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AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF A FUNCTION OF LATENT ROOTS OF THE NONCENTRAL WISHART MATRIX, WHEN $\Omega=O(n)$

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1. Introduction

Let S $(=XX',\ X=(x_{\alpha j}):\ p\times n)$ be a $p\times p$ noncentral Wishart matrix having $W(\Sigma, n; \Omega)$. Here we shall define the matrix of noncentrality parameters Ω by $\Sigma^{-1}MM'/2$ instead of $\Sigma^{-1}MM'$, for $M=(m_{\alpha j})=\mathrm{E}(X)$. In the previous paper [6], we gave the asymptotic expansion of the distribution of the generalized variance under a natural assumption of $\Omega=O(n)$. In this paper we shall derive the asymptotic expansion for the distribution of a function of latent roots of S under the above assumption. The method adopted here is based on a perturbation expansion for latent roots and the work of Nagao [5].

2. Preliminary lemmas

So far as we are concerned with the distribution for latent roots of S, we can take that S has $W(\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_p), n; \Omega)$ and further we assume $\Omega=\Lambda^{-1}MM'/2=m\theta=m\,\mathrm{diag}(\theta_1,\cdots,\theta_p)$, where $m=n-2\Delta$ with $\Delta=O(1)$.

LEMMA 2.1. *Let* $V=(v_{ij})=\sqrt{m}\{S/m-\Lambda(I+2\theta)\}$, *where* $m=n-2\Delta$ *for a correction factor* $\Delta=O(1)$. *Then* $v=(v_{11},\cdots,v_{pp},v_{12},\cdots,v_{p-1,p})$ *converges in law to a* $p(p+1)/2$ *variate normal distribution with mean zero and covariance matrix* Ψ, *where, by putting* $c_\alpha=1+4\theta_\alpha$,

$$(2.1)\qquad \Psi=2\begin{pmatrix} c_1\lambda_1^2 & & & & & \\ & \ddots & & & & \\ & & c_p\lambda_p^2 & & & 0 \\ & & & \frac{1}{4}(c_1+c_2)\lambda_1\lambda_2 & & \\ & & & & \ddots & \\ 0 & & & & & \frac{1}{4}(c_{p-1}+c_p)\lambda_{p-1}\lambda_p \end{pmatrix}.$$

The above proof can be obtained by considering the characteristic function of v.

Jensen [2] showed a similar result above under a different assumption.

LEMMA 2.2. *Let* $v=(v_{11},\cdots,v_{pp},v_{12},\cdots,v_{p-1,p})$ *have a normal distribution with mean zero and covariance matrix* $\Psi=(\psi_{ij;kl})$. *Then we have*

$$\text{(2.2)}\qquad \mathrm{E}\,v_{ij}^2v_{kl}^2=2\psi_{ij;kl}^2+\psi_{ij;ij}\psi_{kl;kl}$$

$$\text{(2.3)}\qquad \mathrm{E}\,v_{ij}^2v_{kk}v_{ll}=2\psi_{ij;kk}\psi_{ij;ll}+\psi_{ij;ij}\psi_{kk;ll}$$

$$\text{(2.4)}\qquad \mathrm{E}\,v_{ii}v_{jj}v_{kk}v_{ll}=\psi_{ii;jj}\psi_{kk;ll}+\psi_{ii;kk}\psi_{jj;ll}+\psi_{ii;ll}\psi_{jj;kk}\,.$$

3. Derivation of asymptotic expansion

Let $l_1>\cdots>l_p$ be latent roots of S/m and $\lambda_1\geqq\cdots\geqq\lambda_p$ be latent roots of Σ. Put $S/m=\Lambda(I+2\theta)+(1/\sqrt{m})V$. If $\tau_\alpha=\lambda_\alpha(1+2\theta_\alpha)$ is simple, the perturbation expansion (see for example Wigner [8], p. 40) for the αth largest root l_α of S/m can be expressed as follows:

$$\text{(3.1)}\qquad l_\alpha=\tau_\alpha+\frac{1}{\sqrt{m}}v_{\alpha\alpha}+\frac{1}{m}\sum\tau_{\alpha\beta}^*v_{\alpha\beta}^2+\frac{1}{m\sqrt{m}}\Big\{\sum\tau_{\alpha\beta}^*\tau_{\alpha\gamma}^*v_{\alpha\beta}v_{\alpha\gamma}v_{\gamma\beta}-v_{\alpha\alpha}\sum\tau_{\alpha\beta}^{*2}v_{\alpha\beta}^2\Big\}+O_p(m^{-2})\,,$$

where the summation $\sum$ stands for the summation with respect to all subscripts appearing in each term except for index α and

$$\text{(3.2)}\qquad \tau_{\alpha\beta}^*=\begin{cases}(\tau_\alpha-\tau_\beta)^{-1} & \text{if } \alpha\neq\beta\\ 0 & \text{if } \alpha=\beta\,.\end{cases}$$

It is noted that the method due to a perturbation expansion was used by Lawley [4], Sugiura [7], Fujikoshi [1] and Konishi [3].

Now we consider an analytic function $f(l_1,\cdots,l_p)$ of $l_1,\cdots,l_p$. Then by using (3.1), we have

$$\text{(3.3)}\qquad f(l_1,\cdots,l_p)=f(\tau_1,\cdots,\tau_p)+\frac{1}{\sqrt{m}}q_0+\frac{1}{m}q_1+\frac{1}{m\sqrt{m}}q_2+O_p(m^{-2})\,,$$

where

$$q_0=\sum v_{\alpha\alpha}f_\alpha$$

$$\text{(3.4)}\qquad q_1=\sum\tau_{\alpha\beta}^*v_{\alpha\beta}^2f_\alpha+\frac{1}{2}\sum v_{\alpha\alpha}v_{\beta\beta}f_{\alpha\beta}$$

$$q_2=\sum\tau_{\alpha\beta}^*\tau_{\alpha\gamma}^*v_{\alpha\beta}v_{\beta\gamma}v_{\gamma\alpha}f_\alpha-\sum\tau_{\alpha\beta}^{*2}v_{\alpha\alpha}v_{\alpha\beta}^2f_\alpha$$

$$+\frac{1}{2}(\sum \tau^{*}_{\beta\gamma}v_{\alpha\alpha}v^2_{\beta\gamma}+\sum \tau^{*}_{\alpha\gamma}v_{\beta\beta}v^2_{\alpha\gamma})f_{\alpha\beta}+\frac{1}{6}\sum v_{\alpha\alpha}v_{\beta\beta}v_{\gamma\gamma}f_{\alpha\beta\gamma}\ .$$

The symbol $\sum$ in (3.4) stands for the summation with respect to all subscripts appearing in each term and from now on we will use this symbol. Also f_α, $f_{\alpha\beta}$ and $f_{\alpha\beta\gamma}$ mean

$$f_\alpha=\frac{\partial}{\partial\tau_\alpha}f(\tau_1,\cdots,\tau_p)\ ,\qquad f_{\alpha\beta}=\frac{\partial^2}{\partial\tau_\alpha\partial\tau_\beta}f(\tau_1,\cdots,\tau_p)\ ,$$

(3.5)

$$f_{\alpha\beta\gamma}=\frac{\partial^3}{\partial\tau_\alpha\partial\tau_\beta\partial\tau_\gamma}f(\tau_1,\cdots,\tau_p)\ .$$

Then from (3.3), the statistic $\sqrt{m}\{f(l_1,\cdots,l_p)-f(\tau_1,\cdots,\tau_p)\}$ converges in law to a normal distribution with mean zero and variance $\tau^2=2\operatorname{tr}(I+4\theta)(\Lambda A)^2$ with $A=\operatorname{diag}(f_1,\cdots,f_p)$. Thus the characteristic function of $\sqrt{m}\{f(l_1,\cdots,l_p)-f(\tau_1,\cdots,\tau_p)\}/\tau$ is given by

$$(3.6)\quad \phi(t)=\mathrm{E}\left[\exp\left(\frac{it}{\tau}q_0\right)\left\{1+\frac{1}{\sqrt{m}}\left(\frac{it}{\tau}\right)q_1+\frac{1}{m}\left[\left(\frac{it}{\tau}\right)q_2+\frac{1}{2}\left(\frac{it}{\tau}\right)^2q_1^2\right]\right\}\right]$$
$$+O(m^{-3/2})\ .$$

Thus we must calculate each term in (3.6). At first the first term is given by

$$(3.7)\quad \mathrm{E}\left[\exp\left[\frac{it}{\tau}q_0\right]\right]$$
$$=\operatorname{etr}\left[-\frac{it}{\tau}\sqrt{m}\,\Lambda A(I+2\theta)\right]\left|I-\frac{2}{\sqrt{m}}\left(\frac{it}{\tau}\right)\Lambda A\right|^{-n/2}$$
$$\times\operatorname{etr}\left[\frac{1}{2}\left(I-\frac{2}{\sqrt{m}}\left(\frac{it}{\tau}\right)\Lambda A\right)^{-1}\Lambda^{-1}MM'-\frac{1}{2}\Lambda^{-1}MM'\right]$$
$$=\exp\left[\frac{(it)^2}{2}\right]\left\{1+\frac{1}{\sqrt{m}}\left[2\Delta\left(\frac{it}{\tau}\right)\operatorname{tr}\Lambda A+\frac{4}{3}\left(\frac{it}{\tau}\right)^3\operatorname{tr}(I+6\theta)(\Lambda A)^3\right]\right.$$
$$+\frac{1}{m}\left[\left(\frac{it}{\tau}\right)^2\{2\Delta^2(\operatorname{tr}\Lambda A)^2+2\Delta\operatorname{tr}(\Lambda A)^2\}+\left(\frac{it}{\tau}\right)^4\right.$$
$$\times\left\{2\operatorname{tr}(I+8\theta)(\Lambda A)^4+\frac{8}{3}\Delta\operatorname{tr}\Lambda A\operatorname{tr}(I+6\theta)(\Lambda A)^3\right\}$$
$$\left.\left.+\frac{8}{9}\left(\frac{it}{\tau}\right)^6[\operatorname{tr}(I+6\theta)(\Lambda A)^3]^2\right]\right\}+O(m^{-3/2})\ .$$

For the second term, expressing q_0 and q_1 as functions of $x_{\alpha j}$ ($\alpha=1,\cdots,p$, $j=1,\cdots,n$), after some algebraic manipulation, it can be expressed as an expectation of a function of new variables $Y=(y_{\alpha j})$ ($\alpha=1,\cdots,p$, $j=1,\cdots,n$), the columns of which are statistically independent and are distributed as normal with mean $(I-(2/\sqrt{m})(it/\tau)\Lambda A)^{-1}M$ and covariance matrix $(\Lambda^{-1}-(2/\sqrt{m})(it/\tau)A)^{-1}$. By noting $\sum_{j=1}^{n} m_{\alpha j}m_{\beta j}=2m\lambda_\alpha\theta_\alpha\delta_{\alpha\beta}$, where the symbol $\delta_{\alpha\beta}$ stands for the Kronecker delta, we have

$$
\begin{aligned}
(3.8)\quad & \mathrm{E}\left[\left(\frac{it}{\tau}\right)q_1 \exp\left[\frac{it}{\tau}q_0\right]\right] \\
&=\exp\left[\frac{(it)^2}{2}\right]\left\{\left(\frac{it}{\tau}\right)\left[\frac{1}{2}\sum \tau^*_{\alpha\beta}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha+\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}\right]\right. \\
&\quad \left.+2\left(\frac{it}{\tau}\right)^3\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}+\frac{1}{\sqrt{m}}\sum_{\alpha=1}^{3} g_{2\alpha}\left(\frac{it}{\tau}\right)^{2\alpha}\right\}+O(m^{-1}),
\end{aligned}
$$

where $c_\alpha=1+4\theta_\alpha$ and

$$
\begin{aligned}
g_2&=2\Delta(\mathrm{tr}\,\Lambda A)\left\{\frac{1}{2}\sum \tau^*_{\alpha\beta}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha+\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}\right\} \\
&\quad +\sum \tau^*_{\alpha\beta}\lambda_\alpha^2\lambda_\beta(2c_\alpha+c_\beta-1)f_\alpha^2+\sum \tau^*_{\alpha\beta}\lambda_\alpha\lambda_\beta^2(c_\alpha+2c_\beta-1)f_\alpha f_\beta \\
&\quad +2\sum \lambda_\alpha^3(3c_\alpha-1)f_\alpha f_{\alpha\alpha}+4\Delta\sum \lambda_\alpha^2\lambda_\beta c_\alpha f_\alpha f_{\alpha\beta}\,,
\end{aligned}
$$

$$
\begin{aligned}
(3.9)\quad g_4&=\frac{4}{3}[\mathrm{tr}\,(I+6\theta)(\Lambda A)^3]\left\{\frac{1}{2}\sum \tau^*_{\alpha\beta}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha+\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}\right\} \\
&\quad +4\Delta(\mathrm{tr}\,\Lambda A)\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}+4\sum \lambda_\alpha^3\lambda_\beta^2(3c_\alpha-1)c_\beta f_\alpha^2 f_\beta f_{\alpha\beta}\,, \\
g_6&=\frac{8}{3}[\mathrm{tr}\,(I+6\theta)(\Lambda A)^3]\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}\,.
\end{aligned}
$$

For the third term, Lemmas 2.1 and 2.2 are useful to calculate them. By Lemma 2.1, the third term can be rewritten as an expectation of a function of $p(p+1)/2$ new variables $W=(w_{11},\cdots,w_{pp},w_{12},\cdots,w_{p-1,p})$ distributed as normal with mean $(2(it/\tau)\lambda_1^2c_1f_1,\cdots,2(it/\tau)\lambda_p^2c_pf_p,0,\cdots,0)$ and covariance matrix Ψ in (2.1). Thus we have

$$
\begin{aligned}
(3.10)\quad & \mathrm{E}\left[\left(\frac{it}{\tau}\right)q_2 \exp\left[\frac{it}{\tau}q_0\right]\right] \\
&=\exp\left[\frac{(it)^2}{2}\right]\left\{\left(\frac{it}{\tau}\right)^2[\sum \tau^*_{\beta\gamma}\lambda_\alpha^2\lambda_\beta\lambda_\gamma c_\alpha(c_\alpha+c_\beta)f_\alpha f_{\alpha\beta}\right. \\
&\quad +2\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\beta f_{\alpha\alpha\beta}+\sum \tau^{*2}_{\alpha\beta}\lambda_\alpha^3\lambda_\beta c_\alpha(c_\alpha+c_\beta)f_\alpha(f_\beta-f_\alpha)] \\
&\quad \left.+\left(\frac{it}{\tau}\right)^4\frac{4}{3}\sum \lambda_\alpha^2\lambda_\beta^2\lambda_\gamma^2 c_\alpha c_\beta c_\gamma f_\alpha f_\beta f_\gamma f_{\alpha\beta\gamma}\right\}+O(m^{-1/2})
\end{aligned}
$$

and

$$
(3.11)\quad \mathrm{E}\left[\frac{1}{2}\left(\frac{it}{\tau}\right)^2 q_1^2 \exp\left[\frac{it}{\tau}q_0\right]\right]=\exp\left[\frac{(it)^2}{2}\right]\sum_{\alpha=1}^{3} g'_{2\alpha}\left(\frac{it}{\tau}\right)^{2\alpha}+O(m^{-1/2})\,,
$$

where

$$g_2'=\frac{1}{2}\left\{\frac{1}{2}\sum \tau_{\alpha\beta}^{*}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha+\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}\right\}^2$$
$$+\frac{1}{4}\sum \tau_{\alpha\beta}^{*2}\lambda_\alpha^2\lambda_\beta^2(c_\alpha+c_\beta)^2 f_\alpha(f_\alpha-f_\beta)+\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_{\alpha\beta}^2\ ,$$

(3.12) $$g_4'=\sum \tau_{\alpha\beta}^{*}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha \sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}$$
$$+2\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}+4\sum \lambda_\alpha^2\lambda_\beta^2\lambda_\gamma^2 c_\alpha c_\beta c_\gamma f_\beta f_\gamma f_{\alpha\beta} f_{\alpha\gamma}\ ,$$

$$g_6'=2\{\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}\}^2 .$$

Therefore the characteristic function $\phi(t)$ in (3.6) is given by

(3.13) $$\phi(t)=\exp\left[\frac{(it)^2}{2}\right]\left\{1+\frac{1}{\sqrt{m}}\left[\left(\frac{it}{\tau}\right)\left\{\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}\right.\right.\right.$$
$$\left.+\frac{1}{2}\sum \tau_{\alpha\beta}^{*}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha+2\Delta\sum \lambda_\alpha f_\alpha\right\}$$
$$\left.+\left(\frac{it}{\tau}\right)^3\left\{\frac{4}{3}\operatorname{tr}(I+6\theta)(\Lambda A)^3+2\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}\right\}\right]$$
$$\left.+\frac{1}{m}\sum_{\alpha=1}^{3}\left(\frac{it}{\tau}\right)^{2\alpha}h_{2\alpha}\right\}+O(m^{-3/2})\ ,$$

where h_2, h_4 and h_6 are given by

$$h_2=\frac{1}{2}\left\{\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}+\frac{1}{2}\sum \tau_{\alpha\beta}^{*}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha+2\Delta\sum \lambda_\alpha f_\alpha\right\}^2$$
$$+\sum \tau_{\alpha\beta}^{*}\lambda_\alpha^2\lambda_\beta(2c_\alpha+c_\beta-1)f_\alpha^2+\sum \tau_{\alpha\beta}^{*}\lambda_\alpha\lambda_\beta^2(c_\alpha+2c_\beta-1)f_\alpha f_\beta$$
$$+\sum \tau_{\beta\gamma}^{*}\lambda_\alpha^2\lambda_\beta\lambda_\gamma c_\alpha(c_\beta+c_\gamma)f_\alpha f_{\alpha\beta}+\frac{1}{4}\sum \tau_{\alpha\beta}^{*2}\lambda_\alpha^2\lambda_\beta^2(c_\alpha+c_\beta)^2$$
$$\times f_\alpha(f_\alpha-f_\beta)+\sum \tau_{\alpha\beta}^{*2}\lambda_\alpha^3\lambda_\beta c_\alpha(c_\alpha+c_\beta)f_\alpha(f_\beta-f_\alpha)$$
$$+\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta(f_{\alpha\beta}^2+2f_\beta f_{\alpha\alpha\beta})+2\sum \lambda_\alpha^3(3c_\alpha-1)f_\alpha f_{\alpha\alpha}$$
$$+4\Delta\sum \lambda_\alpha^2\lambda_\beta c_\alpha f_\alpha f_{\alpha\beta}+2\Delta\sum \lambda_\alpha^2 f_\alpha^2\ ,$$

(3.14) $$h_4=2\operatorname{tr}(I+8\theta)(\Lambda A)^4+\frac{4}{3}[\operatorname{tr}(I+6\theta)(\Lambda A)^3]\left\{\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}\right.$$
$$\left.+\frac{1}{2}\sum \tau_{\alpha\beta}^{*}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha+2\Delta\sum \lambda_\alpha f_\alpha\right\}+\sum \tau_{\alpha\beta}^{*}\lambda_\alpha\lambda_\beta(c_\alpha+c_\beta)f_\alpha$$
$$\times\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}+4\sum \lambda_\alpha^3\lambda_\beta^2(3c_\alpha-1)c_\beta f_\alpha^2 f_\beta f_{\alpha\beta}$$
$$+\frac{4}{3}\sum \lambda_\alpha^2\lambda_\beta^2\lambda_\gamma^2 c_\alpha c_\beta c_\gamma f_\beta f_\gamma(f_\alpha f_{\alpha\beta\gamma}+3f_{\alpha\beta}f_{\alpha\gamma})$$
$$+2\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha}\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}+4\Delta\sum \lambda_\alpha f_\alpha\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}\ ,$$

$$h_6=\frac{8}{9}\left\{\operatorname{tr}(I+6\theta)(\Lambda A)^3+\frac{3}{2}\sum \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}\right\}^2 .$$

Thus we have the following theorem:

THEOREM 3.1. *Let S be a noncentral Wishart matrix having* $W(\Sigma, n; \Omega)$, *and* $l_1 > \cdots > l_p$ *and* $\lambda_1 \geqq \cdots \geqq \lambda_p$ *be latent roots of* S/m *and* Σ, *respectively. For any analytic function* $f(l_1, \cdots, l_p)$, *if* $\tau_\alpha = \lambda_\alpha(1+2\theta_\alpha)$ *is simple, then the distribution function of* $f^* = \sqrt{m}\{f(l_1, \cdots, l_p) - f(\tau_1, \cdots, \tau_p)\}/\tau$ *can be expanded for large* $m = n - 2\Delta$ *under* $\Omega = m\theta$ *as follows:*

$$(3.15)\quad \mathrm{P}(f^* \leqq x) = \Phi(x) - \frac{1}{\sqrt{m}}\left\{\left[\sum \lambda_\alpha^2 c_\alpha f_{\alpha\alpha} + \frac{1}{2}\sum \tau_{\alpha\beta}^* \lambda_\alpha \lambda_\beta (c_\alpha + c_\beta) f_\alpha + 2\Delta \sum \lambda_\alpha f_\alpha\right]\Phi^{(1)}(x)/\tau + \left[\frac{4}{3}\,\mathrm{tr}\,(I+6\theta)(\Lambda A)^3 + 2\sum \lambda_\alpha^2 \lambda_\beta^2 c_\alpha c_\beta f_\alpha f_\beta f_{\alpha\beta}\right]\Phi^{(3)}(x)/\tau^3\right\} + \frac{1}{m}\sum_{\alpha=1}^{3} h_{2\alpha}\Phi^{(2\alpha)}(x)/\tau^{2\alpha} + O(m^{-3/2}),$$

where $\Phi^{(j)}(x)$ *stands for the jth derivative of the standard normal distribution function* $\Phi(x)$, *the symbol* $\sum$ *means the summation with respect to all subscripts appearing in each term and the coefficients* $h_{2\alpha}$ *with* $c_\alpha = 1 + 4\theta_\alpha$ *are given by* (3.14). *The asymptotic variance is* $\tau^2 = 2\,\mathrm{tr}\,(I+4\theta)\cdot(\Lambda A)^2$ *with* $\Lambda = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_p)$ *and* $A = \mathrm{diag}\,(f_1, \cdots, f_p)$ *and* f_α, $f_{\alpha\beta}$ *and* $f_{\alpha\beta\gamma}$ *are given by* (3.5).

In particular, in case $\theta = 0$, the above result agrees with Konishi [3] after some calculation.

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