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THE ASYMPTOTIC EXPANSION OF THE STEIN ESTIMATORS FOR THE VECTOR CASE*

A. J. VAN DER MERWE AND D. J. DE WAAL

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Summary

Charles Stein established the existence of estimators which dominate the maximum likelihood estimators for the problem of simultaneously estimating the means of three or more random variables.

Since the exact distributions of the Stein estimators are not known and because the distributions are of great importance for people studying confidence sets, it was the purpose of this note to derive the asymptotic distributions, means and variances of the Stein estimators, as well as that of the quadratic loss functions for the vector case.

1. Introduction

Suppose $X_1, \cdots, X_N$ is a sample of N observations from a $N(\theta, \Sigma)$ population where $\Sigma = I_p$ and $\theta(p\times 1)$ is unknown, then a Stein estimator (James and Stein [2]) of θ is defined as

$$\Psi(\bar{X}) = \left(1 - \frac{c}{N\bar{X}'\bar{X}}\right)\bar{X} \tag{1.1}$$

where $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i \sim N(\theta, I_p/N)$, the maximum likelihood estimator of θ, and $c = p-2$.

At the Fourth Berkeley Simposium, James and Stein [2] proved that for $p \geqq 3$, $\Psi(\bar{X})$ is a better estimator than $\bar{X}$, because

$$\mathrm{E}\, N(\Psi(\bar{X}) - \theta)'(\Psi(\bar{X}) - \theta) = p - (p-2)^2\, \mathrm{E}\left(\frac{1}{N\bar{X}'\bar{X}}\right) \leqq p \tag{1.2}$$

where

$$\mathrm{E}\, N(\bar{X} - \theta)'(\bar{X} - \theta) = p\ . \tag{1.3}$$

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Let $\bar{X}=\theta+(1/\sqrt{N})Z$ where $Z\sim N(0, I_p)$ and expanding (1.1) in a series (see Sec. 12) we get for large N that

$$(1.4)\qquad \Psi(\bar{X})=h_0(Z)+\frac{1}{\sqrt{N}}h_1(Z)+\frac{1}{N}h_2(Z)+\frac{1}{N\sqrt{N}}h_3(Z)+\frac{1}{N^2}h_4(Z)$$
$$+\frac{1}{N^2\sqrt{N}}h_5(Z)+\frac{1}{N^3}h_6(Z)+R$$

where

$$(1.5)\qquad h_0(Z)=\theta$$

$$(1.6)\qquad h_1(Z)=Z$$

$$(1.7)\qquad h_2(Z)=\frac{-c\theta}{a_0}$$

$$(1.8)\qquad h_3(Z)=\frac{c}{a_0}(\theta g_1(Z)-Z)$$

$$(1.9)\qquad h_4(Z)=\frac{c}{a_0}\{Zg_1(Z)-\theta(g_1^2(Z)-g_2(Z))\}$$

$$(1.10)\qquad h_5(Z)=\frac{-c}{a_0}\{\theta(2g_1(Z)g_2(Z)-g_1^3(Z))+Z(g_1^2(Z)-g_2(Z))\}$$

$$(1.11)\qquad h_6(Z)=\frac{-c}{a_0}\{\theta(g_2^2(Z)-3g_1^2(Z)g_2(Z)+g_1^4(Z))+Z(2g_1(Z)g_2(Z)-g_1^3(Z))\}$$

and

$$(1.12)\qquad a_0=\theta'\theta$$

$$(1.13)\qquad g_1(Z)=\frac{2Z'\theta}{a_0}$$

$$(1.14)\qquad g_2(Z)=\frac{Z'Z}{a_0}\,.$$

Throughout this paper R will indicate a residual term that consists of order terms in N. These order terms will contain the next power terms in N which can be seen from the series under consideration plus additional terms that may arise, because we have ignored the convergence constraints as explained in Sec. 12.

In the next section we are going to give some expected values for further use. In Sec. 3 the asymptotic characteristic function of $V^*=\sqrt{N}(\Psi(\bar{X})-h_0(Z))$ is derived and in Sec. 4 the asymptotic probability density function of V^* is derived. In Sec. 5 the asymptotic values of

$\mathrm{E}\,\Psi(\bar{X})$ and $\mathrm{Var}\,\Psi(\bar{X})$ are given and in Sec. 6 the quadratic loss function U is defined. In Secs. 7 and 8 the asymptotic characteristic function as well as the asymptotic probability density function of U are derived and in Sec. 9, the asymptotic values of $\mathrm{E}(U)$ and $\mathrm{Var}(U)$ are given.

The asymptotic distributions derived in Secs. 3–8 are obtained by using a method similar to the perturbation method, used by Nagao [3].

In Sec. 10 the Stein estimator $\Psi(\bar{X})$ for the case $\Sigma\neq I_p$ is defined and a method for obtaining the asymptotic results is given.

In Sec. 11 the advantages of the Stein estimator over the maximum likelihood estimator are mentioned.

In Sec. 12 the theoretical justification for the convergence of equation (1.4) is given.

2. Expected values

LEMMA 2.1. *Suppose* $Z\sim N(0, I_p)$, $M(p\times p)$, $\theta(p\times 1)$, $s(p\times 1)$, *then*

$$\mathrm{E}\,(ZZ'MZZ')=2M+I_p\,\mathrm{tr}\,M \tag{2.1}$$

$$\mathrm{E}\,(Z'Z)^2=p(p+2) \tag{2.2}$$

$$\mathrm{E}\,(Z'\theta)^4=3(\theta'\theta)^2 \tag{2.3}$$

$$\mathrm{E}(Z'\theta)^2(Z'Z)=(p+2)(\theta'\theta) \tag{2.4}$$

$$\mathrm{E}\,(s'Z)(Z'\theta)(Z'Z)=(p+2)(s'\theta) \tag{2.5}$$

$$\mathrm{E}\,(Z'\theta)^3(Z's)=3(\theta'\theta)(s'\theta)\;. \tag{2.6}$$

The proofs of equations (2.1)–(2.6) follow easily.

3. On the characteristic function

THEOREM 3.1. *The characteristic function of* $V^*=\sqrt{N}\,(\Psi(\bar{X})-h_0(Z))$ *for large* N *is given by*

$$\phi_{V^*}(it)=e^{-t't/2}\left\{1+\frac{1}{\sqrt{N}}\Phi_1(t)+\frac{1}{N}\Phi_2(t)+\frac{1}{N\sqrt{N}}\Phi_3(t)+\frac{1}{N^2}\Phi_4(t)+R\right\} \tag{3.1}$$

where

$$\Phi_1(t)=\frac{-i(p-2)}{a_0}(t'\theta) \tag{3.2}$$

$$\Phi_2(t)=\frac{(p-2)}{a_0}\left\{(t't)-\frac{1}{2a_0}(p+2)(t'\theta)^2\right\} \tag{3.3}$$

(3.4) $$\Phi_3(t)=\frac{(p-2)i}{a_0^2}\left\{(p-2)(t'\theta)-(p+1)(t't)(t'\theta)+\frac{1}{6a_0}(4+8p+p^2)(t'\theta)^3\right\}$$

(3.5) $$\Phi_4(t)=\frac{(p-2)}{a_0^2}\left\{\frac{p}{2}(t't)^2-\frac{3(p-2)}{2}(t't)-\frac{p}{2a_0}(p+6)(t't)(t'\theta)^2+\frac{1}{a_0}(p+2)(p-2)(t'\theta)^2+\frac{1}{24a_0^2}(60p+18p^2+p^3-8)(t'\theta)^4\right\}.$$

PROOF.

(3.6) $$\begin{aligned}\phi_{V^*}(it)=\mathrm{E}\,e^{it'h_1(Z)}\Big\{&1+\frac{1}{\sqrt{N}}(it'h_2(Z))+\frac{1}{N}\left\{it'h_3(Z)+\frac{1}{2}(it'h_2(Z))^2\right\}\\&+\frac{1}{N\sqrt{N}}\left\{it'h_4(Z)+(it'h_2(Z))(it'h_3(Z))+\frac{1}{6}(it'h_2(Z))^3\right\}\\&+\frac{1}{N^2}\Big\{it'h_5(Z)+(it'h_2(Z))(it'h_4(Z))+\frac{1}{2}(it'h_3(Z))^2\\&+\frac{1}{2}(it'h_2(Z))^2(it'h_3(Z))+\frac{1}{24}(it'h_2(Z))^4\Big\}+R\Big\}.\end{aligned}$$

By using Lemma 2.1 and (3.6) and the fact that

(3.7) $$\mathrm{E}\,e^{it'h_1(Z)}h_i(Z)=e^{-t't/2}\,\mathrm{E}\,h_i(Y+it)\,,\qquad i=1,2,\cdots$$

where $Y\sim N(0, I_p)$ we find that

(3.8) $$\mathrm{E}\,it'h_2(Z)e^{it'h_1(Z)}=-\frac{ci}{a_0}(t'\theta)e^{-t't/2}$$

(3.9) $$\mathrm{E}\,it'h_3(Z)e^{it'h_1(Z)}=\frac{c}{a_0}\left\{(t't)-\frac{2}{a_0}(t'\theta)^2\right\}e^{-t't/2}$$

(3.10) $$\mathrm{E}\,it'h_4(Z)e^{it'h_1(Z)}=\frac{ci}{a_0^2}\left\{(p-2)(t'\theta)-3(t't)(t'\theta)+\frac{4}{a_0}(t'\theta)^3\right\}e^{-t't/2}$$

(3.11) $$\mathrm{E}\,it'h_5(Z)e^{it'h_1(Z)}=\frac{-c}{a_0^2}\left\{\frac{8}{a_0}(t't)(t'\theta)^2-\frac{4}{a_0}(p-2)(t'\theta)^2-\frac{8}{a_0^2}(t'\theta)^4+(p-2)(t't)-(t't)^2\right\}e^{-t't/2}$$

(3.12) $$\mathrm{E}\,(it'h_3(Z))^2e^{it'h_1(Z)}=\frac{c^2}{a_0^2}\left\{\frac{4}{a_0^2}(t'\theta)^4-(t't)+(t't)^2-\frac{4}{a_0}(t't)(t'\theta)^2\right\}e^{-t't/2}.$$

Substituting (3.8)–(3.12) in (3.6) and (3.1) follows.

4. The distribution of V^*

THEOREM 4.1. *The probability density function of V^* for large N is given by*

$$(4.1)\qquad f_{V^*}(v^*)=\left(\frac{1}{2\pi}\right)^{p/2}e^{-v^{*\prime}v^*/2}\left\{1+\frac{1}{\sqrt{N}}\tilde{g}_1(v^*)+\frac{1}{N}\tilde{g}_2(v^*)+\frac{1}{N\sqrt{N}}\tilde{g}_3(v^*)+\frac{1}{N^2}\tilde{g}_4(v^*)+R\right\}\qquad -\infty<v^*<\infty$$

where

$$(4.2)\qquad \tilde{g}_1(v^*)=\frac{-1}{a_0}(p-2)(v^{*\prime}\theta)$$

$$(4.3)\qquad \tilde{g}_2(v^*)=\frac{(p-2)}{a_0}\left\{\frac{1}{2}(p-2)-v^{*\prime}v^*+\frac{1}{2a_0}(p+2)(v^{*\prime}\theta)^2\right\}$$

$$(4.4)\qquad \tilde{g}_3(v^*)=\frac{(p-2)}{a_0^2}\left\{\frac{1}{2}(4p-4-p^2)(v^{*\prime}\theta)+(p+1)(v^{*\prime}v^*)(v^{*\prime}\theta)-\frac{1}{6a_0}(4+8p+p^2)(v^{*\prime}\theta)^3\right\}$$

$$(4.5)\qquad \tilde{g}_4(v^*)=\frac{(p-2)}{a_0^2}\left\{\frac{(p-2)}{8}(p^2-8p+20)+\frac{p}{2}(v^{*\prime}v^*)^2+\frac{1}{2}(p-2)(3-p)(v^{*\prime}v^*)+\frac{1}{24a_0^2}(p^3+18p^2+60p-8)(v^{*\prime}\theta)^4-\frac{p}{2a_0}(6+p)(v^{*\prime}v^*)(v^{*\prime}\theta)^2+\frac{(p-2)}{4a_0}(p^2-12)(v^{*\prime}\theta)^2\right\}.$$

PROOF.

$$(4.6)\qquad f_{V^*}(v^*)=\left(\frac{1}{2\pi}\right)^{p}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi_{V^*}(it)e^{-it'v^*}dt_1\cdots dt_p$$

where $\phi_{V^*}(it)$ is the characteristic function of V^*. By using Lemma 2.1 and the fact that $\mathrm{E}\,k^*(t)e^{-itv^*}=e^{-v^{*\prime}v^*/2}\,\mathrm{E}\,k^*(Y-iv^*)$, the proof follows easily. ($k^*(t)$ is a function in t. The first expectation is taken over t and the second over Y where $Y\sim N(0, I_p)$.)

It is clear that if $N\to\infty$ then $V^*\sim N(0, I_p)$.

5. $\mathrm{E}\,\Psi(\bar{X})$ and $\mathrm{Var}\,\Psi(\bar{X})$

THEOREM 5.1.

$$\text{(5.1)} \qquad \mathrm{E}\,\Psi(\bar{X}) = \mathrm{E}\left\{h_0(Z) + \frac{1}{\sqrt{N}} h_1(Z) + \frac{1}{N} h_2(Z) + \frac{1}{N\sqrt{N}} h_3(Z) + \frac{1}{N^2} h_4(Z) + \frac{1}{N^2\sqrt{N}} h_5(Z) + \frac{1}{N^3} h_6(Z) + R\right\}$$

and

$$\text{(5.2)} \qquad \mathrm{Var}\,\Psi(\bar{X}) = \frac{1}{N}\,\mathrm{Var}\,h_1(Z) + \frac{2}{N^2}\,\mathrm{Cov}\,(h_1(Z), h_3(Z)) + \frac{1}{N^3}\{\mathrm{Var}\,h_3(Z) + 2\,\mathrm{Cov}\,(h_1(Z), h_5(Z))\} + R$$

where

$$\text{(5.3)} \qquad \mathrm{E}\,h_0(Z) = \theta$$

$$\text{(5.4)} \qquad \mathrm{E}\,h_1(Z) = 0$$

$$\text{(5.5)} \qquad \mathrm{E}\,h_2(Z) = -\frac{c}{a_0}\theta$$

$$\text{(5.6)} \qquad \mathrm{E}\,h_3(Z) = 0$$

$$\text{(5.7)} \qquad \mathrm{E}\,h_4(Z) = \frac{c(p-2)}{a_0^2}\theta$$

$$\text{(5.8)} \qquad \mathrm{E}\,h_5(Z) = 0$$

$$\text{(5.9)} \qquad \mathrm{E}\,h_6(Z) = \frac{c(p-2)(p-4)}{a_0^3}\theta$$

$$\text{(5.10)} \qquad \mathrm{Var}\,h_1(Z) = I_p$$

$$\text{(5.11)} \qquad \mathrm{Cov}\,(h_1(Z), h_3(Z)) = \frac{c}{a_0}\left\{\frac{2\theta\theta'}{a_0} - I_p\right\}$$

$$\text{(5.12)} \qquad \mathrm{Var}\,h_3(Z) = \frac{c^2}{a_0^2} I_p$$

$$\text{(5.13)} \qquad \mathrm{Cov}\,(h_1(Z), h_5(Z)) = \frac{-c(p-2)}{a_0^2}\left\{\frac{1}{a_0} 4\theta\theta' - I_p\right\}.$$

PROOF. By using Lemma 2.1 the proof follows easily ($c=p-2$).

6. The quadratic loss function

The quadratic loss function for N large is defined as

$$\text{(6.1)} \qquad U = N(\Psi(\bar{X}) - \theta)'(\Psi(\bar{X}) - \theta)$$

$$(6.2)\qquad =V^{*\prime}V^{*}$$

$$(6.3)\qquad =g_0^*(Z)+\frac{1}{\sqrt{N}}g_1^*(Z)+\frac{1}{N}g_2^*(Z)+\frac{1}{N\sqrt{N}}g_3^*(Z)+\frac{1}{N^2}g_4^*(Z)+R$$

where

$$(6.4)\qquad g_0^*(Z)=h_1(Z)'h_1(Z)$$

$$(6.5)\qquad g_1^*(Z)=2h_1(Z)'h_2(Z)$$

$$(6.6)\qquad g_2^*(Z)=h_2(Z)'h_2(Z)+2h_1(Z)'h_3(Z)$$

$$(6.7)\qquad g_3^*(Z)=2\{h_1(Z)'h_4(Z)+h_2(Z)'h_3(Z)\}$$

$$(6.8)\qquad g_4^*(Z)=h_3(Z)'h_3(Z)+2h_2(Z)'h_4(Z)+2h_1(Z)'h_5(Z)\ .$$

7. The characteristic function of U

THEOREM 7.1. *The characteristic function of U is given by*

$$(7.1)\qquad \phi_U(it)=(1-2it)^{-p/2}\left\{1+\frac{1}{\sqrt{N}}\Phi_1^*(t)+\frac{1}{N}\Phi_2^*(t)+\frac{1}{N\sqrt{N}}\Phi_3^*(t)+\frac{1}{N^2}\Phi_4^*(t)+R\right\}$$

where

$$(7.2)\qquad \Phi_1^*(t)=0$$

$$(7.3)\qquad \Phi_2^*(t)=\frac{c^2}{a_0}(it)-\frac{2c(p-2)}{a_0}(it)(1-2it)^{-1}+\frac{2c^2}{a_0}(it)^2(1-2it)^{-1}$$

$$(7.4)\qquad \Phi_3^*(t)=0$$

$$\begin{aligned}(7.5)\qquad \Phi_4^*(t)=&\frac{2c}{a_0^2}(p-4)(p-2)it(1-2it)^{-2}+\frac{c^2}{a_0^2}(4-p)it(1-2it)^{-1}\\&+\frac{c^4}{2a_0^2}(it)^2-\frac{2c^3}{a_0^2}(p-4)(it)^2(1-2it)^{-1}+\frac{2c^2}{a_0^2}(p-4)^2(it)^2(1-2it)^{-2}\\&+\frac{4c^3}{a_0^2}(4-p)(1-2it)^{-2}(it)^3+\frac{2c^4}{a_0^2}(it)^3(1-2it)^{-1}\\&+\frac{2c^4}{a_0^2}(it)^4(1-2it)^{-2}\ .\end{aligned}$$

PROOF. By using the same method as given in Theorem 3.1, the result follows.

8. The distribution of U

THEOREM 8.1. *The probability density function of U defined in* (6.1) *for large N is given by*

$$f_U(u)=\frac{1}{\Gamma(p/2)2^{p/2}}e^{-u/2}u^{p/2-1}\left\{1+\frac{1}{\sqrt{N}}\tilde{h}_1(u)+\frac{1}{N}\tilde{h}_2(u)+\frac{1}{N\sqrt{N}}\tilde{h}_3(u)+\frac{1}{N^2}\tilde{h}_4(u)+R\right\}\qquad u>0 \tag{8.1}$$

where

$$\tilde{h}_1(u)=0 \tag{8.2}$$

$$\tilde{h}_2(u)=\frac{(p-2)^2}{2a_0}\left(1-\frac{1}{p}u\right) \tag{8.3}$$

$$\tilde{h}_3(u)=0 \tag{8.4}$$

$$\tilde{h}_4(u)=\frac{(p-2)^2}{8a_0^2}\left\{(p^2-8p+20)+\frac{2}{p}u(-p^2+6p-12)+\frac{u^2(p-2)^2}{p(p+2)}\right\}. \tag{8.5}$$

PROOF.

$$f_U(u)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi_U(it)e^{-itu}dit \tag{8.6}$$

and equation (8.1) follows. For further algebraic details see van der Merwe and de Waal [5].

9. E(U) and Var(U)

The risk is defined as the expected value of the quadratic loss function.

THEOREM 9.1. *For the density function*

$$f_U(u)=\tilde{f}_U(u)\left\{1+\frac{1}{\sqrt{N}}\tilde{h}_1(u)+\frac{1}{N}\tilde{h}_2(u)+\frac{1}{N\sqrt{N}}\tilde{h}_3(u)+\frac{1}{N^2}\tilde{h}_4(u)\right\}\qquad u>0 \tag{9.1}$$

we find that

$$\mathrm{E}(U)=p-\frac{(p-2)^2}{Na_0}+\frac{(p-2)^2}{N^2a_0^2}(p-4) \tag{9.2}$$

$$\text{(9.3)} \qquad \operatorname{Var}(U)=2p-\frac{4(p-2)^2}{Na_0}+\frac{4(p-2)^2(p-4)}{N^2a_0^2}-\frac{2(p-2)^4(4-p)}{N^3a_0^3}-\frac{(p-2)^4(p-4)^2}{N^4a_0^4}\,.$$

PROOF. $\mathrm{E}(U^r)=\int_0^\infty u^r f_U(u)du$. For $r=2$, we find that

$$\text{(9.4)} \qquad \mathrm{E}(U^2)=p(p+2)-\frac{2(p-2)^2(p+2)}{Na_0}+\frac{(p-2)^2}{N^2a_0^2}(3p^2-8p-12)$$

$\operatorname{Var}(U)=\mathrm{E}(U^2)-\{\mathrm{E}(U)\}^2$ and (9.3) follows (Ullah [4] derived the first four moments as well as their approximations for the marginal distributions of the Stein estimator).

10. The case $\Sigma\neq I_p$

The Stein estimator, for the case Σ unknown, is defined as

$$\text{(10.1)} \qquad \Psi(\bar{X})=\left(1-\frac{c_1}{N\bar{X}'A^{-1}\bar{X}}\right)\bar{X}$$

where

$$c_1=\frac{p-2}{N-p+2}$$

and

$$A=\sum_{i=1}^{N}(X_i-\bar{X})(X_i-\bar{X})'\sim W(\Sigma,\,N-1)\,.$$

By making use of the fact that $N\bar{X}'A^{-1}\bar{X}\sim N\bar{X}'\bar{X}/S$, where $S\sim\chi^2_{N-p}$ and independent of $\bar{X}$, (see Wijsman [6]), (10.1) can be written as:

$$\text{(10.2)} \qquad \Psi(\bar{X})=\left(1-\frac{c^*}{N\bar{X}'\bar{X}}\right)\bar{X}$$

where

$$\text{(10.3)} \qquad c^*=\frac{(p-2)}{(N-p+2)}S$$

and

$$\text{(10.4)} \qquad \bar{X}=D^{-1}\left(D\theta+\frac{1}{\sqrt{N}}Z\right)$$

where $Z\sim N(0,I_p)$ and $D\Sigma D'=I_p$

$$\text{(10.5)} \qquad a_0=\theta'\Sigma^{-1}\theta\,.$$

By making use of the same methods as in Secs. 1–9 the asymptotic distribution, mean and variance of this estimator as well as that of the quadratic loss function can be obtained.

11. Advantages of the Stein estimator over the maximum likelihood estimator

The advantages of the Stein estimator over the maximum likelihood estimator, is that the variance, risk (expected value of the quadratic loss function) and variance of the quadratic loss function in the case of the Stein estimator, are smaller than in the case of the maximum likelihood estimator.

The distribution of the Stein estimator as well as that of the quadratic loss function will be useful for studying confidence sets.

12. Justification for the use of the inverse binomial series

If we put $\bar{X}=\theta+(1/\sqrt{N})Z$ where $Z\sim N(0, I_p)$, then $\Psi(\bar{X})$ as defined in equation (1.1) can be written as

$$\Psi(\bar{X})=\left(1-\frac{p-2}{Na_0(1+(2/a_0\sqrt{N})Z'\theta+(1/Na_0)Z'Z)}\right)\left(\theta+\frac{1}{\sqrt{N}}Z\right) . \tag{12.1}$$

By making use of the inverse binomial theorem (12.1) can be written as equation (1.4) which will converge, if

$$1<\frac{2}{a_0\sqrt{N}}Z'\theta+\frac{1}{Na_0}Z'Z<1 \tag{12.2}$$

where

$$a_0=\theta'\theta .$$

However by taking expected values in the previous sections, we integrated Z over the entire real space. This can be done if N and θ are large enough because the difference in the integration taken over the region for convergence and the whole space is small, as explained by de Bruijn [1].

By calculation of exact values of the mean and variance of the Stein estimator and by comparing them with the asymptotic values this justification was confirmed. The asymptotic distribution of the quadratic loss function also compared well with the exact distribution obtained from Monte Carlo simulation experiments. See van der Merwe and de Waal [5].

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