n^{-p}$ for some $p > 0$, then $r$ is an integer $\geq 2p(1+\alpha)/\alpha$. Then, $b^{1+}(n)n^{\alpha r} \geq n^{p(1+\alpha)/\alpha} \to \infty$, and (3.1) is satisfied.

**Case 3.** \hspace{1cm} $\alpha < -1$. Again, if $b(n) = n^{-p}$ for some $p > 0$, then $r$ is an integer $\geq -p(\alpha+1)/2\alpha$. Then $b^{1+}(n)n^{\alpha r} \geq n^{-p(1+\alpha)/2} \to \infty$, and (3.1) is satisfied.
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