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SOME GENERALIZED METHODS OF OPTIMAL SCALING AND THEIR ASYMPTOTIC THEORIES: THE CASE OF MULTIPLE RESPONSES-MULTIPLE FACTORS

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Abstract

Seven generalized criteria are proposed with the corresponding formulations for optimal scaling of multiple responses. Then, on the basis of a natural probabilistic model, the asymptotic theories are derived concerning the test statistics on factor-response relationships as well as the distributions of sample criteria (eigenvalues of the determinantal equation) and optimal scores (eigenvectors) by means of so-called δ-method. A numerical example is provided for illustrations.

1. Introduction

The methods of usual multivariate analysis can be applied on quantitative observations to do with inference for a factor-response relationship. These methods, however, are not available, when only qualitative observations are obtained. For such cases there exist the methods of optimal scaling or quantification, and the present author [6] have proposed several generalized methods of optimal scaling and their asymptotic theories for the case of single response-multiple factors.

The purpose of the present paper is to extend them to the case of multiple responses-multiple factors. In Section 2 several generalized criteria are introduced, including two criteria proper to the multi-response situation. Hayashi's second method of quantification [1] is included in the method based on one of the criteria, when there exists only one factor item and it is selected as the external criterion. In Section 3 some asymptotic properties of sample criteria and optimal scores are investigated corresponding to each criterion, and in Section 4 an asymptotic statistical method is given for inference concerning effects of factors. Finally in Section 5 a numerical example is provided to illustrate the proposed procedures.

2. Some generalized criteria for optimal scaling

Let us suppose that every subject under study can be classified as falling into one and only one of r_i categories of ith response item for $i=1, 2, \cdots, q$, and of c_k categories of kth factor item for $k=1, 2, \cdots, I$. Let us introduce the following dummy variables to express the data of a sample of size n.

$$z_\alpha(ij)=\begin{cases} 1 & \text{if subject } \alpha \text{ belongs to category } j \text{ of the } i\text{th response item}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

$$x_\alpha(kl)=\begin{cases} 1 & \text{if subject } \alpha \text{ belongs to category } l \text{ of the } k\text{th response item}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

Define

$\hat{\Sigma}=\begin{bmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \end{bmatrix}=$the sample variance-covariance matrix of the dummy variables,

$\hat{\Sigma}_{11}=$the $\sum_{i=1}^{q} r_i \times \sum_{i=1}^{q} r_i$ matrix with $(1/n)[h(ij, st)-n_{ij}n_{st}/n]$ as its (ij, st) element, where ij (or st) denotes jth (or tth) category of ith (or sth) response item,

$\hat{\Sigma}_{22}=$the $\sum_{k=1}^{I} c_k \times \sum_{k=1}^{I} c_k$ matrix with $(1/n)[f(kl, uv)-n'_{kl}n'_{uv}/n]$ as its (kl, uv) element, where kl (or uv) denotes lth (or vth) category of kth (uth) factor item,

$\hat{\Sigma}_{12}=$the $\sum_{i=1}^{q} r_i \times \sum_{k=1}^{I} c_k$ matrix with $(1/n)[g^{ij}(kl)-n_{ij}n'_{kl}/n]$ as its (ij, kl) elements,

$r_i=$the number of categories of ith response item,

$r=\sum_{i=1}^{q}(r_i-1)$,

$c_k=$the number of categories of kth factor item,

$c=\sum_{k=1}^{I}(c_k-1)$,

$n_{ij}=$the frequency of occurrence of category j of ith response item,

$n'_{kl}=$the frequency of occurrence of category l of kth factor item,

$h(ij, st)=$the frequency of joint occurrence of category j of ith response item and category t of sth response item,

$f(kl, uv)=$the frequency of joint occurrence of category l of kth factor item and category v of uth factor item,

$g^{ij}(kl)$ = the frequency of joint occurrence of category j of ith response item and category l of kth factor item.

Let t_{ij} be a score assigned to category j of ith response item, then the responses of subject α are expressed by q quantities

$$w_\alpha(i)=\sum_{j=1}^{r_i} z_\alpha(ij)t_{ij}\ , \qquad i=1, 2,\cdots, q\ . \tag{2.3}$$

Moreover, consider a composite variable with weights $\{d_i,\ i=1, 2,\cdots, q\}$

$$w_\alpha=\sum_{i=1}^{q} d_i w_\alpha(i)=\sum_{i=1}^{q}\sum_{j=1}^{r_i} z_\alpha(ij)t^{*}_{ij}\ , \qquad \text{where } t^{*}_{ij}=d_i t_{ij}\ . \tag{2.4}$$

In order to analyze a factor-response relationship by using the quantities w_α and $\{w_\alpha(i),\ i=1, 2,\cdots, q\}$, we define seven criteria for optimal scaling and give formulations based upon them. The mathematical model assumed is given by

(a univariate linear model)

$$w_\alpha=\theta_0+\sum_k \sum_l x_\alpha(kl)\theta_{kl}+e_\alpha\ , \qquad \alpha=1, 2,\cdots, n\ , \tag{2.5}$$

where θ_0 is the constant term, θ_{kl} the effect of category l of kth factor item, and e_α the error term, or

(a multivariate linear model)

$$w_\alpha(i)=\theta_0^{(i)}+\sum_k \sum_l x_\alpha(kl)\theta_{kl}^{(i)}+e_\alpha^{(i)}\ , \qquad i=1, 2,\cdots, q,\ \alpha=1, 2,\cdots, n\ , \tag{2.6}$$

where $\theta_0^{(i)}$ is the constant term, $\theta_{kl}^{(i)}$ the effect of category l of kth factor item and $e_\alpha^{(i)}$ the error term for ith response item.

Since there exist the conditions of exclusive and exhaustive categories such that

$$\begin{aligned} &\sum_{j=1}^{r_i} z_\alpha(ij)=1\ , \qquad i=1, 2,\cdots, q\ ,\\ &\sum_{l=1}^{c_k} x_\alpha(kl)=1\ , \qquad k=1, 2,\cdots, I\ , \end{aligned} \tag{2.7}$$

we may omit the dummy variables for an arbitrary category per item and the corresponding rows and columns of the variance-covariance matrix $\hat{\Sigma}$, but for the sake of simplicity we shall use the same notations $\hat{\Sigma}$, $\hat{\Sigma}_{11}$, $\hat{\Sigma}_{12}$, $\hat{\Sigma}_{22}$ for such abbreviated matrices. The omitting of dummy variables as described above is obviously equivalent to equating the scores or effects of the selected categories to zero.

On the basis of the univariate or multivariate model described above, we shall introduce seven criteria summarized in Table 2.1 for

Table 2.1 Generalized criteria proposed for optimal scaling in the case of multiple responses

(i) Criteria based on the relationships between factors and responses

	Criteria for optimal scaling	Determinantal equations
CM-1	Maximization of the variation due to the effects of all factors relative to the total variation	$(\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}-\hat{\lambda}\hat{\Sigma}_{11})\boldsymbol{t}=0$ (*1) where $\boldsymbol{t}=[t_{ij}^{*}]$: $r\times 1$
CM-2	Maximization of the squared canonical correlation coefficient between the two sets of dummy variables corresponding to the responses and the factors	$(\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}-\hat{\lambda}\hat{\Sigma}_{11})\boldsymbol{t}=0$ (*1) or $(\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\hat{\Sigma}_{12}-\hat{\lambda}\hat{\Sigma}_{22})\boldsymbol{s}=0$ (*2) where $\boldsymbol{s}=[s_{kl}]$: $c\times 1$
CM-3	Maximization of the correlation ratio	$(\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}-\hat{\lambda}\hat{\Sigma}_{11})\boldsymbol{t}=0$ (*1)
CM-4	Maximization of the variation due to an arbitrary testable hypothesis H_0: $K'\boldsymbol{\theta}=0$	$\{\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}K(K'\hat{\Sigma}_{22}^{-1}K)^{-1}K'\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}-\hat{\lambda}\hat{\Sigma}_{11}\}\boldsymbol{t}=0$ (*3)
CM-5	Maximization of the squared partial canonical correlation coefficient between the response items and the factors of interest	$(\tilde{\Sigma}_{12}\tilde{\Sigma}_{22}^{-1}\tilde{\Sigma}_{21}-\hat{\lambda}\tilde{\Sigma}_{11})\boldsymbol{t}=0$ (*4) where $\tilde{\Sigma}_{ij}=\hat{\Sigma}_{ij}-\hat{\Sigma}_{i3}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{3j}$

Notice: 1) In the case of CM-3 we assume that the factor is constructed with a single item. The formulation is obtained by taking the factor item as the external criterion in Hayashi's second methods of quantification.

2) Subscripts 2 and 3 in the formulation of CM-5 correspond to the factors of interest and nuisance.

3) The vector $\boldsymbol{\theta}$, which indicates the effects of factors, is constructed with $\{\theta_{kl}\}$ other than the omitted categories.

(ii) Criteria based on the relationships among responses

	Criteria for optimal scaling	Determinantal equations
CM-6	Maximization of the sum of covariance between response items relative to the sum of variances	$\begin{bmatrix} 0 & \hat{\Sigma}_{(12)} & \cdots & \hat{\Sigma}_{(1q)} \\ \hat{\Sigma}_{(21)} & 0 & \cdots & \hat{\Sigma}_{(2q)} \\ \vdots & & \ddots & \vdots \\ \hat{\Sigma}_{(q1)} & \hat{\Sigma}_{(q2)} & \cdots & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t}_{(1)} \\ \boldsymbol{t}_{(2)} \\ \vdots \\ \boldsymbol{t}_{(q)} \end{bmatrix} -\hat{\lambda}\begin{bmatrix} \hat{\Sigma}_{(11)} & & & 0 \\ & \hat{\Sigma}_{(22)} & & \\ & & \ddots & \\ 0 & & & \hat{\Sigma}_{(qq)} \end{bmatrix}\begin{bmatrix} \boldsymbol{t}_{(1)} \\ \boldsymbol{t}_{(2)} \\ \vdots \\ \boldsymbol{t}_{(q)} \end{bmatrix}=0$ (*5)
CM-7	Maximization of the squared multiple correlation coefficient between a response item and the others	$\{\hat{\Sigma}_{(i\bar{i})}\hat{\Sigma}_{(\bar{i}\bar{i})}^{-1}\hat{\Sigma}_{(\bar{i}i)}-\hat{\lambda}_i\hat{\Sigma}_{(ii)}\}\boldsymbol{t}_{(i)}=0$ (*6) $i=1, 2, \cdots, q$ q: the number of response items

Notice: 1) The criteria CM-6 and 7 are derived from the basic idea such that, if we seek separate scoring systems for the two categorical variables such as to maximize their correlation, we are basically trying to produce a bivariate normal distribution by operating upon the margins of the cross table [3].

2) The matrix $\hat{\Sigma}_{(ii')}$ denotes the covariance matrix of the dummy variables

corresponding to the ith and i'th response items.

3) The criteria based on the relationships among the residuals are derived by substituting $\hat{\Phi}_{(ii')}=\hat{\Sigma}_{(ii')}-\hat{\Sigma}_{(if)}\hat{\Sigma}_{(ff)}^{-1}\hat{\Sigma}_{(fi')}$ into $\hat{\Sigma}_{(ii')}$, where the subscript f corresponds to the factors.

4) The subscript $\bar{i}$ corresponds to the response items excepting the ith item.

optimal scaling of multiple responses. The optimal scores are obtained as the eigenvector corresponding to the largest eigenvalue of each determinantal equation. Furthermore, when the amount of information is poor by assigning a unidimensional score to each response category, we may use a multidimensional score. In such cases the eigenvectors corresponding to the eigenvalues smaller than the largest should be used. The criterion becomes the maximization of $\prod_i \lambda_i$ instead of λ under the orthogonarity conditions. Hayashi [2] discussed precisely the multidimensional case.

3. Asymptotic properties of the sample criteria and optimal scores

3.1. *Preparations*

The optimal score vector $\boldsymbol{t}$ based on the CM-1~5 criteria, $\boldsymbol{t}=[\boldsymbol{t}'_{(1)}, \cdots, \boldsymbol{t}'_{(q)}]'$ or $\boldsymbol{t}_{(i)}$ based on the CM-6 or CM-7 criterion is determined by the same type of the eigenvalue problem such as

$$(\hat{A}-\hat{\lambda}\hat{B})\boldsymbol{t}=\boldsymbol{0} \,. \tag{3.1}$$

In this section we shall derive the asymptotic joint distribution of the sample eigenvalues (criteria) and eigenvectors (optimal scores) by means of so-called δ-method [4]. Now define

$p(j_1,\cdots,j_q|i_1,\cdots,i_I)$ or $p(j|i)$=the probability of the response combination $(j_1,\cdots,j_q)$ or the jth response combination in the sample of the treatment combination $(i_1,\cdots,i_I)$ or the ith treatment combination,

$n(i_1,\cdots,i_I;j_1,\cdots,j_q)$ or $n(i;j)$=the frequency of occurrence corresponding to $p(j_1,\cdots,j_q|i_1,\cdots,i_I)$ or $p(j|i)$,

$n(i_1,\cdots,i_I)$ or $n(i)=\sum_{j_1=1}^{r_1}\cdots\sum_{j_q=1}^{r_q} n(i_1,\cdots,i_I;j_1,\cdots,j_q)$: given,

$\hat{p}(j_1,\cdots,j_q|i_1,\cdots,i_I)$ or $\hat{p}(j|i)=n(i_1,\cdots,i_I;j_1,\cdots,j_q)/n(i_1,\cdots,i_I)$,

$\pi(i_1,\cdots,i_I)$ or $\pi(i)=n(i_1,\cdots,i_I)/n$: the relative sample size of the treatment combination $(i_1,\cdots,i_I)$,

m=the number of treatment combinations, $m\leqq\prod_{k=1}^{I} c_k$,

R=the number of response combinations, $R\leqq\prod_{j=1}^{q} r_j$.

In order to introduce the probability measure, we shall assume a natural probabilistic model as follows.

DEFINITION 3.1 (Probabilistic model). Let the sample sizes for the treatment combinations $\{(i_1, \cdots, i_I),\ i_1=1, 2, \cdots, c_1, \cdots, i_I=1, 2, \cdots, c_I\}$ be fixed, and suppose the frequencies of response combinations $\{n(i; j), j=1, 2, \cdots, R\}$ be distributed as a multinomial distribution with probabilities $\{p(j|i),\ j=1, 2, \cdots, R,\ \sum_{j=1}^{R} p(j|i)=1\}$.

According to the above probabilistic model, we obtain

$$\hat{\Sigma}_{11}=L(\hat{D}-\hat{P}'Q\hat{P})L' , \tag{3.2}$$

$$\hat{\Sigma}_{12}=L\hat{P}'(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J' , \tag{3.3}$$

$$\hat{\Sigma}_{22}=J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J' , \tag{3.4}$$

where

$$\hat{D}=\mathrm{diag}\left[\sum_{i=1}^{m} \pi(i)p(1|i), \cdots, \sum_{i=1}^{m} \pi(i)p(R|i)\right] : R\times R , \tag{3.5}$$

$$\hat{P}=[\hat{p}(j|i)] : m\times R , \tag{3.6}$$

$$L=[l_{st}]=[\delta(st|j)] : r\times R , \tag{3.7}$$

$$\delta(st|j)=\begin{cases} 1 & \text{for } j \in \mathcal{S}_1(st) , \\ 0 & \text{for } j \notin \mathcal{S}_1(st) , \end{cases} \tag{3.8}$$

$$J=[\delta(kl|i)] : c\times m , \tag{3.9}$$

$$\delta(kl|i)=\begin{cases} 1 & \text{for } i \in \mathcal{S}_2(kl) , \\ 0 & \text{for } i \notin \mathcal{S}_2(kl) , \end{cases} \tag{3.10}$$

$$\boldsymbol{\pi}=[\pi(i)] : m\times 1 , \tag{3.11}$$

$$\Pi=\mathrm{diag}\,[\pi(1), \cdots, \pi(m)] : m\times m , \tag{3.12}$$

$$Q=[q_{ii'}]=\boldsymbol{\pi}\boldsymbol{\pi}' : m\times m , \tag{3.13}$$

$\mathcal{S}_1(st)$ and $\mathcal{S}_2(kl)$ denoting a set of response combinations which contains tth category of sth response item and a set of treatment combinations which contains lth category of kth factor item, respectively.

Now consider the p-dimensional eigenvalue problem with population values A and B such that

$$AY=BY\Lambda \quad \text{subject to} \quad Y'BY=I , \tag{3.14}$$

where the matrices $\Lambda=\mathrm{diag}\,[\lambda_1, \lambda_2, \cdots, \lambda_p]$ and $Y=[\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_p]$ are constructed with the eigenvalues and eigenvectors of A relative to B, and assume that the eigenvalues are all distinct, namely $\lambda_1 > \lambda_2 > \cdots > \lambda_p$. Let the sample values be distinguished from the population values by superimposing the symbol ($\hat{}$), and define

$$A^* = \hat{A} - A\,, \quad B^* = \hat{B} - B\,, \quad \Lambda^* = \hat{\Lambda} - \Lambda\,, \quad Y^* = \hat{Y} - Y\,, \tag{3.15}$$

$$Y^* = YZ^*\,. \tag{3.16}$$

Then, from (3.14)-(3.16), neglecting the higher order of the quantities with asterisk, we obtain

$$\begin{aligned} &\lambda_s^* \sim \sum_j \sum_k a_{jk}^* y_{js} y_{ks} - \lambda_s \sum_j \sum_k b_{jk}^* y_{js} y_{ks}\,, \qquad s = 1, 2, \cdots, p \\ &z_{vu}^* \sim (\lambda_v - \lambda_u)^{-1} \{\lambda_u \sum_j \sum_k b_{jk}^* y_{jv} y_{ku} - \sum_j \sum_k a_{jk}^* y_{jv} y_{ku}\}\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad u, v = 1, 2, \cdots, p,\; u \neq v\,, \\ &z_{vv}^* \sim -1/2 \sum_j \sum_k b_{jk}^* y_{jv} y_{kv}\,, \qquad v = 1, 2, \cdots, p\,. \end{aligned} \tag{3.17}$$

Thus the small deviations of eigenvalues and eigenvectors can be expressed asymptotically by linear equations of the deviations of the matrices A and B. On the basis of these relations the asymptotic properties of sample eigenvalues and eigenvectors are investigated corresponding to each criterion for optimal scaling, as in the previous paper [6]. In order to use the expansions (3.17) we assume that all nonzero eigenvalues of the eigenvalue problem derived from each criterion are distinct.

3.2. *The case of the* CM-1$\sim$3 *criteria*

The relations (3.17) are valid under the assumption that the p eigenvalues are all distinct. Therefore, suppose that the population covariance matrices Σ_{11}, Σ_{12}, Σ_{22} are of full rank after omitting an arbitrary category per item. Let us consider the following two cases separately, in order to investigate the asymptotic properties of the eigenvalues and eigenvectors of (*1) in Table 2.1.

a) The case where the number of nonzero eigenvalues is determined by the number of response categories, i.e.

$$\operatorname{rank}(\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) = r \leqq c\,. \tag{3.18}$$

b) The case where the number of nonzero eigenvalues is determined by the number of categories of factor items, i.e.

$$\operatorname{rank}(\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) = \operatorname{rank}(\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}) = c < r\,. \tag{3.19}$$

Consider first the case where the condition a) holds, and put

$$\hat{A} = \hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}\,, \qquad \hat{B} = \hat{\Sigma}_{11}\,, \qquad \hat{Y} = T = [\boldsymbol{t}_1, \cdots, \boldsymbol{t}_r]\,, \tag{3.20}$$

in the eigenvalue problem (3.14) with the sample values. Substitution of (3.2)~(3.4) into (3.20) yields

$$\hat{A}=L\hat{P}'E\hat{P}L' , \tag{3.21}$$

$$\hat{B}=L(\hat{D}-\hat{P}'Q\hat{P})L' , \tag{3.22}$$

where

$$E=[e_{ii'}]=(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'[J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']^{-1}J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}') : m\times m . \tag{3.23}$$

Moreover, applying the Taylor expansion to a_{jk}^* and b_{jk}^* about $p^*=0$, we obtain

$$a_{kk'}^*=a_{k'k}^*=\sum_i\sum_j \xi_{ij}^{(kk')}p^*(j\,|\,i)+o(p^*) , \qquad k, k'=1, 2,\cdots, r , \tag{3.24}$$

$$b_{kk'}^*=b_{k'k}^*=\sum_i\sum_j \eta_{ij}^{(kk')}p^*(j\,|\,i)+o(p^*) , \qquad k, k'=1, 2,\cdots, r , \tag{3.25}$$

with

$$\xi_{ij}^{(kk')}=\sum_{i'}\sum_{j'} (l_{kj}l_{k'j'}+l_{kj'}l_{k'j})e_{ii'}p(j'\,|\,i') , \tag{3.26}$$

$$\eta_{ij}^{(kk')}=l_{kj}l_{k'j}\pi(i)-\sum_{i'}\sum_{j'} (l_{kj}l_{k'j'}+l_{kj'}l_{k'j})q_{ii'}p(j'\,|\,i') . \tag{3.27}$$

Substituting (3.24)~(3.25) in (3.17) and transforming z^* to t^*, we have the following asymptotic expansions.

$$\lambda_s^*=\sum_i\sum_j \alpha_{ij}^{(s)}p^*(j\,|\,i)+o(p^*) , \qquad s=1, 2,\cdots, r \tag{3.28}$$

$$t_{ks}^*=\sum_i\sum_j \beta_{ij}^{(ks)}p^*(j\,|\,i)+o(p^*) , \qquad k, s=1, 2,\cdots, r \tag{3.29}$$

where

$$\alpha_{ij}^{(s)}=\sum_k\sum_{k'} (\xi_{ij}^{(kk')}-\lambda_s\eta_{ij}^{(kk')})\tau_{ks}^{\sharp}\tau_{k's}^{\sharp} , \tag{3.30}$$

$$\beta_{ij}^{(ks)}=\sum_{k'}\sum_{k''}\left\{\sum_{v\neq s}\frac{\tau_{kv}^{\sharp}}{\lambda_v-\lambda_s}[\lambda_s\eta_{ij}^{(k'k'')}\tau_{k'v}^{\sharp}\tau_{k''s}^{\sharp}-\xi_{ij}^{(k'k'')}\tau_{k'v}^{\sharp}\tau_{k''s}^{\sharp}]\right.$$
$$\left.-\frac{1}{2}\eta_{ij}^{(k'k'')}\tau_{k's}^{\sharp}\tau_{k''s}^{\sharp}\tau_{ks}^{\sharp}\right\} , \tag{3.31}$$

λ_v and $\{\tau_{kv}^{\sharp}, k=1, 2,\cdots, r\}$ denoting a population eigenvalue and the corresponding eigenvector.

Next, consider the case where the condition b) holds. In this case we should put

$$\hat{A}=\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21} , \qquad \hat{B}=\hat{\Sigma}_{22} , \qquad \hat{Y}=S=[\boldsymbol{s}_1,\cdots, \boldsymbol{s}_c] , \tag{3.32}$$

in the eigenvalue problem (3.14) with the sample values at the first place, and then apply the δ-method again to the relation

$$t_u = \lambda_u^{-1/2} \hat{\Sigma}_{11}^{-1} \hat{\Sigma}_{12} s_u . \tag{3.33}$$

Substitution of (3.2)~(3.4) into (3.32) yields

$$\hat{A} = \hat{\Sigma}_{21} \hat{\Sigma}_{11}^{-1} \hat{\Sigma}_{12} = J(\Pi - \pi\pi')\hat{P}L'\{L(\hat{D} - \hat{P}'Q\hat{P})L'\}^{-1} L\hat{P}'(\Pi - \pi\pi')J' , \tag{3.34}$$

$$\hat{B} = \hat{\Sigma}_{22} = J(\Pi - \pi\pi')J' , \tag{3.35}$$

$$\hat{Y} = S = [s_1, s_2, \cdots, s_c] . \tag{3.36}$$

Then, obviously

$$b_{uv}^* = 0 , \tag{3.37}$$

while a_{uv}^*'s are obtained as follows.

$$A^* = H'P^*H^{(1)} - H^{(1)\prime}(D - P'QP)^* H^{(1)} + H^{(1)\prime}P^{*\prime}H + o(p^*) , \tag{3.38}$$

where

$$H = [h_{iu}] = (\Pi - \pi\pi')J' : m \times c , \tag{3.39}$$

$$H^{(1)} = [h_{ju}^{(1)}] = L'\{L(D - P'QP)L'\}^{-1} LP'H , \tag{3.40}$$

or elementwisely

$$a_{uv}^* = \sum_i \sum_j \zeta_{ij}^{(uv)} p^*(j \mid i) + o(p^*) , \tag{3.41}$$

where

$$\begin{aligned} \zeta_{ij}^{(uv)} = {} & h_{iu} h_{jv}^{(1)} + h_{iv} h_{ju}^{(1)} - h_{ju}^{(1)} h_{jv}^{(1)} \pi(i) \\ & + \sum_{i'} \sum_{j'} \{ h_{ju}^{(1)} h_{j'v}^{(1)} q_{ii'} + h_{j'u}^{(1)} h_{jv}^{(1)} q_{i'i} \} p(j' \mid i') . \end{aligned} \tag{3.42}$$

Hence by substituting (3.37) and (3.41) into (3.17) and transforming z^* to s^*, we obtain

$$\lambda_u^* = \sum_i \sum_j \alpha_{ij}^{(u)} p^*(j \mid i) + o(p^*) , \qquad u = 1, 2, \cdots, c , \tag{3.43}$$

$$s_{vu}^* = \sum_{u'} \theta_{vu'} z_{u'u}^* = \sum_i \sum_j \phi_{ij}^{(vu)} p^*(j \mid i) + o(p^*) , \qquad v, u = 1, 2, \cdots, c , \tag{3.44}$$

where

$$\alpha_{ij}^{(u)} = \sum_{u'} \sum_{u''} \zeta_{ij}^{(u'u'')} \theta_{u'u} \theta_{u''u} , \tag{3.45}$$

$$\phi_{ij}^{(vu)} = \sum_{u' \neq u} \sum_{v'} \sum_{v''} (\lambda_u - \lambda_{u'})^{-1} \zeta_{ij}^{(v'v'')} \theta_{vu'} \theta_{v'u'} \theta_{v''u} . \tag{3.46}$$

Since the sample optimal score vector t_u is expressed as

(3.47) $$t_u=\hat{\lambda}_u^{-1/2}\hat{\Sigma}_{11}^{-1}\hat{\Sigma}_{12}s_u=\lambda_u^{-1/2}\{L(\hat{D}-\hat{P}'Q\hat{P})L'\}^{-1}L\hat{P}'Hs_u\ ,$$

then we have

(3.48) $$t_{ku}^*=\sum_i\sum_j\beta_{ij}^{(ku)}p^*(j\,|\,i)+o(p^*)\ ,$$

where

(3.49) $$\beta_{ij}^{(ku)}=-0.5\lambda_u^{-3/2}h_{ku}^{(2)}\alpha_{ij}^{(u)}-\lambda_u^{-1/2}h_{kj}^{(3)}h_{ju}^{(4)}\pi(i)+\lambda_u^{-1/2}\sum_{i'}\sum_{j'}(h_{kj}^{(3)}h_{j'u}^{(4)}q_{ii'}$$
$$+h_{kj'}^{(3)}h_{ju}^{(4)}q_{i'i})p(j'\,|\,i')+\lambda_u^{-1/2}h_{kj}^{(3)}h_{j'u}^{(5)}+\lambda_u^{-1/2}\sum_{u'}h_{ku'}^{(6)}\phi_{ij}^{(u'u)}\ ,$$

(3.50) $$H^{(2)}=[h_{ku}^{(2)}]=\{L(D-P'QP)L'\}^{-1}LP'H\Theta : r\times c\ , \qquad \Theta=[\theta_{uv}] : c\times c$$

(3.51) $$H^{(3)}=[h_{kj}^{(3)}]=\{L(D-P'QP)L'\}^{-1}L : r\times R\ ,$$

(3.52) $$H^{(4)}=[h_{ju}^{(4)}]=L'\{L(D-P'QP)L'\}^{-1}LP'H\Theta : R\times c\ ,$$

(3.53) $$H^{(5)}=[h_{ju}^{(5)}]=H\Theta : m\times c\ ,$$

(3.54) $$H^{(6)}=[h_{ku}^{(6)}]=\{L(D-P'QP)L'\}^{-1}LP'H : r\times c\ .$$

Thus we obtain the formulas that the small deviations of eigenvalues and eigenvectors are approximated asymptotically by linear functions of small deviations of the multinomial probabilities $p^*(j\,|\,i)$, whether which of the conditions a) and b) holds.

3.3. *The case of the* CM-4 *criterion*

In the case of the CM-4 criterion the optimal score vector $t=[t_{ij}^*]$ is obtained as a solution of the eigenvalue problem (*3). Here again we must consider the following two cases.

a) The case where the number of nonzero eigenvalues is determined by the number of response categories, i.e.

(3.55) $$\text{rank}\,\{\Sigma_{12}\Sigma_{22}^{-1}K(K'\Sigma_{22}^{-1}K)^{-1}K'\Sigma_{22}^{-1}\Sigma_{21}\}=r\leqq p\ ,$$

where p denotes the number of independent contrasts in the hypothesis $H_0 :\ K'\theta=0$.

b) The case where the number of nonzero eigenvalues is determined by the number of linearly independent contrasts in the hypothesis, i.e.

(3.56) $$\text{rank}\,\{\Sigma_{12}\Sigma_{22}^{-1}K(K'\Sigma_{22}^{-1}K)^{-1}K'\Sigma_{22}^{-1}\Sigma_{21}\}=p<r\ .$$

At the first place consider the case where the condition a) holds. Put

(3.57) $$\hat{A}=\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}K(K'\hat{\Sigma}_{22}^{-1}K)^{-1}K'\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}\ , \qquad \hat{B}=\hat{\Sigma}_{11}\ ,$$
$$\hat{Y}=T=[t_1,\cdots,t_r]\ ,$$

then we have

$$\hat{A}=L\hat{P}'E\hat{P}L' , \tag{3.58}$$

$$\hat{B}=L(\hat{D}-\hat{P}'Q\hat{P})L' , \tag{3.59}$$

where

$$\begin{aligned} E=[e_{ii'}]=&(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'[J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']^{-1}K\{K'[J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']^{-1}K\}^{-1} \\ &\cdot K'[J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']^{-1}J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}') : m\times m . \end{aligned} \tag{3.60}$$

As the equations (3.58)$\sim$(3.59) accord with (3.21)$\sim$(3.22) excepting the definition of E, we obtain the asymtotic expansions (3.28)$\sim$(3.31) with E defined by (3.60).

Next, consider the case where the condition b) holds. By analogy with the formulation in the case of the CM-1$\sim$3 criteria, let us put

$$\begin{aligned} &\hat{A}=K'\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}K , \qquad \hat{B}=K'\hat{\Sigma}_{22}^{-1}K , \\ &\hat{Y}=S=[\boldsymbol{s}_1,\cdots,\boldsymbol{s}_p] , \end{aligned} \tag{3.61}$$

where $\boldsymbol{s}_i$ denotes an artificial variable vector obtained as a solution of the eigenvalue problem which is derived from (*3), just as (*2) is derived from (*1). After obtaining the asymptotic expansions of λ's and $\boldsymbol{s}$'s, we apply the δ-method again to the relation

$$\boldsymbol{t}_u=\hat{\lambda}_u^{-1/2}\hat{\Sigma}_{11}^{-1}\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}K\boldsymbol{s}_u . \tag{3.62}$$

Substituting (3.2)$\sim$(3.4) into (3.61), we obtain

$$\begin{aligned} \hat{A}=&K'[J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']^{-1}J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')\hat{P}L'[L(\hat{D}-\hat{P}'Q\hat{P})L']^{-1} \\ &\cdot L\hat{P}'(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'[J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']^{-1}K , \end{aligned} \tag{3.63}$$

$$\hat{B}=K'[J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']K . \tag{3.64}$$

Then obviously

$$b_{uv}^*=0 , \tag{3.65}$$

while a_{uv}^*'s are obtained as follows.

$$A^*=H'P^*H^{(1)}-H^{(1)\prime}(D-P'QP)^*H^{(1)}+H^{(1)\prime}P^{*\prime}H+o(p^*) , \tag{3.66}$$

where

$$H=[h_{iu}]=(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'[J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']^{-1}K , \tag{3.67}$$

$$H^{(1)}=[h_{ju}^{(1)}]=L'[L(D-P'QP)L']^{-1}LP'H . \tag{3.68}$$

The equation (3.66) is just the same with (3.38) excepting the definitions of H and $H^{(1)}$. Moreover, by substituting (3.2)$\sim$(3.4) in (3.62), we obtain the same formula to (3.47) with H defined in (3.67) instead of (3.39).

Thus we have the results similar to the case of CM-1~3 criteria except for the definitions of E, H and $H^{(1)}$.

3.4. *The case of the* CM-5 *criterion*

In this case the optimal score vector $\boldsymbol{t}=[t_{ij}^{\dagger}]$ is obtained through the eigenvalue problem (*4). Again consider the following two cases.

a) The case where the number of nonzero eigenvalues is determined by the number of response categories, i.e.

$$\text{rank}\,\{(\Sigma_{12}-\Sigma_{13}\Sigma_{33}^{-1}\Sigma_{32})(\Sigma_{22}-\Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32})^{-1}(\Sigma_{21}-\Sigma_{23}\Sigma_{33}^{-1}\Sigma_{31})\}=r\leqq c' , \tag{3.69}$$

where

$$c'=\sum_{k\in\mathcal{I}_1}(c_k-1) ,$$

$\mathcal{I}_1$ indicating a set of item numbers for factors of interest.

b) The case where the number of nonzero eigenvalues is determined by the number of categories of factors of interest, i.e.

$$\text{rank}\,\{(\Sigma_{12}-\Sigma_{13}\Sigma_{33}^{-1}\Sigma_{32})(\Sigma_{22}-\Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32})^{-1}(\Sigma_{21}-\Sigma_{23}\Sigma_{33}^{-1}\Sigma_{31})\}=c'<r . \tag{3.70}$$

Consider first that the condition a) holds, and put

$$\begin{aligned}\hat{A}&=(\hat{\Sigma}_{12}-\hat{\Sigma}_{13}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{32})(\hat{\Sigma}_{22}-\hat{\Sigma}_{23}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{32})^{-1}(\hat{\Sigma}_{21}-\hat{\Sigma}_{23}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{31})\\ \hat{B}&=\hat{\Sigma}_{11}-\hat{\Sigma}_{13}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{31} ,\\ \hat{Y}&=T=[\boldsymbol{t}_1, \boldsymbol{t}_2, \cdots, \boldsymbol{t}_r] ,\end{aligned} \tag{3.71}$$

to apply the asymptotic expansion (3.17), under a constraint for normalization,

$$\boldsymbol{t}'(\hat{\Sigma}_{11}-\hat{\Sigma}_{13}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{31})\boldsymbol{t}=1 . \tag{3.72}$$

Define J_1 and J_2 by dividing J into two part as follows.

$$J_1=[\delta(uv\,|\,i)] : c'\times m , \qquad \text{for } u\in\mathcal{I}_1 , \tag{3.73}$$

$$J_2=[\delta(uv\,|\,i)] : (c-c')\times m , \qquad \text{for } u\notin\mathcal{I}_1 , \tag{3.74}$$

$$\delta(uv\,|\,i)=\begin{cases}1 & \text{for } i\in\mathcal{S}_2(uv) ,\\ 0 & \text{for } i\notin\mathcal{S}_2(uv) .\end{cases} \tag{3.75}$$

Then we obtain

$$\begin{aligned}&\hat{\Sigma}_{11}=L(\hat{D}-\hat{P}'Q\hat{P})L' , \quad \hat{\Sigma}_{12}=L\hat{P}'(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1' , \quad \hat{\Sigma}_{13}=L\hat{P}'(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2' ,\\ &\hat{\Sigma}_{22}=J_1(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1' , \quad \hat{\Sigma}_{23}=J_1(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2' , \quad \hat{\Sigma}_{33}=J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2' ,\end{aligned} \tag{3.76}$$

Substituting (3.76) into (3.71), we obtain

$$\hat{A}=L\hat{P}'E\hat{P}L' , \tag{3.77}$$

$$\hat{B}=L(\hat{D}-\hat{P}'Q\hat{P})L' , \tag{3.78}$$

where

$$\begin{aligned} E=[e_{ii'}]= &\{(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1'-(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2'[J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2']^{-1}J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1'\} \\ &\cdot \{J_1(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1'-J_1(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2'[J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2']^{-1}J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1'\}^{-1} \\ &\cdot \{J_1(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')-J_1(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2'[J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2']^{-1}J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')\} : \\ &m\times m , \end{aligned} \tag{3.79}$$

$$Q=[q_{ii'}]=\boldsymbol{\pi}\boldsymbol{\pi}'+(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2'[J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2']^{-1}J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}') : m\times m . \tag{3.80}$$

Applying the Taylor expansion, we have

$$a^*_{kk'}=a^*_{k'k}=\sum_i\sum_j \xi_{ij}^{(kk')}p^*(j\,|\,i)+o(p^*) , \qquad k, k'=1, 2,\cdots, r , \tag{3.81}$$

$$b^*_{kk'}=b^*_{k'k}=\sum_i\sum_j \eta_{ij}^{(kk')}p^*(j\,|\,i)+o(p^*) , \qquad k, k'=1, 2,\cdots, r , \tag{3.82}$$

where

$$\xi_{ij}^{(kk')}=\sum_{i'}\sum_{j'} (l_{kj}l_{k'j'}+l_{kj'}l_{k'j})e_{ii'}p(j'\,|\,i') , \tag{3.83}$$

$$\eta_{ij}^{(kk')}=l_{kj}l_{k'j}\pi(i)-\sum_{i'}\sum_{j'} (l_{kj}l_{k'j'}+l_{kj'}l_{k'j})q_{ii'}p(j'\,|\,i') . \tag{3.84}$$

Next, consider the case where the condition b) holds. On the basis of the analogy to the formulation in the case of the CM–1$\sim$3 criteria, let us put

$$\begin{aligned} &\hat{A}=(\hat{\Sigma}_{21}-\hat{\Sigma}_{23}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{31})(\hat{\Sigma}_{11}-\hat{\Sigma}_{13}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{31})^{-1}(\hat{\Sigma}_{12}-\hat{\Sigma}_{13}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{32}) , \\ &\hat{B}=\hat{\Sigma}_{22}-\hat{\Sigma}_{23}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{32} , \\ &\hat{Y}=S=[\boldsymbol{s}_1,\cdots, \boldsymbol{s}_{c'}] , \end{aligned} \tag{3.85}$$

considering artificial variables $\boldsymbol{s}$'s. Then the optimal score vector $\boldsymbol{t}_u$ is given by

$$\boldsymbol{t}_u=\hat{\lambda}_u^{-1/2}(\hat{\Sigma}_{11}-\hat{\Sigma}_{13}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{31})^{-1}(\hat{\Sigma}_{12}-\hat{\Sigma}_{13}\hat{\Sigma}_{33}^{-1}\hat{\Sigma}_{32})\boldsymbol{s}_u . \tag{3.86}$$

By substituting (3.76) into (3.85) and (3.86) we can easily find that the equations (3.37), (3.38) and (3.47) are valid under the definitions in (3.80) and

$$\begin{aligned} H=[h_{iu}]=&(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1'-(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2'[J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_2']^{-1} \\ &\cdot J_2(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1' : m\times c' . \end{aligned} \tag{3.87}$$

Hence also in the case of the CM–5 criterion we have the results similar to the case of CM–1$\sim$3 criteria except for the definitions of $E=[e_{ii'}]$, $Q=[q_{ii'}]$ and $H=[h_{iu}]$.

3.5. *The case of the* CM–6 *criterion*

In this case the optimal score vector $\boldsymbol{t}=[\boldsymbol{t}'_{(1)}, \boldsymbol{t}'_{(2)}, \cdots, \boldsymbol{t}'_{(q)}]'$ is obtained as a solution of the eigenvalue problem (*5) in its original form or by substituting $\hat{\Phi}_{(ii')}$ for $\hat{\Sigma}_{(ii')}$. Now consider the asymptotic properties of $\boldsymbol{t}$ under the constraint

$$\sum_{i=1}^{q} \boldsymbol{t}'_{(i)}\hat{\Sigma}_{(ii)}\boldsymbol{t}_{(i)}=1 \qquad \text{or} \qquad \sum_{i=1}^{q} \boldsymbol{t}'_{(i)}\hat{\Phi}_{(ii)}\boldsymbol{t}_{(i)}=1 \,.$$

According to the probabilistic model defined by Definition 3.1, we obtain

$$(3.88)\quad \hat{\Sigma}_{(ii')}=L_i(\hat{D}-\hat{P}'\boldsymbol{\pi}\boldsymbol{\pi}'\hat{P})L'_{i'} \,,$$

$$(3.89)\quad \hat{\Phi}_{(ii')}=L_i\{\hat{D}-\hat{P}'[\boldsymbol{\pi}\boldsymbol{\pi}'+(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'\{J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'\}^{-1}J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')]\hat{P}\}L'_{i'} \,,$$

where L_i denotes a $(r_i-1)\times R$ submatrix of L in (3.7), i.e.

$$(3.90)\qquad L'=[L'_1, L'_2, \cdots, L'_q] \,.$$

Thus we can again apply (3.17). Using the Taylor expansions, we obtain (3.24) and (3.25), where

$$(3.91)\quad \xi_{ij}^{(kk')}=\begin{cases} 0\,, & \text{if } k \text{ and } k' \text{ belong to the same item}\,,\\ l_{kj}l_{k'j}\pi(i)-\sum\limits_{i'}\sum\limits_{j'}(l_{kj}l_{k'j'}+l_{kj'}l_{k'j})q_{ii'}p(j'\,|\,i')\,, & \text{otherwise}\,, \end{cases}$$

$$(3.92)\quad \eta_{ij}^{(kk')}=\begin{cases} l_{kj}l_{k'j}\pi(i)-\sum\limits_{i'}\sum\limits_{j'}(l_{kj}l_{k'j'}+l_{kj'}l_{k'j})q_{ii'}p(j'\,|\,i')\,, & \\ & \text{if } k \text{ and } k' \text{ belongs to the same item}\,,\\ 0\,, & \text{otherwise}\,, \end{cases}$$

and Q is defined as follows

$$(3.93)\quad Q=[q_{ii'}]=\begin{cases} \boldsymbol{\pi}\boldsymbol{\pi}'\,, & \text{in the case based on } \hat{\Sigma}_{(ii')}\,,\\ \boldsymbol{\pi}\boldsymbol{\pi}'+(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'\{J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'\}^{-1}J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}') & \\ & \text{in the case based on } \hat{\Phi}_{(ii')}\,. \end{cases}$$

Thus, under the assumption that all eigenvalues are distinct, we have the asymptotic expansions (3.28)$\sim$(3.31) with $\xi_{ij}^{(kk')}$ and $\eta_{ij}^{(kk')}$ defined in (3.91)$\sim$(3.92).

3.6. *The case of the* CM–7 *criterion*

In this case the optimal score vector $\boldsymbol{t}_{(i)}$ is obtained as a solution of the eigenvalue problem (*6) in its original form or by substituting $\hat{\Phi}_{(ii')}$ for $\hat{\Sigma}_{(ii')}$. Now consider the asymptotic properties of $\boldsymbol{t}_{(i)}$ under the constraint $\boldsymbol{t}'_{(i)}\hat{\Sigma}_{(ii)}\boldsymbol{t}_{(i)}=1$ or $\boldsymbol{t}'_{(i)}\hat{\Phi}_{(ii')}\boldsymbol{t}_{(i)}=1$. For the sake of simplicity put $i=1$ and

$$A=[a_{kk'}]=\Sigma_{(1\bar{1})}\Sigma^{-1}_{(\bar{1}\bar{1})}\Sigma_{(\bar{1}1)}\ , \qquad B=[b_{kk'}]=\Sigma_{(11)}\ , \tag{3.94}$$

or

$$A=[a_{kk'}]=\Phi_{(1\bar{1})}\Phi^{-1}_{(\bar{1}\bar{1})}\Phi_{(\bar{1}1)}\ , \qquad B=[b_{kk'}]=\Phi_{(11)}\ . \tag{3.95}$$

Using the Taylor expansions, we obtain (3.24) and (3.25), where

$$\begin{aligned}\xi_{ij}^{(kk')}=&\,l_{kj}^{(1)}h_{k'j}^{(1)}\pi(i)-h_{kj}^{(1)}h_{k'j}^{(1)}\pi(i)+h_{kj}^{(1)}l_{k'j}^{(1)}\pi(i)\\&+\sum_{i'}\sum_{j'}(l_{kj}^{(1)}h_{k'j'}^{(1)}+l_{kj'}^{(1)}h_{k'j}^{(1)}-h_{kj}^{(1)}h_{k'j'}^{(1)}-h_{kj'}^{(1)}h_{k'j}^{(1)}+h_{kj}^{(1)}l_{k'j'}^{(1)}+h_{kj'}^{(1)}l_{k'j}^{(1)})\\&\cdot q_{ii'}p(j'\,|\,i')\ .\end{aligned} \tag{3.96}$$

$$\eta_{ij}^{(kk')}=l_{kj}^{(1)}l_{k'j}^{(1)}\pi(i)-\sum_{i}\sum_{j}(l_{kj}^{(1)}l_{k'j'}^{(1)}+l_{kj'}^{(1)}l_{k'j}^{(1)})q_{ii'}p(j'\,|\,i')\ , \tag{3.97}$$

$l_{kj}^{(1)}$ or $h_{kj}^{(1)}$ indicating a (k, j) element of $L_1=[l_{kj}^{(1)}]$ or

$$H_1=[h_{kj}^{(1)}]=L_1(D-P'QP)L'_{\bar{1}}\{L_{\bar{1}}(D-P'QP)L'_{\bar{1}}\}^{-1}L_{\bar{1}}\ . \tag{3.98}$$

The matrix Q is defined as follows.

$$Q=\begin{cases}\boldsymbol{\pi}\boldsymbol{\pi}'\ , & \text{in the case based on } \hat{\Sigma}_{(ii')}\ ,\\ \boldsymbol{\pi}\boldsymbol{\pi}'+(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'\{J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J'\}^{-1}J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')\ , & \\ \qquad\qquad \text{in the case based on } \hat{\Phi}_{(ii')}\ . &\end{cases} \tag{3.99}$$

Thus, under the assumption that all eigenvalues are distinct, we have the asymptotic expansions (3.28)~(3.31) with $\xi_{ij}^{(kk')}$ and $\eta_{ij}^{(kk')}$ defined in (3.96)~(3.97).

3.7. *Theorem and corollary*

From the above derivation, we have the following theorem.

THEOREM 3.1. *Assume that the eigenvalue problem* (*1) *based on the* CM-1~3, (*3) *based on the* CM-4, (*4) *based on the* CM-5, (*5) *based on the* CM-6, *or* (*6) *based on the* CM-7 *criterion has distinct eigenvalues* $\lambda_1>\lambda_2>\cdots>\lambda_p$ *(p: the rank of the eigenvalue problem). Then the asymptotic joint distribution of sample eigenvalues* $\hat{\boldsymbol{\lambda}}=[\hat{\lambda}_1,\cdots,\hat{\lambda}_p]'$ *and eigenvectors* $\boldsymbol{t}_u$, $u=1, 2,\cdots, p$ is given by

$$\sqrt{n}\begin{bmatrix}\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\\ \boldsymbol{t}_1-\boldsymbol{\tau}_1\\ \vdots\\ \boldsymbol{t}_p-\boldsymbol{\tau}_p\end{bmatrix}\sim N\left(0, \begin{bmatrix}\Phi^{(0,0)} & \Phi^{(0,1)}\cdots\Phi^{(0,p)}\\ & \Phi^{(1,1)}\cdots\Phi^{(1,p)}\\ & \cdots\\ \text{symmetric} & \Phi^{(p,p)}\end{bmatrix}\right) \tag{3.100}$$

where

$$(3.101)\qquad \begin{aligned} \Phi^{(0,0)}&=(\phi_{uv}^{(0,0)})\ , & \phi_{uv}^{(0,0)}&=\sum_i \sum_{i'} \sum_j \sum_{j'} \alpha_{ij}^{(u)}\alpha_{i'j'}^{(v)}\pi_{ij,i'j'}\ ,\\ \Phi^{(0,t)}&=(\phi_{uv}^{(0,t)})\ , & \phi_{uv}^{(0,t)}&=\sum_i \sum_{i'} \sum_j \sum_{j'} \alpha_{ij}^{(u)}\beta_{i'j'}^{(vt)}\pi_{ij,i'j'}\ ,\\ \Phi^{(s,t)}&=(\phi_{uv}^{(s,t)})\ , & \phi_{uv}^{(s,t)}&=\sum_i \sum_{i'} \sum_j \sum_{j'} \beta_{ij}^{(us)}\beta_{i'j'}^{(vt)}\pi_{ij,i'j'}\ , \end{aligned}$$

$$(3.102)\qquad \pi_{ij,i'j'}=\begin{cases} \{1-p(j\mid i)\}p(j\mid i)/\pi(i) & \textit{for } i=i',\ j=j'\ ,\\ -p(j\mid i)p(j'\mid i)/\pi(i) & \textit{for } i=i',\ j\neq j'\ ,\\ 0 & \textit{for } i\neq i'\ , \end{cases}$$

and where α*'s and* β*'s in* (3.101) *are given by* (3.30) *and* (3.31) *or* (3.45) *and* (3.49) *according to whether the rank* p *is determined by response or by factor, with right-hand sides obtained corresponding to each of the* CM-1$\sim$7 *criteria as described in Subsections* 3.2$\sim$3.6.

Although in the above we specify that the score for an arbitrary category per item is zero for normalization of location, the optimal score vector may be sometimes required to satisfy

$$(3.103)\qquad \sum_{i=1}^{m}\sum_{l=1}^{r_j}\left\{\sum_{j'\in S_1(j)} n(i;\,j')\right\} t^{\sharp}_{(m)[jl]}=0\ ,$$

where $t^{\sharp}_{(m)[jl]}$ denotes an adjusted score for category l of response item j. Concerning to the asymptotic distribution of the score vector $\boldsymbol{t}_{(m)}$ under (3.103), the following corollary is derived from Theorem 3.1, by means of the δ-method applying to the relation

$$(3.104)\qquad t^{\sharp}_{(m)[jl]}=t^{\sharp}_{[jl]}-\sum_{i=1}^{m}\sum_{l'=1}^{r_j}\left\{\sum_{j'\in S_1(jl')} n(i;\,j')/n\right\} t^{\sharp}_{[jl']}\ .$$

COROLLARY. *The asymptotic distribution of the sample eigenvector* $\boldsymbol{t}_{(m)}$ *under the constraint* (3.103) *is multivariate normal as follows.*

$$(3.105)\qquad \sqrt{n}(\boldsymbol{t}_{(m)}-\boldsymbol{\tau}_{(m)})\sim N(\boldsymbol{0},\,\Phi^{(1,1)}_{(m)})\ ,$$

where

$$(3.106)\qquad \begin{aligned} \Phi^{(1,1)}_{(m)}&=[\phi^{(1,1)}_{(m)[jl][j'l']}]\ ,\\ \phi^{(1,1)}_{(m)[jl][j'l']}&=\sum_i \sum_k \sum_{i'} \sum_{k'} \beta^{([jl]1)}_{(m)ik}\beta^{([j'l']1)}_{(m)i'k'}\pi_{ik,i'k'}\ ,\\ \beta^{([jl]1)}_{(m)ik}&=\beta^{([jl]1)}_{ik}-\sum_{i'}\sum_{l'}\sum_{j'\in S_1(jl')}\pi(i')p(j'\mid i')\beta^{([jl']1)}_{ik}\\ &\qquad -\sum_{l'}\delta(k:\,[jl'])\pi(i)\tau^{\sharp}_{[jl']1}\ , \end{aligned}$$

$$\delta(k:[jl'])=\begin{cases} 1 & \text{for } k\in \mathcal{S}_1(jl') ,\\ 0 & \text{for } k\notin \mathcal{S}_1(jl') . \end{cases}$$

4. An asymptotic method of statistical inference on effects of factors

Using a similar procedure as applied in the previous paper [6], we can easily obtain the following theorems.

THEOREM 4.1. *Under the restriction that the effects of the categories omitted are zero, the asymptotic distribution of the least square estimate* $\hat{\boldsymbol{\theta}}_s=[\hat{\theta}_{ks}]$ *in the model* (2.5) *based on the optimal score vector* $\boldsymbol{t}_s$ *is multivariate normal with mean* $\boldsymbol{\theta}_s$ *and covariance matrix* $n^{-1}\Omega_{(s)}$, *where*

$$\Omega_{(s)}=[\omega_{uu'}^{(s)}] , \qquad \omega_{uu'}^{(s)}=\sum_i \sum_{i'} \sum_j \sum_{j'} \gamma_{ij}^{(us)}\gamma_{i'j'}^{(u's)}\pi_{ij,i'j'} , \tag{4.1}$$

$$\gamma_{ij}^{(us)}=\sum_k \{v_{ui}l_{kj}\tau_{ks}^{*}+\sum_{i'}\sum_{j'} v_{ui'}l_{kj'}p(j'\,|\,i')\beta_{ij}^{(ks)}\} \tag{4.2}$$

$$V=[v_{ui}]=\begin{cases} [J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J']^{-1}J(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}') , & \text{for CM-1}\sim 4 ,\\ [J_1(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}')J_1']^{-1}J_1(\Pi-\boldsymbol{\pi}\boldsymbol{\pi}') , & \text{for CM-5} , \end{cases} \tag{4.3}$$

and where $\beta_{ij}^{(ks)}$ *is defined according to each criterion as shown in Section* 3.

THEOREM 4.2. *Under an arbitrary testable hypothesis*

$$H_0 : Q'\boldsymbol{\theta}=\boldsymbol{0} , \tag{4.4}$$

the statistic

$$\chi_0^2=n\hat{\boldsymbol{\theta}}_s'Q(Q'\hat{\Omega}_{(s)}Q)^{-1}Q'\hat{\boldsymbol{\theta}}_s \tag{4.5}$$

has asymptotically a chi-square distribution with p $(=\operatorname{rank} Q)$ *degrees of freedom, where* $\Omega_{(s)}$ *is obtained by substituting the corresponding estimates into* (4.1), *and* Q *is a* $c\times p$ (*for* CM-1$\sim$4) *or* $c'\times p$ (*for* CM-5) *matrix.*

THEOREM 4.3. *Consider a sequence of alternatives*

$$H_1 : p(j\,|\,i)=p^0(j\,|\,i)+d_{ij}/\sqrt{n} , \qquad \sum_{i=1}^{R} d_{ij}=0 , \tag{4.6}$$

where $p^0(j\,|\,i)$'s *indicate the proportions satisfying the null hypothesis* (4.4). *Then the statistic* χ_0^2 *has asymptotically a noncentral chi-square distribution with* p *degrees of freedom and the following noncentrality parameter,*

$$\Delta=\boldsymbol{d}'\Gamma_{(s)}Q(Q'\Omega_{(s)}Q)^{-1}Q'\Gamma_{(s)}\boldsymbol{d} , \tag{4.7}$$

where $\Gamma_{(s)}=[\gamma_{ij}^{(us)}]$ *indicates a* $c\times mR$ *matrix with the elements defined in* (4.2), *and* $\boldsymbol{d}=[d_{ij}]$ *an* mR *dimensional vector with the elements defined in* (4.6).

5. Numerical example

As an illustration of the results in the preceding sections, we shall analyze the data shown in Table 5.1. These data are made artificially by categorizing continuous responses $y_{jk}^{(i)}$'s generated according to a multivariate linear model

$$y_{jk}^{(i)}=\mu_j^{(i)}+e_{jk}^{(i)}, \qquad k=1,2,\cdots,500,\ j=1,2,\cdots,6,\ i=1,2,$$

where $(\mu_j^{(1)},\mu_j^{(2)})$'s are $(-5,0)$, $(-5,1)$, $(-6,0)$, $(-5,-1)$, $(-4,0)$, $(-6,-1)$ and the normally distributed errors $(e_{jk}^{(1)},e_{jk}^{(2)})$'s are generated by Box-Muller method so as to satisfy $\mu_1=\mu_2=0$, $\sigma_1=\sigma_2=10$, $r=0.3$.

Table 5.1 Numerical example

Level of factor	Response 2 \ Response 1	Response 1			Total
		1	2	3	
1	1	37	48	13	500
	2	81	221	100	
2	1	27	43	19	500
	2	50	223	138	
3	1	23	41	12	500
	2	104	219	101	
4	1	50	56	5	500
	2	104	211	74	
5	1	55	74	25	500
	2	76	170	100	
6	1	34	30	2	500
	2	132	231	71	

From Table 5.1, the multinomial probabilities are estimated as

$$\hat{P}=\begin{array}{c} \begin{array}{cccccc} (1,1) & (1,2) & (2,1) & (2,2) & (3,1) & (3,2) \end{array} \\ \begin{bmatrix} 0.0740 & 0.1620 & 0.0960 & 0.4420 & 0.0260 & 0.2000 \\ 0.0540 & 0.1000 & 0.0860 & 0.4460 & 0.0380 & 0.2760 \\ 0.0460 & 0.2080 & 0.0820 & 0.4380 & 0.0240 & 0.2020 \\ 0.1000 & 0.2080 & 0.1120 & 0.4220 & 0.0100 & 0.1480 \\ 0.1100 & 0.1520 & 0.1480 & 0.3400 & 0.0500 & 0.2000 \\ 0.0680 & 0.2640 & 0.0600 & 0.4620 & 0.0040 & 0.1420 \end{bmatrix} \end{array} \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \end{array} .$$

On the basis of the CM-1$\sim$3 criteria, the optimal scores are given as a solution of the eigenvalue problem such that

$$(\hat{A}-\hat{\lambda}\hat{B})\boldsymbol{t}=\boldsymbol{0} ,$$

where

$$\hat{A}=\hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}=\begin{bmatrix} 0.0032 & -0.0002 & -0.0002 \\ -0.0002 & 0.0003 & -0.0006 \\ -0.0002 & -0.0006 & 0.0033 \end{bmatrix},$$

$$\hat{B}=\hat{\Sigma}_{11}=\begin{bmatrix} 0.1913 & -0.1346 & 0.0243 \\ -0.1346 & 0.2495 & -0.0061 \\ 0.0243 & -0.0061 & 0.1588 \end{bmatrix},$$

omitting a last category in each item. The eigenvalues and corresponding eigenvectors are

$$\hat{\boldsymbol{\lambda}}=[0.0325 \quad 0.0170 \quad 0.0006]' ,$$

$$[\boldsymbol{t}_1 \quad \boldsymbol{t}_2 \quad \boldsymbol{t}_3]=\begin{bmatrix} 2.5482 & 1.4600 & 0.0646 \\ 1.4542 & 0.5417 & 2.0210 \\ -1.5780 & 1.9537 & 0.3844 \end{bmatrix},$$

while the asymptotic covariance matrices of $\hat{\boldsymbol{\lambda}}$ and $\boldsymbol{t}_1$ are obtained as

$$\widehat{\mathrm{Cov}}\,(\hat{\boldsymbol{\lambda}})=\frac{1}{3000}\begin{bmatrix} 0.1152 & 0.0025 & -0.0000 \\ 0.0025 & 0.0712 & -0.0006 \\ -0.0000 & -0.0006 & 0.0026 \end{bmatrix},$$

$$\widehat{\mathrm{Cov}}\,(\boldsymbol{t}_1)=\frac{1}{3000}\begin{bmatrix} 432.5618 & 167.5566 & 574.8704 \\ 167.5565 & 189.8284 & 240.9344 \\ 574.8704 & 240.9347 & 776.9349 \end{bmatrix},$$

by using Theorem 4.1. Thus the estimates and standard errors are given as follows.

(criterion)	Estimate	S.E.
$\hat{\lambda}$	0.0325	0.0062
(optimal scores $\boldsymbol{t}_1$)		
$t^{\#}_{11}$	2.5482	0.3797
$t^{\#}_{12}$	1.4542	0.2515
$t^{\#}_{21}$	-1.5780	0.5089
(adjusted optimal scores $\boldsymbol{t}_{(m)1}$)		
$t^{\#}_{(m)11}$	1.1320	0.2314
$t^{\#}_{(m)12}$	0.0380	0.1027
$t^{\#}_{(m)13}$	-1.4162	0.2054
$t^{\#}_{(m)21}$	-1.2656	0.4080
$t^{\#}_{(m)22}$	0.3124	0.1017

It is noted that the optimal scores satisfy the order relation $t^{\#}_{(m)11} > t^{\#}_{(m)12} > t^{\#}_{(m)13}$.

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