REGIONS OF AUTOCORRELATION COEFFICIENTS
IN AR (p) AND EX (p) PROCESSES

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(Received Oct. 31, 1977; revised June 7, 1978)

1. Introduction

In the earlier paper [3] we discussed about the regions of autocorrelation coefficients \((\rho_1, \ldots, \rho_p)\) for various sets of the spectral distribution functions of the stationary time series. In this paper we obtain the regions of \((\rho_1, \ldots, \rho_p)\) of the processes with the following spectral densities of \([0, \pi]\).

Autoregressive (AR (p)) type:

\[
\begin{align*}
\varphi(\lambda | \theta, \sigma^2) &= \frac{\sigma^2}{\pi} \left| 1 - \sum_{i=1}^{p} \theta_i e^{i\lambda} \right|^{-2}. \\
\end{align*}
\]

Exponential (EX (p)) type:

\[
\begin{align*}
\varphi(\lambda | \theta, \sigma^2) &= \frac{\sigma^2}{\pi} \exp \left( \sum_{i=1}^{p} \theta_i \cos s\lambda \right). \\
\end{align*}
\]

Let \(\mathcal{F}\) be the set of all probability distribution functions on \([0, \pi]\). For an arbitrary set \(\mathcal{U}\) in \(\mathcal{F}\), we let

\[
\mathcal{R}_p(\mathcal{U}) = \left\{ (\rho_1, \ldots, \rho_p) : \rho_\pi = \int_0^\pi \cos s\lambda dG(\lambda), \ G \in \mathcal{U} \right\}.
\]

We use notations \(\text{AR} (p)\) and \(\text{EX} (p)\) also as the sets of all normalized spectral distribution functions with the spectral densities (1.1) and (1.2). Let \(\partial A\) mean the set of all boundary points of a set \(A\).

2. Autoregressive type

**Theorem 2.1.** For any positive integer \(p\),

\[
\mathcal{R}_p(\text{AR} (p)) = \mathcal{R}_p(\mathcal{F}) - \partial \mathcal{R}_p(\mathcal{F}).
\]

**Proof.** From Theorem 3.2 of [3], \(\mathcal{R}_p(\text{AR} (p)) \subset \mathcal{R}_p(\mathcal{F}) - \partial \mathcal{R}_p(\mathcal{F}).\) In order to prove the converse half, we will first prove the convexity of \(\mathcal{R}_p(\text{AR} (p))\).
Let \( \{\rho_{s,t}: s=1, 2, \ldots \} \) and \( \{\rho_{s,t}: s=1, 2, \ldots \} \) be the sequences of the autocorrelations of two AR\((p)\) processes. From Theorems 3.1 and 3.2 of [3], for any number \( \nu \) on the interval \([0, 1]\), there is a stationary process with the absolutely continuous spectral distribution function whose autocorrelations \( \rho_{s,t} \) satisfy

\[
\rho_{s,t} = \nu \rho_{s,t} + (1-\nu)\rho_{s,t}, \quad s=1, \ldots, p.
\]

Let \( \{\Phi_s\} \) be the sequence of the partial autocorrelations of this process, and let

\[
\Phi^*_s = \begin{cases} 
\Phi_s, & s=1, \ldots, p \\
0, & s \geq p+1.
\end{cases}
\]

From Ramsey's [4] Theorem 1, \( |\Phi^*_s| < 1 \) for all \( s \) and \( \{\Phi^*_s\} \) defines a unique positive definite sequence \( \{\rho^*_s\} \) via the relation between autocorrelations and partial autocorrelations. Therefore, there is a stationary process with the autocorrelations \( \rho^*_s \) and the partial autocorrelations \( \Phi^*_s \). That process is found to be AR\((p)\) process from Ramsey's [4] Theorem 3, so that the convexity of \( R_p(AR(p)) \) holds.

It is obvious that \( R_p(AR(1)) = R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) \). When \( p \geq 2 \), it is easily found that for any number \( \lambda \) on \([0, \pi]\) there is a sequence \( \{G_j\} \) in AR\((p)\) converging weakly to the one point distribution on \( \lambda \). Therefore the closure of \( R_p(AR(p)) \) contains the curve \( \{(\cos \lambda, \ldots, \cos p\lambda): 0 \leq \lambda \leq \pi\} \). Since \( R_p(AR(p)) \) is convex, this means from Theorem 3.1 of [2] that \( R_p(AR(p)) = R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) \). Q.E.D.

The following corollary shows that the special Yule-Walker estimates exist in the stationary region of parameters.

**Corollary 2.1.** Let \( \hat{\rho}_r = \sum_{i=1}^{n-r} x_i x_{i+r} / \sum_{i=1}^{n} x_i^2 \) for observations \( x_1, \ldots, x_n \).

Let \( \hat{\theta} = (\theta_1, \ldots, \theta_p)' \) be the Yule-Walker estimates of structural parameters \( \theta \) of the \( p \)th order autoregressive process obtained by using the \( \hat{\rho}_s \)'s, i.e.

\[
\begin{pmatrix} 
\hat{\theta}_1 \\
\vdots \\
\hat{\theta}_p 
\end{pmatrix} = \text{Toepl}_p \left[ 1, \hat{\rho}_1, \ldots, \hat{\rho}_{p-1} \right]^{-1} \begin{pmatrix} 
\hat{\rho}_1 \\
\vdots \\
\hat{\rho}_p 
\end{pmatrix}
\]

where \( \text{Toepl}_p[\ldots] \) is the \( p \times p \) Toeplitz matrix. Then, all the roots of \( z^p - \hat{\theta}_1 z^{p-1} - \cdots - \hat{\theta}_p = 0 \) lie inside the unit circle.

**Proof.** By Theorem 4.1 of [3], \( (\hat{\rho}_1, \ldots, \hat{\rho}_p) \in R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) \), so that the corollary follows from Theorem 2.1. Q.E.D.
3. Exponential type

The density of exponential type is represented such as \( f(\lambda | \theta) = \frac{1}{\pi} \exp \left( \sum_{j=0}^{p} \theta_j \cos s \lambda \right) \), where \( \theta_0 = \log a^2 \) and \( \theta = (\theta_1, \ldots, \theta_p)' \). Let \( \gamma(s) = \int_0^s \cos s \lambda f(\lambda | \theta) d\lambda \) and \( \gamma(\theta) = (\gamma_0(\theta), \ldots, \gamma_p(\theta))' \). The autocorrelation \( \rho_s = \gamma_s(\theta)/\gamma_0(\theta) \) does not depend on \( \theta_s \), so that we can define the function \( \rho_s = \rho_s(\xi) \) on \( \mathbb{R}^p \) and the mapping \( \rho(\xi) = (\rho_1(\xi), \ldots, \rho_p(\xi))' \) where \( \xi = (\theta_1, \ldots, \theta_p)' \in \mathbb{R}^p \).

For any \( m \times n \) matrix \( A(\lambda) = (a_{ij}(\lambda)) \) whose components are measurable functions on the real line and for any measurable set \( E \), we define \( \int_E A(\lambda)d\lambda \) or \( \int_E A\lambda d\lambda \) as the \( m \times n \) matrix \( \left( \int_E a_{ij}(\lambda)d\lambda \right) \). Then the following lemma holds.

**Lemma 3.1.** \( R_p(\text{EX}(p)) \) is an open set in \( \mathbb{R}^p \).

**Proof.** First we prove that the Jacobian \( |\partial \rho_s/\partial \theta_j| \) is not zero for any \( s \). Let \( P(\lambda) = (1, \cos \lambda, \cdots, \cos p\lambda)' \). Then the Jacobian matrix of the mapping \( \theta \to \gamma \) is

\[
J(\theta) = \left( \frac{\partial \gamma_j}{\partial \theta_k} \right) = \int_0^s P(\lambda)P(\lambda)'f(\lambda | \theta)d\lambda.
\]

For any non-zero vector \( x \), the equation \( P(\lambda)'x = 0 \) has at most \( p \) real roots, so that

\[
x'(\int_0^s PP'f d\lambda)x = \int_0^s (P'x)'f d\lambda > 0.
\]

This means that \( J(\theta) \) is positive definite. Since \( \partial \rho_s/\partial \theta_s = 0 \) and \( \partial \gamma_s/\partial \theta_s = \gamma_s \),

\[
\begin{vmatrix}
\frac{\partial \rho_1}{\partial \theta_1} & \cdots & \frac{\partial \rho_1}{\partial \theta_p} \\
\vdots & \ddots & \vdots \\
\frac{\partial \rho_p}{\partial \theta_1} & \cdots & \frac{\partial \rho_p}{\partial \theta_p}
\end{vmatrix}
= \gamma_s \begin{vmatrix}
\frac{\partial \gamma_1}{\partial \theta_1} & \cdots & \frac{\partial \gamma_1}{\partial \theta_p} \\
\vdots & \ddots & \vdots \\
\frac{\partial \gamma_p}{\partial \theta_1} & \cdots & \frac{\partial \gamma_p}{\partial \theta_p}
\end{vmatrix}^{-1}
\begin{vmatrix}
\frac{\partial \gamma_0}{\partial \theta_1} & \cdots & \frac{\partial \gamma_0}{\partial \theta_p} \\
\vdots & \ddots & \vdots \\
\frac{\partial \gamma_p}{\partial \theta_1} & \cdots & \frac{\partial \gamma_p}{\partial \theta_p}
\end{vmatrix}.
\]

The Jacobian of the mapping from \( (\gamma_0, \ldots, \gamma_p) \) to \( (\gamma_0, \rho_1, \ldots, \rho_p) \) is \( \gamma_0^{-p} \), so that above Jacobian is \( \gamma_0^{-p} |J(\theta)| (\neq 0) \).

Suppose that \( R_p(\text{EX}(p)) \) is not open. Then there is a point \( \xi \) such that \( \rho(\xi) \in \partial R_p(\text{EX}(p)) \). Since the Jacobian \( |\partial \rho_s/\partial \theta_s| \) at \( \xi \) is not zero, some neighborhood of \( \rho(\xi) \) is contained in \( R_p(\text{EX}(p)) \) by the inverse
function theorem (e.g. [5], p. 68, Theorem 7A). This contradicts the fact that \( \rho(\xi) \in \partial R_p(\mathbb{E}(p)) \).

**Theorem 3.1.** For any positive integer \( p \),

\[
R_p(\mathbb{E}(p)) = R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) .
\]

**Proof.** It is clear that \( R_p(\mathbb{E}(p)) \subset R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) \). In order to prove the converse part, we assume that there is a vector \( a \) such that \( a \in R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) - R_p(\mathbb{E}(p)) \). Let \( b \) be an arbitrary point in \( R_p(\mathbb{E}(p)) \). Then there is a boundary point of \( R_p(\mathbb{E}(p)) \) on the line segment connecting \( a \) and \( b \), and that point is contained in \( R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) \) because of the convexity of \( R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) \). Therefore, the contradiction is derived by proving that if \( c \in \partial R_p(\mathbb{E}(p)) \), then \( c \in \partial R_p(\mathcal{F}) \).

Let \( \rho_j = \rho(\xi_j) \) be the sequence in \( R_p(\mathbb{E}(p)) \) which converges to a boundary point \( c \). Suppose that the sequence \( \xi_j \) is bounded. Then, there is a subsequence \( \xi_{j_n} \) which converges to some point \( \xi^* \in R^p \). Since \( \rho(\xi) \) is continuous, the equation \( c = \rho(\xi^*) \) holds. Hence, \( c \in R_p(\mathbb{E}(p)) \). This contradicts the Lemma 3.1. Therefore \( \xi_j = (\theta_{j,1}, \ldots, \theta_{j,p})' \) disperses.

Without loss of generality we can assume that \( |\theta_{j,k}| \geq |\theta_{j,i}| \) (\( t = 1, \ldots, p \)) for some integer \( k \) which is not dependent on \( j \), the \( \theta_{j,k} \)'s are all positive or all negative and \( \nu_{j,i} = \theta_{j,i}/\theta_{j,k} \) (\( t = 1, \ldots, p \)) converge to values \( \nu_i \) respectively. First we shall consider the case where the \( \theta_{j,k} \)'s are all positive.

The sequence of the probability distribution functions with the densities \( f(\lambda|\xi_j, \sigma^2)/\int_0^\infty f(\mu|\xi_j, \sigma^2) d\mu \) has the subsequence converging weakly to some probability distribution function \( G \), and \( c \) is an autocorrelation vector for \( G \). In order to show that \( G \) has probability masses only on the points which maximize the function \( \sum_{i=1}^p \nu_i \cos s\lambda \), we select arbitrary numbers \( \alpha, \beta \) and \( \varepsilon > 0 \) such that \( \alpha - \varepsilon > \beta + \varepsilon, \varepsilon > 0 \) and that the Lebesgue measures of the sets

\[
E = \left\{ \lambda \in [0, \pi]: \sum_{i=1}^p \nu_i \cos s\lambda \geq \alpha \right\}
\]

and

\[
D = \left\{ \lambda \in [0, \pi]: \sum_{i=1}^p \nu_i \cos s\lambda \leq \beta \right\}
\]

are both positive. Then

\[
\int_0^\infty \int_0^\pi f(\lambda|\xi_j, \sigma^2) d\lambda / \int_0^\pi f(\mu|\xi_j, \sigma^2) d\mu \quad \text{and} \quad \int_0^\pi \int_0^\infty f(\lambda|\xi_j, \sigma^2) d\lambda / \int_0^\pi f(\mu|\xi_j, \sigma^2) d\mu
\]
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\[ \int_E \left\{ \exp \left( \sum_{i=1}^p \nu_i \cos s \lambda \right) \right\}^{\ell_j, k} d\lambda / \int_D \left\{ \exp \left( \sum_{i=1}^p \nu_i \cos s \lambda \right) \right\}^{\ell_j, k} d\lambda \]

and for sufficiently large \( j \),

\[ \geq \left\{ \exp (\alpha - \beta - 2s) \right\}^{\ell_j, k} \int_E d\lambda / \int_D d\lambda \xrightarrow{f \to \infty} \infty \]

which shows that \( G \) has probability masses only on the maximizing points of \( \sum_{i=1}^p \nu_i \cos s \lambda \). The function \( \sum_{i=1}^p \nu_i \cos s \lambda \) is the polynomial of \( \cos \lambda \) whose degree is not less than \( k \) and is at most \( p \), because \( \nu_k = 1 \). Hence, if we count \( \lambda = 0 \) and \( \pi \) as half points and other \( \lambda \) as one point, the number of the maximizing points of the function \( \sum_{i=1}^p \nu_i \cos s \lambda \) on \([0, \pi]\) is at most \( p/2 \). Therefore by Theorem 2.1 of [2] the vector \( c \) is located on \( \partial R_p(\mathcal{E}) \). Similarly we can prove this in the case when the \( \theta_j, k \)'s are all negative.

Q.E.D.

Bloomfield [1] expected a good fit of the exponential model. A part of such expectation is justified by this theorem.

Acknowledgement

The author wishes to thank a referee for valuable comments.

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REFERENCES


