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STEPWISE TEST PROCEDURES AND APPROXIMATE CHI-SQUARE ANALYSIS

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Abstract

Let an overall null hypothesis H be factored in a certain stepwise manner into k subhypotheses as $H=\bigcap_{i=1}^{k} H_{i|1,\cdots,i-1}$. Suppose the test statistic w for H be correspondingly expressed as $w=w_1 w_2\cdots w_k$ where w_i is the test statistic for H_i. We consider the case where the Box method [2] is applicable for the distributions of w and w_i's. If w_i's are independent under H, we obtain a stepwise test procedure for H on the basis of an approximate chi-square analysis. To demonstrate the procedure of this sort, the testing hypotheses of equality of several covariance matrices and of the multiple independence are discussed. Finally the related asymptotic distributions are shortly noted.

1. Introduction

We concern in this paper with the simultaneous test procedures performed step by step. Suppose for example an experimental center is interested in testing an overall hypothesis concerning with the homogeneity of experiments conducted at $k+1$ different experimental spots. If the data arrive the center in different times over a certain period, the center might want to conduct the test stepwise at each time when the data arrive there from each experimental spot so as to complete the overall test with the arrival of the last data. In some other cases as an example given by Roy and Bargmann [8], the experimenter might be interested in the step-down test for the multiple independence in order to find out which part of components of a vector variate differs from others.

Suppose an overall null hypothesis H be expressed as

$$(1.1)\qquad H=H_1\cap H_{2|1}\cap\cdots\cap H_{k|1,\cdots,k-1}\ , \qquad (H_{1|0}=H_1)$$

Key words: Stepwise test procedure, approximate chi-square analysis, factorization of hypothesis and statistic, equality of covariance matrices, multiple independence.

where $H_{i|1,\dots,i-1}$ ($i=1,\cdots,k$) is the conditional hypothesis given preceding $i-1$ hypotheses are all true. Suppose we have a situation such that the test statistic w for H is correspondingly factored as

$$w=w_1 w_2 \cdots w_k .$$

If $w_1,\cdots,w_k$ are statistically independent when H is true, then the overall test with a preassigned significance level α ($0<\alpha<1$) will be performed by a series of tests of subhypotheses $H_{i|1,\dots,i-1}$ based on w_i in the following manner: we first test H_1 based on w_1 with a significance level α_1. If H_1 is rejected, then we reject H. If H is accepted, we proceed to test $H_{2|1}$ based on w_2 with significance level α_2. Proceeding in this way, H is rejected immediately after the first rejection of a subhypothesis found in the sequence of tests. The overall hypothesis H is accepted when and only when all subhypotheses are accepted. For this case, α_i's must be chosen so that $\alpha=1-\prod_{i=1}^{k}(1-\alpha_i)$. When w and w_i's are the (modified) likelihood ratio criteria for the corresponding hypotheses, we shall be able to make use of the Box method (Box [2]; Anderson [1], Section 8.6.1) to obtain the asymptotic distribution of those criteria and to obtain a stepwise test procedure based on a good approximation to the analysis of the chi-square.

The purpose of this paper is to discuss two testing hypotheses in the multivariate analysis along this line;

(i) test for the equality of covariance matrices of $k+1$ p-variate normal populations $N_p(\mu_i,\Sigma_i)$, $i=1,\cdots,k+1$, i.e., $H:\ \Sigma_1=\Sigma_2=\cdots=\Sigma_{k+1}$,

(ii) test for the multiple independence of $k+1$ sets of components in a normal random vector.

2. Testing the equality of covariance matrices

Consider the test of the hypothesis

$$H:\ \Sigma_1=\Sigma_2=\cdots=\Sigma_{k+1}\,(=\Sigma) \tag{2.1}$$

on the basis of random samples $\{x_1^{(i)}, x_2^{(i)},\cdots, x_{N_i}^{(i)}\}$, $i=1,\cdots,k+1$ drawn from the $k+1$ p-variate normal populations $N_p(\mu_i,\Sigma_i)$, $i=1,\cdots,k+1$, respectively. It is well known that the modified likelihood ratio criterion (LR-criterion) is

$$w=\prod_{i=1}^{k+1}\left(\frac{n}{n_i}\right)^{pn_i/2}\frac{\prod_{i=1}^{k+1}|V_i|^{n_i/2}}{|V|^{n/2}} \tag{2.2}$$

where $n_i=N_i-1$, $n=n_1+\cdots+n_{k+1}$, $V=V_1+\cdots+V_{k+1}$, $V_i=\sum_{\alpha=1}^{N_i}(x_\alpha^{(i)}-\bar{x}^{(i)})\cdot$

$(x_\alpha^{(i)}-\bar{x}^{(i)})'$, and $\bar{x}^{(i)}=(1/N_i)\sum_{\alpha=1}^{N_i} x_\alpha^{(i)}$. The asymptotic expansion for the null distribution of w is also well known (Anderson [1], Section 10.5); that is, under the condition that $d_i=n_i/n$, $i=1,\cdots,k+1$ are fixed

$$(2.3)\qquad -2\tau\log w=-2\left[1-\left(\sum_{i=1}^{k+1}\frac{1}{n_i}-\frac{1}{n}\right)\frac{2p^2+3p-1}{6k(p+1)}\right]\log w$$

is asymptotically distributed according to the chi-square distribution with $f=kp(p+1)/2$ degrees of freedom. More accurately we have

$$(2.4)\qquad \mathrm{P}(-2\tau\log w\leqq z)=\mathrm{P}(\chi_f^2\leqq z)+\gamma_2[\mathrm{P}(\chi_{f+4}^2\leqq z)-\mathrm{P}(\chi_f^2\leqq z)]+O(n^{-3})$$

where

$$(2.5)\qquad \gamma_2=\frac{p(p+1)}{48\tau^2}\left[(p-1)(p+2)\left(\sum_{i=1}^{k+1}\frac{1}{n_i^2}-\frac{1}{n^2}\right)-6k(1-\tau)^2\right].$$

Roy [7] and Krishnaiah ([4], [5]) have considered the simultaneous test procedure for H in several ways including the step-down method. We here consider a step-by-step procedure with respect to samples in the following way: let

$$(2.6)\qquad \begin{aligned} &H_1:\ \Sigma_1=\Sigma_2\,,\\ &H_{2|1}:\ \Sigma_1=\Sigma_2=\Sigma_3 \text{ given } H_1\,,\\ &\cdots\cdots\cdots\cdots\\ &H_{k|1,\cdots,k-1}:\ \Sigma_1=\cdots=\Sigma_k=\Sigma_{k+1} \text{ given } H_{k-1|1,\cdots,k-2}\,,\end{aligned}$$

then we have the relation (1.1). Let

$$(2.7)\qquad \begin{aligned} &V_{(i)}=V_1+\cdots+V_i\,, && V_{(1)}=V_1\,, && V_{(k+1)}=V\,,\\ &n_{(i)}=n_1+\cdots+n_i\,, && n_{(1)}=n_1\,, && n_{(k+1)}=n\,,\end{aligned}$$

and

$$(2.8)\qquad w_i=c_i\frac{|V_{(i)}|^{n_{(i)}/2}|V_{i+1}|^{n_{i+1}/2}}{|V_{(i+1)}|^{n_{(i+1)}/2}}\,,\qquad i=1,\cdots,k$$

where

$$(2.9)\qquad c_i=\left[\frac{n_{(i+1)}}{n_{(i)}}\right]^{pn_{(i)}/2}\left[\frac{n_{(i+1)}}{n_{i+1}}\right]^{pn_{i+1}/2}\,,\qquad i=1,\cdots,k\,.$$

Then we can easily see the following lemmas:

LEMMA 2.1. $w=w_1w_2\cdots w_k$.

LEMMA 2.2. *The statistic w_i is the modified LR-criterion for testing the hypothesis $H_{i|1,\cdots,i-1}$: $\Sigma_{i+1}=\Sigma$ on the basis of $i+1$ random samples* *drawn from* $N_p(\mu_1, \Sigma), \cdots, N_p(\mu_i, \Sigma)$ *and* $N_p(\mu_{i+1}, \Sigma_{i+1})$, *respectively.*

Hence $0<w_i<1$, so the distribution of w_i is uniquely determined by its moments. We can also prove the independence of w_i's.

LEMMA 2.3. *Under the overall null hypothesis* H, w_i, $i=1,\cdots,k$ *are statistically independent.*

PROOF. For $i<j$ and for any non-negative integers h_i and h_j, consider

$$\begin{aligned} \text{(2.10)} \quad \mathrm{E}(w_i^{h_i} w_j^{h_j}) = c_i^{h_i} c_j^{h_j} \mathrm{E}\,[&|V_1+\cdots+V_i|^{h_i n_{(i)}/2} |V_{i+1}|^{h_i n_{i+1}/2} \\ &\cdot |V_1+\cdots+V_{i+1}|^{-h_i n_{(i+1)}/2} |V_1+\cdots+V_j|^{h_j n_{(j)}/2} \\ &\cdot |V_{j+1}|^{h_j n_{j+1}/2} |V_1+\cdots+V_{j+1}|^{-h_j n_{(j+1)}/2}] \end{aligned}$$

and we wish to show that $\mathrm{E}(w_i^{h_i} w_j^{h_j}) = \mathrm{E}(w_i^{h_i})\,\mathrm{E}(w_j^{h_j})$ under H. Let $w_p(U; m, \Sigma)$ be the density of the Wishart matrix U with m degrees of freedom and covariance matrix Σ. Since V_α's are independent Wishart matrices with a common covariance matrix Σ under H, $V_{(i)}=V_1+\cdots+V_i$ is also Wishart matrix with $n_{(i)}$ degrees of freedom and covariance matrix Σ. The expectation in (2.10) is calculated on the basis of the distributions of $V_{(i)}, V_{i+1}, \cdots, V_{j+1}$. We combine the powers of $|V_{(i)}|$ and $|V_{i+1}|$ in the two places in that expectation for each of them to obtain

$$\begin{aligned} \text{(2.11)} \quad \mathrm{E}(w_i^{h_i} w_j^{h_j}) = c_i^{h_i} c_j^{h_j} &\frac{k[p, n_{(i)}, \Sigma]}{k[p, n_{(i)}(1+h_i), \Sigma]} \frac{k[p, n_{i+1}, \Sigma]}{k[p, n_{i+1}(1+h_i), \Sigma]} \\ &\cdot \mathrm{E}\,[V_{(i)}+V_{i+1}|^{-h_i n_{(i+1)}/2} |V_{(i)}+V_{i+1}+\cdots+V_j|^{h_j n_{(j)}/2} \\ &\cdot |V_{(i)}+V_{i+1}+\cdots+V_{j+1}|^{-h_j n_{(j+1)}/2} |V_{j+1}|^{h_j n_{j+1}/2}]\,, \end{aligned}$$

where $k(p, m, \Sigma)=[2^{pm/2}\Gamma_p(m/2)|\Sigma|^{m/2}]^{-1}$, the normalizing constant of Wishart density $w_p(U; m, \Sigma)$ and the expectation is now with respect to $w_p\{V_{(i)}; n_{(i)}(1+h_i), \Sigma\}$, $w_p\{V_{i+1}; n_{i+1}(1+h_i), \Sigma\}$, $w_p\{V_\alpha; n, \Sigma\}$, $\alpha=i+2, \cdots, j+1$. Since $V_{(i)}+V_{i+1}=V_{(i+1)}$ is again the Wishart matrix with the density $w_p\{V_{(i+1)}; n_{(i+1)}(1+h_i), \Sigma\}$, the similar calculation as for (2.11) gives

$$\begin{aligned} \text{(2.12)} \quad \mathrm{E}(w_i^{h_i} w_j^{h_j}) = c_i^{h_i} c_j^{h_j} &\frac{k[p, n_{(i)}, \Sigma]}{k[p, n_{(i)}(1+h_i), \Sigma]} \frac{k[p, n_{i+1}, \Sigma]}{k[p, n_{i+1}(1+h_i), \Sigma]} \\ &\cdot \frac{k[p, n_{(i+1)}(1+h_i), \Sigma]}{k[p, n_{(i+1)}, \Sigma]} \\ &\cdot \mathrm{E}\,[|V_{(i+1)}+V_{i+2}+\cdots+V_j|^{h_j n_{(j)}/2} |V_{j+1}|^{h_j n_{j+1}/2} \\ &\cdot |V_{(i+1)}+V_{i+2}+\cdots+V_{j+1}|^{-h_j n_{(j+1)}/2}] \\ =\, &\mathrm{E}(w_i^{h_i})\,\mathrm{E}(w_j^{h_j})\,. \end{aligned}$$

Since the joint distribution of (w_i, w_j), and the marginal distributions of w_i and w_j are uniquely determined by their moments, it follows from the above that w_i and w_j are statistically independent. Thus we have proved that w_i's are pairwise independent, which implies that $w_1, \cdots, w_k$ are independent. Q.E.D.

By Lemma 2.2 and the Box method, the statistic

$$-2\tau_i \log w_i = -2\left[1-\left(\frac{1}{n_{(i)}}+\frac{1}{n_{i+1}}-\frac{1}{n_{(i+1)}}\right)\frac{2p^2+3p-1}{6(p+1)}\right]\log w_i \tag{2.13}$$

is approximately distributed according to the chi-square distribution with $f_i = p(p+1)/2$ degrees of freedom. More accurate approximation can be obtained if we use the further expansion for the distribution of $-2\tau_i \log w_i$.

Let $d(\alpha)$ and $d(\alpha_i)$ be the upper 100α and $100\alpha_i$ percent points of $-2\tau \log w$ and $-2\tau_i \log w_i$, respectively, where $\prod_{i=1}^{k}(1-\alpha_i)=1-\alpha$. Then

$$\begin{aligned} &P(-2\tau_i \log w_i \leqq d(\alpha_i),\ i=1,\cdots,k \mid H) \\ &= \prod_{i=1}^{k} P(-2\tau_i \log w_i \leqq d(\alpha_i) \mid H) \\ &= \prod_{i=1}^{k}(1-\alpha_i) = 1-\alpha = P(-2\tau \log w \leqq d(\alpha) \mid H) . \end{aligned} \tag{2.14}$$

If we use the chi-square approximation, then $d(\alpha)=\chi^2_{kp(p+1)/2}(\alpha)$ and $d(\alpha_i) = \chi^2_{p(p+1)/2}(\alpha_i)$, where $\chi^2_m(\beta)$ is the upper 100β percent point of the chi-square distribution with m degrees of freedom. The stepwise test procedure with the overall significance level α is then given by

$$\begin{aligned} &\text{Accept } H \text{ if } \chi^2_{(i)} \equiv -2\tau_i \log w_i \leqq \chi^2_{p(p+1)/2}(\alpha_i) \text{ for all } i=1,\cdots,k , \\ &\text{Reject } H \text{ otherwise} . \end{aligned} \tag{2.15}$$

For this case we can summarize the test procedure in Table 2.1. In the table, $c=\prod_{i=1}^{k+1}(n/n_i)^{pn_i/2}$ and c_i, τ, τ_i are given by (2.9), (2.3), and (2.14), respectively. It is noted that

$$\begin{aligned} f=\frac{1}{2}kp(p+1)=f_1+\cdots+f_k , \qquad c=\prod_{i=1}^{k} c_i , \\ \tau=\frac{1}{k}(\tau_1+\cdots+\tau_k) . \end{aligned} \tag{2.16}$$

Remark. Gleser and Olkin [3] have given the asymptotic expansion formula for the null distribution of the product of k independent random variables $v_1, \cdots, v_k$ $(0 \leqq v_i \leqq 1,\ i=1,\cdots,k)$, each of which has the

Table 2.1 Analysis of the approximate chi-square for the test of equality of covariance matrices

Hypotheses	w_i	$\chi^2_{(i)}$	f_i (d.f.)
H_1: $\Sigma_1=\Sigma_2$	$c_1\dfrac{\|V_1\|^{n_1/2}\|V_2\|^{n_2/2}}{\|V_1+V_2\|^{(n_1+n_2)/2}}$	$-2\tau_1 \log w_1$	$\frac{1}{2}p(p+1)$
$H_{2\|1}$: $\Sigma_1=\Sigma_2=\Sigma_3$ given $\Sigma_1=\Sigma_2$	$c_2\dfrac{\|V_{(2)}\|^{n_{(2)}/2}\|V_3\|^{n_3/2}}{\|V_{(3)}\|^{n_{(3)}/2}}$	$-2\tau_2 \log w_2$	$\frac{1}{2}p(p+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H_{k\|1,\dots,k-1}$: $\Sigma_1=\cdots=\Sigma_{k+1}$ given $\Sigma_1=\cdots=\Sigma_k$	$c_k\dfrac{\|V_{(k)}\|^{n_{(k)}/2}\|V_{k+1}\|^{n_{k+1}/2}}{\|V_{(k+1)}\|^{n_{(k+1)}/2}}$	$-2\tau_k \log w_k$	$\frac{1}{2}p(p+1)$
H: $\Sigma_1=\cdots=\Sigma_{k+1}$	$c\dfrac{\|V_1\|^{n_1/2}\cdots\|V_{k+1}\|^{n_{k+1}/2}}{\|V_1+\cdots+V_{k+1}\|^{(n_1+\cdots+n_{k+1})/2}}$	$-2\tau \log w$	$\frac{1}{2}kp(p+1)$

moments of the form such that the Box method is applicable. Although the direction of consideration is opposite, we can obtain the same test procedure as the one obtained above by employing the Gleser and Olkin formulae, once we prove the independence of w_i's.

3. The stepwise test procedure for the multiple independence

Let $x\sim N_p(\mu, \Sigma)$ and $V\sim W_p(n, \Sigma)$. Let

$$x: p\times 1=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+1}\end{bmatrix}\begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_{k+1}\end{matrix} \qquad \Sigma: p\times p=\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1,k+1} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2,k+1} \\ \vdots & \vdots & & \vdots \\ \Sigma_{k+1,1} & \Sigma_{k+1,2} & \cdots & \Sigma_{k+1,k+1}\end{bmatrix}\begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_{k+1}\end{matrix}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad p_1 \qquad p_2 \qquad \cdots p_{k+1}$$

and

$$V: p\times p=\begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1,k+1} \\ V_{21} & V_{22} & \cdots & V_{2,k+1} \\ \vdots & \vdots & & \vdots \\ V_{k+1,1} & V_{k+1,2} & \cdots & V_{k+1,k+1}\end{bmatrix}\begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_{k+1}\end{matrix}$$
$$\qquad\qquad p_1 \qquad p_2 \qquad \cdots p_{k+1}$$

where $p=p_1+\cdots+p_{k+1}$. The overall null hypothesis we are interested in is

$$H:\ \Sigma_{ij}=0 \qquad \text{for all } i\neq j\,; \tag{3.1}$$

that is, x_i's are statistically independent. This overall hypothesis H is decomposed into the subhypotheses in various ways. Let for example

H_i : independence between x_i and $(x_{i+1}, \cdots, x_{k+1})$,
i.e., $\Sigma_{i,i+1} = \cdots = \Sigma_{i,k+1} = 0$

for $i=1, \cdots, k$. Then it is obvious that $H = \bigcap_{i=1}^{k} H_i$; that is, H is accepted if and only if all H_i's are accepted; otherwise H is rejected. Roy and Bargmann [8] gave the step-down test procedure using the largest canonical correlation between x_i and $(x_{i+1}, \cdots, x_{k+1})$ as the criterion for testing H_i against K_i : not H_i. Here we use the modified LR-criterion to obtain the analysis of approximate chi-square.

Let

$$q_j = p_j + \cdots + p_{k+1} , \qquad x'_{[j]} = (x'_j, \cdots, x'_{k+1}) , \tag{3.2}$$

$$\Sigma_{[j]} : q_j \times q_j = \begin{bmatrix} \Sigma_{jj} & \Sigma_{j,j+1} & \cdots \Sigma_{j,k+1} \\ \Sigma_{j+1,j} & \Sigma_{j+1,j+1} & \cdots \Sigma_{j+1,k+1} \\ \vdots & \vdots & \vdots \\ \Sigma_{k+1,j} & \Sigma_{k+1,j+1} & \cdots \Sigma_{k+1,k+1} \end{bmatrix}$$

and $V_{[j]}$ is the corresponding notation for V. As well known, the modified LR-criterion for H is given by

$$w = \frac{|V|}{|V_{11}||V_{22}| \cdots |V_{k+1,k+1}|} \tag{3.3}$$

while the modified LR-criterion for the test of H_i is

$$w_i = \frac{|V_{[i]}|}{|V_{ii}||V_{[i+1]}|} , \qquad (i=1, \cdots, k) . \tag{3.4}$$

It has been shown by Anderson ([1], p. 243) that

$$w = w_1 w_2 \cdots w_k \tag{3.5}$$

and that under H, w_i's are independent. We further know that

$$\chi^2_{(i)} = -2\tau_i \log w_i = -\left[n - \frac{1}{2} - \frac{1}{2} q_i\right] \log w_i , \tag{3.6}$$

$$\chi^2 = -2\tau \log w = -\left[n - \frac{1}{2} - \frac{1}{3}\left(p^3 - \sum_{j=1}^{k+1} p_j^3\right)\left(p^2 - \sum_{j=1}^{k+1} p_j^2\right)^{-1}\right] \log w \tag{3.7}$$

are the approximate chi-square variates with $f_i = p_i q_{i+1}$ and $f = p^2 - \sum_{j=1}^{k+1} p_j^2$ degrees of freedom, respectively. Thus we obtain Table 3.1 similar to Table 2.1. We observe again that $f = f_1 + \cdots + f_k$ and $\tau = (1/f)(f_1\tau_1 + \cdots + f_k\tau_k)$ which are formulae given by Gleser and Olkin. As in the last section, we can obtain a more accurate procedure if necessary by

Table 3.1 Analysis of the approximate chi-square for the test of the multiple independence

Hypotheses: H_i	Criteria w_i	Chi-square $\chi^2_{(i)}$	d.f. f_i
H_1: $\Sigma_{1\alpha}=0$, $\alpha=2,\cdots,k+1$	$\dfrac{\|V\|}{\|V_{11}\|\|V_{[2]}\|}$	$-2\tau_1 \log w_1$	p_1q_2
H_2: $\Sigma_{2\alpha}=0$, $\alpha=3,\cdots,k+1$	$\dfrac{\|V_{[2]}\|}{\|V_{22}\|\|V_{[3]}\|}$	$-2\tau_2 \log w_2$	p_2q_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
H_k: $\Sigma_{k,k+1}=0$	$\dfrac{\|V_{[k]}\|}{\|V_{kk}\|\|V_{k+1,k+1}\|}$	$-2\tau_k \log w_k$	p_kq_{k+1}
H: $\Sigma_{ij}=0$ for all $i\neq j$	$\dfrac{\|V\|}{\|V_{11}\|\cdots\|V_{k+1,k+1}\|}$	$-2\tau \log w$	$\frac{1}{2}(p^2-\sum_i p_i^2)$

using the further expansions of the asymptotic distributions of $-2\tau \log w$ and $-2\tau_i \log w_i$.

4. Some note on the nonnull joint distributions of criteria

Let us consider in this section the distributional behavior of w_i, $i=1,\cdots,k$ treated in the previous sections when the overall null hypothesis is not true. The limiting nonnull distribution will be considered by using the following lemma which is a direct extension of Olkin and Siotani [6], Siotani and Hayakawa [9], and Sugiura [10].

LEMMA 4.1. *Let $m_\alpha U_\alpha$ have independent Wishart distribution $W_p(m_\alpha, \Lambda_\alpha)$, $\alpha=1,\cdots,q$, $m=\sum_{\alpha=1}^q m_\alpha$, and $l_\alpha=m_\alpha/m$. If $g_i=g_i(U_1,\cdots,U_q)$, $i=1,\cdots,t$ be real-valued linearly independent functions of U_α's continuously differentiable with respect to each variable, then the limiting distribution of*

$$(4.1)\qquad m^{1/2}[\{g_1(U_1,\cdots,U_q),\cdots,g_t(U_1,\cdots,U_q)\} - \{g_1(\Lambda_1,\cdots,\Lambda_q),\cdots,g_t(\Lambda_1,\cdots,\Lambda_q)\}]$$

is the t-variate normal distribution with zero mean vector and covariance matrix $\Phi=(\phi_{ij})$ obtained by the following formulae:

$$(4.2)\qquad \phi_{ii}=\mathrm{Var}\,(m^{1/2}g_i)=2[g_i(\Lambda_1,\cdots,\Lambda_q)]^2 \sum_{\alpha=1}^q \frac{1}{l_\alpha}\,\mathrm{tr}\,\{(\partial^{(\alpha)}\log g_i)\Lambda_\alpha\}^2,$$

$$(4.3)\qquad \phi_{ij}=\mathrm{Cov}\,(m^{1/2}g_i, m^{1/2}g_j)=2[g_i(\Lambda_1,\cdots,\Lambda_q)g_j(\Lambda_1,\cdots,\Lambda_q)] \cdot \sum_{\alpha=1}^q \frac{1}{l_\alpha}\,\mathrm{tr}\,\{(\partial^{(\alpha)}\log g_i)\Lambda_\alpha(\partial^{(\alpha)}\log g_j)\Lambda_\alpha\}$$

where $\partial^{(\alpha)}\log g_i$ *is the symmetric matrix with elements* $(1/2)(1+\delta_{ab})\{\partial \log g_i/\partial\lambda_{ab}^{(\alpha)}\}$, $g_i=g_i(\Lambda_1,\cdots,\Lambda_q)$, $\Lambda_\alpha=(\lambda_{ab}^{(\alpha)})$ *and* δ_{ab} *is the kronecker delta.*

4.1. *For the criteria in Section* 2

First of all, we note the following

LEMMA 4.2. *Suppose that the first* $r-1$ *subhypotheses* $H_1, H_{2|1},\cdots, H_{r-1|1,\cdots,r-2}$ *are true and the rth and hence the subsequent subhypotheses are not true. Then* $(w_1,\cdots,w_{r-1})$ *and* $(w_r,\cdots,w_k)$ *are statistically independent.*

Proof of this lemma is easily obtained by the similar calculation as in the proof of Lemma 2.3.

From this, it is enough for us to consider the joint distribution of $w_r,\cdots,w_k$ under the assumption that

$$V_\alpha\sim W_p(n_\alpha,\Sigma) \qquad \text{for } \alpha=1,\cdots,r$$
$$V_\beta\sim W_p(n_\beta,\Sigma_\beta) \qquad \text{for } \beta=r+1,\cdots,k+1 \tag{4.4}$$

and they are independent. It is noted that for $\beta\geqq r+1$, $V_{(\beta)}=V_{(r)}+V_{r+1}+\cdots+V_\beta$, where $V_{(r)}\sim W_p(n_{(r)},\Sigma)$. Let us use the notations;

$$U_r=V_{(r)}/n_{(r)}, \qquad U_\alpha=V_\alpha/n_\alpha, \quad \alpha=r+1,\cdots,k+1,$$
$$l_\alpha=n_\alpha/n, \qquad l_{(i)}=n_{(i)}/n,$$
$$\tilde{U}_{(i)}=\frac{1}{l_{(i)}}[l_{(r)}U_r+l_{r+1}U_{r+1}+\cdots+l_iU_i], \qquad i\geqq r$$
$$\tilde{\Sigma}_{(i)}=\frac{1}{l_{(i)}}[l_{(r)}\Sigma+l_{r+1}\Sigma_{r+1}+\cdots+l_i\Sigma_i], \qquad i\geqq r.$$

Then w_i, $(i\geqq r)$, is expressed as

$$w_i^{1/n}=g_i(U_r,\cdots,U_{k+1})=|\tilde{U}_{(i)}|^{l_{(i)}/2}|U_{i+1}|^{l_{i+1}/2}|\tilde{U}_{(i+1)}|^{-l_{(i+1)}/2} \tag{4.5}$$

and the corresponding population quantity is

$$\omega_i=g_i(\Sigma,\Sigma_{r+1},\cdots,\Sigma_{i+1})=|\tilde{\Sigma}_{(i)}|^{l_{(i)}/2}|\Sigma_{i+1}|^{l_{i+1}/2}|\tilde{\Sigma}_{(i+1)}|^{-l_{(i+1)}/2}. \tag{4.6}$$

Applying Lemma 4.1, we obtain the limiting distribution of

$$n^{1/2}\{(w_r^{1/n},\cdots,w_k^{1/n})-(\omega_r,\cdots,\omega_k)\} \tag{4.7}$$

as the normal distribution with zero mean vector and covariance matrix obtained by the formulae (4.2) and (4.3). From this, the limiting distribution of

$$2n^{-1/2}(\log w_r,\cdots,\log w_k)-2n^{1/2}(\log\omega_r,\cdots,\log\omega_k) \tag{4.8}$$

is the $(k-r+1)$-variate normal with zero mean vector and covariance matrix $\Phi=(\phi_{ij})$, where $i, j=r,\cdots,k$ and

$$\phi_{ii}=2\operatorname{tr}\left[\sum_{\alpha=r}^{i} l_\alpha\{(\tilde{\Sigma}_{(i)}^{-1}-\tilde{\Sigma}_{(i+1)}^{-1})\Sigma_\alpha\}^2+l_{i+1}\{I-\tilde{\Sigma}_{(i+1)}^{-1}\Sigma_{i+1}\}^2\right], \tag{4.9}$$

$$\phi_{ij}=2\operatorname{tr}\left[\sum_{\alpha=r}^{i} l_\alpha\{(\tilde{\Sigma}_{(i)}^{-1}-\tilde{\Sigma}_{(i+1)}^{-1})\Sigma_\alpha(\tilde{\Sigma}_{(j)}^{-1}-\tilde{\Sigma}_{(j+1)}^{-1})\Sigma_\alpha\}\right.$$
$$\left.+l_{i+1}(I-\tilde{\Sigma}_{(i+1)}^{-1}\Sigma_{i+1})(\tilde{\Sigma}_{(j)}^{-1}-\tilde{\Sigma}_{(j+1)}^{-1})\Sigma_{i+1}\right], \qquad (i<j). \tag{4.10}$$

4.2. *For the criteria in Section* 3

Since we assume again that the first $r-1$ subhypotheses are true, $(x_1,\cdots,x_{r-1})$ are independent each other and they are independent of other subvectors $x_r,\cdots,x_{k+1}$; hence the structure of population covariance matrix Σ is now given by

$$\Sigma=\left[\begin{array}{ccc|c} \Sigma_{11} & & 0 & \\ & \ddots & & 0 \\ 0 & & \Sigma_{r-1,r-1} & \\ \hline & 0 & & \Sigma_{[r]} \end{array}\right]. \tag{4.11}$$

Furthermore the same argument as in Anderson ([1], p. 243) tells us that $(w_1,\cdots,w_{r-1})$ are statistically independent of $(w_r,\cdots,w_k)$, so that we have only to consider the joint distribution of w_i, $i=r,\cdots,k$, starting with $(x_r,\cdots,x_{k+1})$ and $\Sigma_{[r]}$.

Thus there is no loss of generality if we let $r=1$. By Lemma 4.1 with $q=1$, the limiting distribution of

$$n^{1/2}\{(w_1, w_2,\cdots,w_k)-(\theta_1,\theta_2,\cdots,\theta_k)\} \tag{4.12}$$

is normal with mean vector $(0,\cdots,0)$, where

$$\theta_i=|\Sigma_{[i]}||\Sigma_{ii}|^{-1}|\Sigma_{[i+1]}|^{-1}, \qquad i=1,\cdots,k. \tag{4.13}$$

The variance of $n^{1/2}w_i$ in this limiting distribution is given by Olkin and Siotani [6] and Siotani and Hayakawa [9] as

$$\phi_{ii}=2\theta_i^2\operatorname{tr}\{\Sigma_{[i]}D_\Sigma^{(i)}-I_{q_i}\}^2, \qquad i=1,\cdots,k \tag{4.14}$$

where $D_\Sigma^{(i)}=\operatorname{diag}(\Sigma_{ii}^{-1},\Sigma_{[i+1]}^{-1})$. The covariance of $n^{1/2}w_i$ and $n^{1/2}w_j$ is, from (4.3) with $q=1$, obtained in the following form:

$$\phi_{ij}=2\theta_i\theta_j\operatorname{tr}[\Sigma_{ii}^{-1}\{\Sigma_{ij}\Sigma_{jj}^{-1}\Sigma_{ji}-\Sigma_{ij\cdot[j+1]}\Sigma_{jj\cdot[j+1]}^{-1}\Sigma_{ji\cdot[j+1]}\}] \tag{4.15}$$

where $i<j$ and

$$\Sigma_{jj\cdot[j+1]}=\Sigma_{jj}-(\Sigma_{j,j+1},\cdots,\Sigma_{j,k+1})\Sigma_{[j+1]}^{-1}\begin{bmatrix}\Sigma_{j+1,j}\\ \vdots \\ \Sigma_{k+1,j}\end{bmatrix} \tag{4.16}$$

$$\Sigma_{ij\cdot[j+1]}=\Sigma'_{ji\cdot[j+1]}=\Sigma_{ij}-(\Sigma_{i,j+1},\cdots,\Sigma_{i,k+1})\Sigma^{-1}_{[j+1]}\begin{bmatrix}\Sigma_{j+1,j}\\ \vdots\\ \Sigma_{k+1,j}\end{bmatrix}. \tag{4.17}$$

If we use the above notations, ϕ_{ii} is expressed as

$$\phi_{ii}=4\theta_i^2 \operatorname{tr}\{\Sigma_{ii}^{-1}(\Sigma_{ii}-\Sigma_{ii\cdot[j+1]})\}. \tag{4.18}$$

It is noted that $\Sigma_{ii\cdot[i+1]}$ is the population residual covariance matrix of x_i after removing the effect of $(x_{i+1},\cdots,x_{k+1})$ by linear regression and $\Sigma_{ij\cdot[j+1]}$ is the covariance matrix between the residual of x_i and the residual of x_j.

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