Ann. Inst. Statist. Math.
30 (1978), Part A, 315-319

REGIONS OF AUTOCORRELATION COEFFICIENTS IN AR (p) AND EX (p) PROCESSES

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(Received Oct. 31, 1977; revised June 7, 1978)

1. Introduction

In the earlier paper [3] we discussed about the regions of autocorrelation coefficients $(\rho_1, \cdots, \rho_p)$ for various sets of the spectral distribution functions of the stationary time series. In this paper we obtain the regions of $(\rho_1, \cdots, \rho_p)$ of the processes with the following spectral densities of $[0, \pi]$.
Autoregressive (AR (p)) type:

$$f(\lambda | \boldsymbol{\theta}, \sigma^2) = \frac{\sigma^2}{\pi} \left| 1 - \sum_{s=1}^{p} \theta_s e^{is\lambda} \right|^{-2} . \tag{1.1}$$

Exponential (EX (p)) type:

$$f(\lambda | \boldsymbol{\theta}, \sigma^2) = \frac{\sigma^2}{\pi} \exp\left(\sum_{s=1}^{p} \theta_s \cos s\lambda \right) . \tag{1.2}$$

Let $\mathcal{F}$ be the set of all probability distribution functions on $[0, \pi]$. For an arbitrary set $\mathcal{U}$ in $\mathcal{F}$, we let

$$\boldsymbol{R}_p(\mathcal{U}) = \left\{ (\rho_1, \cdots, \rho_p) :\ \rho_s = \int_0^\pi \cos s\lambda dG(\lambda),\ G \in \mathcal{U} \right\} .$$

We use notations AR (p) and EX (p) also as the sets of all normalized spectral distribution functions with the spectral densities (1.1) and (1.2). Let ∂A mean the set of all boundary points of a set A.

2. Autoregressive type

THEOREM 2.1. *For any positive integer p,*

$$\boldsymbol{R}_p(\text{AR}\,(p)) = \boldsymbol{R}_p(\mathcal{F}) - \partial \boldsymbol{R}_p(\mathcal{F}) .$$

PROOF. From Theorem 3.2 of [3], $\boldsymbol{R}_p(\text{AR}\,(p)) \subset \boldsymbol{R}_p(\mathcal{F}) - \partial \boldsymbol{R}_p(\mathcal{F})$. In order to prove the converse half, we will first prove the convexity of $\boldsymbol{R}_p(\text{AR}\,(p))$.

Let $\{\rho_{1,s}: s=1, 2, \cdots\}$ and $\{\rho_{2,s}: s=1, 2, \cdots\}$ be the sequences of the autocorrelations of two AR(p) processes. From Theorems 3.1 and 3.2 of [3], for any number ν on the interval $[0, 1]$, there is a stationary process with the absolutely continuous spectral distribution function whose autocorrelations $\rho_{3,s}$ satisfy

$$\rho_{3,s}=\nu\rho_{1,s}+(1-\nu)\rho_{2,s}\,, \qquad s=1, \cdots, p\,.$$

Let $\{\Phi_s\}$ be the sequence of the partial autocorrelations of this process, and let

$$\Phi_s^*=\begin{cases} \Phi_s\,, & s=1, \cdots, p \\ 0\,, & s\geqq p+1\,. \end{cases}$$

From Ramsey's [4] Theorem 1, $|\Phi_s^*|<1$ for all s and $\{\Phi_s^*\}$ defines a unique positive definite sequence $\{\rho_s^*\}$ via the relation between autocorrelations and partial autocorrelations. Therefore, there is a stationary process with the autocorrelations ρ_s^* and the partial autocorrelations Φ_s^*. That process is found to be AR(p) process from Ramsey's [4] Theorem 3, so that the convexity of $\boldsymbol{R}_p(\mathrm{AR}(p))$ holds.

It is obvious that $\boldsymbol{R}_1(\mathrm{AR}(1))=\boldsymbol{R}_1(\mathcal{F})-\partial\boldsymbol{R}_1(\mathcal{F})$. When $p\geqq 2$, it is easily found that for any number λ on $[0, \pi]$ there is a sequence $\{G_j\}$ in AR(p) converging weakly to the one point distribution on λ. Therefore the closure of $\boldsymbol{R}_p(\mathrm{AR}(p))$ contains the curve $\{(\cos\lambda, \cdots, \cos p\lambda): 0\leqq \lambda\leqq\pi\}$. Since $\boldsymbol{R}_p(\mathrm{AR}(p))$ is convex, this means from Theorem 3.1 of [2] that $\boldsymbol{R}_p(\mathrm{AR}(p))=\boldsymbol{R}_p(\mathcal{F})-\partial\boldsymbol{R}_p(\mathcal{F})$. Q.E.D.

The following corollary shows that the special Yule-Walker estimates exist in the stationary region of parameters.

COROLLARY 2.1. *Let* $\hat\rho_s=\sum_{t=1}^{n-s} x_t x_{t+s}\Big/\sum_{t=1}^{n} x_t^2$ *for observations* $x_1, \cdots, x_n$. *Let* $\hat{\boldsymbol{\theta}}=(\theta_1, \cdots, \theta_p)'$ *be the Yule-Walker estimates of structural parameters* $\boldsymbol{\theta}$ *of the pth order autoregressive process obtained by using the* $\hat\rho_s$*'s, i.e.*

$$\begin{pmatrix} \hat\theta_1 \\ \vdots \\ \hat\theta_p \end{pmatrix}=\mathrm{Toepl}_p\,[1, \hat\rho_1, \cdots, \hat\rho_{p-1}]^{-1}\begin{pmatrix} \hat\rho_1 \\ \vdots \\ \hat\rho_p \end{pmatrix}$$

where $\mathrm{Toepl}_p[\cdots]$ *is the* $p\times p$ *Toeplitz matrix. Then, all the roots of* $z^p-\hat\theta_1 z^{p-1}-\cdots-\hat\theta_p=0$ *lie inside the unit circle.*

PROOF. By Theorem 4.1 of [3], $(\hat\rho_1, \cdots, \hat\rho_p)\in\boldsymbol{R}_p(\mathcal{F})-\partial\boldsymbol{R}_p(\mathcal{F})$, so that the corollary follows from Theorem 2.1. Q.E.D.

3. Exponential type

The density of exponential type is represented such as $f(\lambda|\boldsymbol{\theta})=1/\pi \cdot\exp\left(\sum_{s=0}^{p}\theta_s\cos s\lambda\right)$, where $\theta_0=\log\sigma^2$ and $\boldsymbol{\theta}=(\theta_0,\cdots,\theta_p)'$. Let $\gamma_s(\boldsymbol{\theta})=\int_0^\pi \cos s\lambda f(\lambda|\boldsymbol{\theta})d\lambda$ and $\boldsymbol{\gamma}(\boldsymbol{\theta})=(\gamma_0(\boldsymbol{\theta}),\cdots,\gamma_p(\boldsymbol{\theta}))'$. The autocorrelation $\rho_s=\gamma_s(\boldsymbol{\theta})/\gamma_0(\boldsymbol{\theta})$ does not depend on θ_0, so that we can define the function $\rho_s=\rho_s(\boldsymbol{\xi})$ on R^p and the mapping $\boldsymbol{\rho}(\boldsymbol{\xi})=(\rho_1(\boldsymbol{\xi}),\cdots,\rho_p(\boldsymbol{\xi}))'$ where $\boldsymbol{\xi}=(\theta_1,\cdots,\theta_p)'\in R^p$.

For any $m\times n$ matrix $A(\lambda)=(a_{ij}(\lambda))$ whose components are measurable functions on the real line and for any measurable set E, we define $\int_E A(\lambda)d\lambda$ or $\int_E Ad\lambda$ as the $m\times n$ matrix $\left(\int_E a_{ij}(\lambda)d\lambda\right)$. Then the following lemma holds.

LEMMA 3.1. *$\boldsymbol{R}_p(\mathrm{EX}\,(p))$ is an open set in R^p.*

PROOF. First we prove that the Jacobian $|\partial\rho_i/\partial\theta_j|$ is not zero for any $\boldsymbol{\xi}$. Let $\boldsymbol{P}(\lambda)=(1,\cos\lambda,\cdots,\cos p\lambda)'$. Then the Jacobian matrix of the mapping $\boldsymbol{\theta}\to\boldsymbol{\gamma}$ is

$$J(\boldsymbol{\theta})=\left(\frac{\partial\gamma_j}{\partial\theta_k}\right)=\int_0^\pi \boldsymbol{P}(\lambda)\boldsymbol{P}(\lambda)'f(\lambda|\boldsymbol{\theta})d\lambda\ .$$

For any non-zero vector $\boldsymbol{x}$, the equation $\boldsymbol{P}(\lambda)'\boldsymbol{x}=0$ has at most p real roots, so that

$$\boldsymbol{x}'\left(\int_0^\pi \boldsymbol{P}\boldsymbol{P}'fd\lambda\right)\boldsymbol{x}=\int_0^\pi(\boldsymbol{P}'\boldsymbol{x})^2fd\lambda>0\ .$$

This means that $J(\boldsymbol{\theta})$ is positive definite. Since $\partial\rho_j/\partial\theta_0=0$ and $\partial\gamma_0/\partial\theta_0=\gamma_0$,

$$\begin{vmatrix}\dfrac{\partial\rho_1}{\partial\theta_1},&\cdots,&\dfrac{\partial\rho_1}{\partial\theta_p}\\ \vdots&&\vdots\\ \dfrac{\partial\rho_p}{\partial\theta_1},&\cdots,&\dfrac{\partial\rho_p}{\partial\theta_p}\end{vmatrix}=\gamma_0^{-1}\begin{vmatrix}\dfrac{\partial\gamma_0}{\partial\theta_0},&\dfrac{\partial\gamma_0}{\partial\theta_1},&\cdots,&\dfrac{\partial\gamma_0}{\partial\theta_p}\\ \dfrac{\partial\rho_1}{\partial\theta_0},&\dfrac{\partial\rho_1}{\partial\theta_1},&\cdots,&\dfrac{\partial\rho_1}{\partial\theta_p}\\ \vdots&\vdots&&\vdots\\ \dfrac{\partial\rho_p}{\partial\theta_0},&\dfrac{\partial\rho_p}{\partial\theta_1},&\cdots,&\dfrac{\partial\rho_p}{\partial\theta_p}\end{vmatrix}\ .$$

The Jacobian of the mapping from $(\gamma_0,\cdots,\gamma_p)$ to $(\gamma_0,\rho_1,\cdots,\rho_p)$ is γ_0^{-p}, so that above Jacobian is $\gamma_0^{-p-1}|J(\boldsymbol{\theta})|$ $(\neq 0)$.

Suppose that $\boldsymbol{R}_p(\mathrm{EX}\,(p))$ is not open. Then there is a point $\boldsymbol{\xi}$ such that $\boldsymbol{\rho}(\boldsymbol{\xi})\in\partial\boldsymbol{R}_p(\mathrm{EX}\,(p))$. Since the Jacobian $|\partial\rho_i/\partial\theta_j|$ at $\boldsymbol{\xi}$ is not zero, some neighborhood of $\boldsymbol{\rho}(\boldsymbol{\xi})$ is contained in $\boldsymbol{R}_p(\mathrm{EX}\,(p))$ by the inverse function theorem (e.g. [5], p. 68, Theorem 7A). This contradicts the fact that $\boldsymbol{\rho}(\boldsymbol{\xi}) \in \partial \boldsymbol{R}_p(\mathrm{EX}(p))$. Q.E.D.

THEOREM 3.1. *For any positive integer p,*

$$\boldsymbol{R}_p(\mathrm{EX}(p)) = \boldsymbol{R}_p(\mathcal{F}) - \partial \boldsymbol{R}_p(\mathcal{F}) .$$

PROOF. It is clear that $\boldsymbol{R}_p(\mathrm{EX}(p)) \subset \boldsymbol{R}_p(\mathcal{F}) - \partial \boldsymbol{R}_p(\mathcal{F})$. In order to prove the converse part, we assume that there is a vector $\boldsymbol{a}$ such that $\boldsymbol{a} \in \boldsymbol{R}_p(\mathcal{F}) - \partial \boldsymbol{R}_p(\mathcal{F}) - \boldsymbol{R}_p(\mathrm{EX}(p))$. Let $\boldsymbol{b}$ be an arbitrary point in $\boldsymbol{R}_p \cdot (\mathrm{EX}(p))$. Then there is a boundary point of $\boldsymbol{R}_p(\mathrm{EX}(p))$ on the line segment connecting $\boldsymbol{a}$ and $\boldsymbol{b}$, and that point is contained in $\boldsymbol{R}_p(\mathcal{F}) - \partial \boldsymbol{R}_p(\mathcal{F})$ because of the convexity of $\boldsymbol{R}_p(\mathcal{F}) - \partial \boldsymbol{R}_p(\mathcal{F})$. Therefore, the contradiction is derived by proving that if $\boldsymbol{c} \in \partial \boldsymbol{R}_p(\mathrm{EX}(p))$, then $\boldsymbol{c} \in \partial \boldsymbol{R}_p(\mathcal{F})$.

Let $\boldsymbol{\rho}_j = \boldsymbol{\rho}(\boldsymbol{\xi}_j)$ be the sequence in $\boldsymbol{R}_p(\mathrm{EX}(p))$ which converges to a boundary point $\boldsymbol{c}$. Suppose that the sequence $\boldsymbol{\xi}_j$ is bounded. Then, there is a subsequence $\boldsymbol{\xi}_{j_n}$ which converges to some point $\boldsymbol{\xi}^* \in R^p$. Since $\boldsymbol{\rho}(\boldsymbol{\xi})$ is continuous, the equation $\boldsymbol{c} = \boldsymbol{\rho}(\boldsymbol{\xi}^*)$ holds. Hence, $\boldsymbol{c} \in \boldsymbol{R}_p(\mathrm{EX}(p))$. This contradicts the Lemma 3.1. Therefore $\boldsymbol{\xi}_j = (\theta_{j1}, \cdots, \theta_{jp})'$ disperses.

Without loss of generality we can assume that $|\theta_{j,k}| \geqq |\theta_{j,t}|$ $(t=1, \cdots, p)$ for some integer k which is not dependent on j, the $\theta_{j,k}$'s are all positive or all negative and $\nu_{j,t} \equiv \theta_{j,t}/\theta_{j,k}$ $(t=1, \cdots, p)$ converge to values ν_t respectively. First we shall consider the case where the $\theta_{j,k}$'s are all positive.

The sequence of the probability distribution functions with the densities $f(\lambda | \boldsymbol{\xi}_j, \sigma^2) \Big/ \int_0^\pi f(\mu | \boldsymbol{\xi}_j, \sigma^2) d\mu$ has the subsequence converging weakly to some probability distribution function G, and $\boldsymbol{c}$ is an autocorrelation vector for G. In order to show that G has probability masses only on the points which maximize the function $\sum_{s=1}^{p} \nu_s \cos s\lambda$, we select arbitrary numbers α, β and ε such that $\alpha - \varepsilon > \beta + \varepsilon$, $\varepsilon > 0$ and that the Lebesgue measures of the sets

$$E = \left\{ \lambda \in [0, \pi] : \sum_{s=1}^{p} \nu_s \cos s\lambda \geqq \alpha \right\}$$

and

$$D = \left\{ \lambda \in [0, \pi] : \sum_{s=1}^{p} \nu_s \cos s\lambda \leqq \beta \right\}$$

are both positive. Then

$$\int_E \frac{f(\lambda | \boldsymbol{\xi}_j, \sigma^2)}{\int_0^\pi f(\mu | \boldsymbol{\xi}_j, \sigma^2) d\mu} d\lambda \Big/ \int_D \frac{f(\lambda | \boldsymbol{\xi}_j, \sigma^2)}{\int_0^\pi f(\mu | \boldsymbol{\xi}_j, \sigma^2) d\mu} d\lambda$$

$$=\int_E \left\{\exp\left(\sum_{s=1}^{p} \nu_{j,s} \cos s\lambda\right)\right\}^{\theta_{j,k}} d\lambda \Big/ \int_D \left\{\exp\left(\sum_{s=1}^{p} \nu_{j,s} \cos s\lambda\right)\right\}^{\theta_{j,k}} d\lambda$$

and for sufficiently large j,

$$\geqq \{\exp(\alpha-\beta-2\varepsilon)\}^{\theta_{j,k}} \int_E d\lambda \Big/ \int_D d\lambda \xrightarrow{j\to\infty} \infty$$

which shows that G has probability masses only on the maximizing points of $\sum_{s=1}^{p} \nu_s \cos s\lambda$. The function $\sum_{s=1}^{p} \nu_s \cos s\lambda$ is the polynomial of $\cos\lambda$ whose degree is not less than k and is at most p, because $\nu_k=1$. Hence, if we count $\lambda=0$ and π as half points and other λ as one point, the number of the maximizing points of the function $\sum_{s=1}^{p} \nu_s \cos s\lambda$ on $[0, \pi]$ is at most $p/2$. Therefore by Theorem 2.1 of [2] the vector $\boldsymbol{c}$ is located on $\partial \boldsymbol{R}_p(\mathcal{F})$. Similarly we can prove this in the case when the $\theta_{j,k}$'s are all negative. Q.E.D.

Bloomfield [1] expected a good fit of the exponential model. A part of such expectation is justified by this theorem.

Acknowledgement

The author wishes to thank a referee for valuable comments.

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