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LINEAR STATISTICS AND EXPONENTIAL FAMILIES

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1. Introduction

Several authors have considered the problem of characterizing those distributions which are closed under the formation of certain linear statistics ([2], [4], [5], [8], [9], [10]). In this paper we consider the related problem of characterizing those one-parameter exponential families of probability distributions which are closed under the formation of certain linear statistics. Let $\mathfrak{P}$ denote a one-parameter exponential family in x and θ with natural parameter space Θ and dominating σ-finite measure μ:

$$\mathfrak{P}=\{P_\theta:\ \theta\in\Theta,\ dP_\theta=C(\theta)\exp(\theta x)d\mu\} \tag{1}$$

where Θ is a non-degenerate interval in R.

If we denote the distribution of a random variable Y by P^Y then the problem may be formulated as follows: characterize those one-parameter exponential families $\mathfrak{P}$ for which $P^X\in\mathfrak{P}$ where (i) $X=\sum_{j=1}^{p} a_jX_j-\sum_{j=p+1}^{n} a_jX_j$, (ii) $P^{X_j}\in\mathfrak{P}$, $j=1,\cdots,n$ and (iii) the a_j, $j=1,\cdots,m$, are fixed real numbers satisfying $0<a_j<1$, $j=1,\cdots,n$.

As the number of cases which have to be considered separately is quite large we restrict ourselves to the case $p=n$ or equivalently $X=\sum_{j=1}^{n} a_jX_j$ where $0<a_j<1$, $j=1,\cdots,n$.

We remark at this point that we shall always assume that the dominating measure μ is non-degenerate and that the points of Θ which correspond to the distributions of X and X_j are interior points.

2. Preliminary results and further notation

2.1. Without loss of generality we may assume

(2) (a) the origin is an interior point of Θ,
(b) the dominating measure μ is a probability measure,
(c) $\mu=P_0$, the distribution corresponding to the param-

eter value $\theta=0\in\Theta$,

(d) P_0 is the distribution of $X=\sum_{j=1}^{n} a_j X_j$.

We leave it as an exercise to the reader to check the above statements.

We denote the characteristic function of X by φ so that

$$\varphi(z)=\int_{(-\infty,\infty)} \exp(izt)P^X(dt)=\int_{(-\infty,\infty)} \exp(izt)P_0(dt)\ . \tag{3}$$

The maximal horizontal strip of analyticity of φ is given by

$$\mathfrak{S}=\{z:\ -\rho_1<\operatorname{Im} z<\rho_2\}$$

where

$$0<\rho_1=\sup\left\{r:\ r>0,\ \int_{(-\infty,\infty)} \exp(rx)P_0(dx)<\infty\right\}\leqq\infty \tag{4}$$

and

$$0<\rho_2=\sup\left\{r:\ r>0,\ \int_{(-\infty,\infty)} \exp(-rx)P_0(dx)<\infty\right\}\leqq\infty\ . \tag{5}$$

The fact that ρ_1 and ρ_2 are both non-zero follows from (2-a) and the natural parameter space Θ is then either $(-\rho_2,\rho_1)$, $(-\rho_2,\rho_1]$, $[-\rho_2,\rho_1)$ or $[-\rho_2,\rho_1]$.

The distributions P^{X_j} of the X_j are assumed to belong to P so that for each j there exists a $b_j\in\Theta$ such that $P^{X_j}=P_{b_j}$. Thus if φ_j is the characteristic function of X_j we have

$$\varphi_j(z)=\varphi(z-ib_j)/\varphi(-ib_j) \tag{6}$$

and φ_j is analytic in the strip

$$\mathfrak{S}_j=\{z:\ -\rho_1+b_j<\operatorname{Im} z<\rho_2+b_j\}$$

which is the maximal horizontal strip of analyticity of φ_j.

2.2. As the $(X_j)_1^n$ are assumed to be independent the relation $X=\sum_{j=1}^{n} a_j X_j$ leads to the functional equation

$$\varphi(t)=\prod_{j=1}^{n}\varphi_j(a_j t)=\prod_{j=1}^{n}(\varphi(a_j t-ib_j)/\varphi(-ib_j))\ ,\qquad -\infty<t<\infty\ . \tag{7}$$

LEMMA 1. *The functional equation* (7) *holds for all z in $\mathfrak{S}$ i.e.*

$$\varphi(z)=\prod_{j=1}^{n}(\varphi(a_j z-ib_j)/\varphi(-ib_j)) \tag{8}$$

for all z satisfying

$$-\rho_1<\operatorname{Im} z<\rho_2 \tag{9}$$

where ρ_1 *and* ρ_2, $0<\rho_1, \rho_2\leqq\infty$, *are given by* (4) *and* (5).

PROOF. This is a special case of a theorem concerning the factorization of analytic characteristic functions (see [7], p. 43). Moreover this theorem implies that every factor $\varphi(a_j z-ib_j)/\varphi(-ib_j)$ is also analytic in $\mathfrak{S}$. However the maximal strip of analyticity $\mathfrak{S}'_j$ of $\varphi_j(a_j z)=\varphi(a_j z-ib_j)/\varphi(-ib_j)$ is given by

$$\mathfrak{S}'_j=\{z: (-\rho_1+b_j)a_j^{-1}<\operatorname{Im} z<(\rho_2+b_j)a_j^{-1}\}$$

and as $\mathfrak{S}\subset\mathfrak{S}'_j$ we may conclude

$$r_1=\min_{1\leqq j\leqq n}(\rho_1-b_j)a_j^{-1}\geqq\rho_1$$

and

$$r_2=\min_{1\leqq j\leqq n}(\rho_2+b_j)a_j^{-1}\geqq\rho_2 .$$

If we set $\mathfrak{D}=\{z: -r_1<\operatorname{Im} z<r_2\}$ then the $\varphi_j(a_j z)$ are analytic in $\mathfrak{D}$ and hence $\mathfrak{S}\subset\mathfrak{D}$. This implies $\mathfrak{S}=\mathfrak{D}$ as otherwise the functional equation (8) would yield an analytic continuation of φ into $\mathfrak{D}$ contradicting the fact that $\mathfrak{S}$ is the maximal horizontal strip of analyticity of φ. We have therefore proved

THEOREM 1.
(a) *If* $\rho_1<\infty$ *then* $\rho_1=\max_{1\leqq j\leqq n}(b_j/(1-a_j))$.
(b) *If* $\rho_2<\infty$ *then* $\rho_2=\max_{1\leqq j\leqq n}(-b_j/(1-a_j))$.

COROLLARY. *If* $\theta\in\Theta$ *then*

$$\theta a_{j_1}\cdots a_{j_m}+b_{j_1}a_{j_2}\cdots a_{j_m}+\cdots+b_{j_m}\in\Theta$$

for all $m\geqq 1$ *and* $(j_1,\cdots,j_m)$ *with* $1\leqq j_k\leqq n$, $1\leqq k\leqq m$.

2.3. If $\rho_1<\infty$ we define $p_1=\#\{j: b_j/(1-a_j)=\rho_1\}$ and if $\rho_2<\infty$ we define $p_2=\#\{j: -b_j/(1-a_j)=\rho_2\}$ where $\#\{:\}$ denotes the number of elements in the set $\{:\}$. Theorem 1 implies that $p_1\geqq 1$ and $p_2\geqq 1$. Without loss of generality we can and shall assume that if $p_1\geqq 1$ then $\rho_1=b_j/(1-a_j)$, $j=1,\cdots,p_1$, and that if $p_2\geqq 1$ then $\rho_2=-b_j/(1-a_j)$, $j=n-p_2+1,\cdots,n$.

If $p\geqq 2$ is an integer we define $\mathfrak{A}(0,p)$ and $\mathfrak{A}(\alpha,p)$ for $\alpha\geqq 0$ by

(10) $\mathfrak{A}(0,p)=\{(c_1,\cdots,c_p)$: c_i real and positive, $\log c_i/\log c_j$ is is irrational $1\leqq i<j\leqq p\}$

and

(11) $\mathfrak{A}(\alpha, p)=\{(c_1,\cdots, c_p)\colon c_j=\alpha^{m_j},\ j=1,\cdots, p,$ where $\alpha>0$ and $m_1,\cdots, m_p$ are positive integers with highest common factor $1\}$.

Each of the different cases we consider may now be represented by an ordered set of the form $(m\colon \Gamma_1 : \Gamma_2)$ where

(12) (a) $m=\{n=1\}+2\{n\geqq 2\}$.

(b) $\Gamma_1=\infty\{\rho_1=\infty\}+\rho_1\{\rho_1<\infty$ and $p_1=1\}+(\rho_1, p_1, \mathfrak{A}(\alpha_1, p_1)) \cdot \{\rho_1<\infty,\ p_1\geqq 2$ and $(a_1,\cdots, a_{p_1})\in\mathfrak{A}(\alpha_1, p_1)\}$.

(c) $\Gamma_2=\infty\{\rho_2=\infty\}+\rho_2\{\rho_2<\infty$ and $p_2=1\}+(\rho_2, p_2, \mathfrak{A}(\alpha_2, p_2)) \cdot \{\rho_2<\infty,\ p_2\geqq 2$ and $(a_{n+1-p_2},\cdots, a_n)\in\mathfrak{A}(\alpha_2, p_2)\}$.

Thus for example $(2:\rho_1, p_1, \mathfrak{A}(\alpha_1, p_1)\colon \infty)$ denotes the case $n\geqq 2$, $\rho_1<\infty$, $p_1\geqq 2$, $(a_1,\cdots, a_{p_1})\in\mathfrak{A}(\alpha_1, p_1)$ and $\rho_2=\infty$.

3. The cases $(1:\infty\colon \infty)$ and $(2:\infty\colon \infty)$

THEOREM 2.

(a) *The case* $(1:\infty\colon \infty)$ *cannot occur.*

(b) *The case* $(2:\infty\colon \infty)$ *corresponds to a family of normal distributions with common variance* $\sigma^2>0$. *Furthermore, the* $(a_j)_1^n$ *satisfy*

(13) $$\sum_{j=1}^{n} a_j^2=1 .$$

PROOF. As $\rho_1=\rho_2=\infty$, φ is an entire function and we denote the maximum modulus of φ in the circle $|z|\leqq r$ by

$$M(r)=\max_{|z|\leqq r}|\varphi(z)|=\max(\varphi(-ir), \varphi(ir)) .$$

On iterating the functional equation (8) m times we obtain

$$\varphi(z)=\prod_{j_1=1}^{n}\cdots\prod_{j_m=1}^{n}\left(\frac{\varphi(a_{j_1}\cdots a_{j_m}z-i(b_{j_1}a_{j_2}\cdots a_{j_m}+\cdots+b_{j_m}))}{\varphi(-i(b_{j_1}a_{j_2}\cdots a_{j_m}+\cdots+b_{j_m}))}\right)$$

which implies

$$M(c^m)\leqq(\max(\varphi(ic_mc^m-id_m)/\varphi(-id_m), \varphi(-ic_mc^m-id_m)/\varphi(-id_m)))^{n^m}$$

where $1<c<1/(\max_{1\leqq j\leqq n} a_j)$ and where the c_m and d_m are such that $\lim_{m\to\infty} c_mc^m=0$ and $|d_m|\leqq\max_{1\leqq j\leqq n}|b_j|/(1-\max_{1\leqq j\leqq n} a_j)$. Thus for sufficiently large m we have

$$\log M(c^m)\leqq n^m\cdot\text{constant}$$

which implies that φ is of finite order.

Suppose now that φ has a zero at z_0. From (8) we conclude that $\varphi(a_{j_1}z_0 - ib_{j_1})=0$ for some j_1, and on repeating this we obtain a sequence $(z_k)_1^\infty$ of zeros of φ of the form

$$z_k = a_{j_1}\cdots a_{j_k}z_0 - i(b_{j_1}a_{j_2}\cdots a_{j_k}+\cdots+b_{j_k}) = c_k z_0 - id_k$$

where $\lim_{k\to\infty} c_k = 0$ and d_k is a bounded sequence of real numbers. The sequence $(z_k)_0^\infty$ therefore has a point of accumulation iy on the imaginary axis and we conclude that $\varphi(iy)=0$. This is inconsistent with (3) and therefore φ can have no zeros.

From Hadamard's factorization theorem and the theorem of Marcinkiewicz (see [7], pp. 14 and 63) it follows that φ is the characteristic function of a normal distribution and hence $\varphi(z)=\exp(i\mu z-(1/2)\sigma^2 z^2)$ where $\sigma^2>0$. On substituting this into (8) and equating powers of z we conclude that $\sum_{j=1}^{n} a_j^2 = 1$ which proves (13). As $0<a_j<1$ we see from (13) that $n\geqq 2$ and hence the case $(1:\infty:\infty)$ is impossible. This completes the proof of the theorem.

4. The cases $(1:\rho_1:\infty)$ and $(1:\infty:\rho_2)$

THEOREM 3.

(a) *In the case* $(1:\infty:\rho_2)$ *the distribution function* F *of* X *has the form*

$$F(t)=\left(1-\rho_2\int_{(t,0)} e^{\rho_2 s}|s|^c g(s)ds - e^{\rho_2 t}|t|^c g(t)\right)\{t<0\}+\{t\geqq 0\}$$

where $\rho_2=-b_1/(1-a_1)$, $c>0$ *is a constant and* g *is a non-negative right continuous function satisfying*

(i) $g(s)=g(a_1 s)$ *for all* $s<0$,

(ii) $|s|^c g(s)$ *is non-increasing for all* $s<0$

and

(iii) $\rho_2\int_{(-\infty,0)} e^{\rho_2 s}|s|^c g(s)ds=1$.

(b) *In the case* $(1:\rho_1:\infty)$ *the distribution function* F *if* X *has the form*

$$F(t)=\left(\rho_1\int_{(0,t]} e^{-\rho_1 s}s^c g(s)ds + e^{-\rho_1 t}t^c g(t)\right)\{t\geqq 0\}$$

where $\rho_1=b_1/(1-a_1)$, $c>0$ *is a constant and* g *is a non-negative right continuous function satisfying*

(i) $g(s)=g(a_1 s)$ *for all* $s>0$,

(ii) $s^c g(s)$ *is non-decreasing for all* $s>0$

and

(iii) $\rho_1 \int_{(0,\infty)} e^{\rho_1 s} s^c g(s) ds = 1$.

PROOF. As the two cases are similar we restrict ourselves to a proof of (a).

The fact $\rho_2 = -b_1/(1-a_1)$ follows from Theorem 1. We define

$$\psi(x) = \varphi(-ix) = \int_{(-\infty,\infty)} \exp(xt) F(dt) \tag{14}$$

so that $\psi(x)$ is defined for all $x > -\rho_2$ and satisfies the functional equation

$$\psi(x) = \psi(a_1 x + b_1)/\psi(b_1)\ , \qquad x > -\rho_2\ . \tag{15}$$

Furthermore, as $\psi(x) > 0$ for all $x > -\rho_2$ we may write

$$\zeta(x) = \zeta(a_1 x) - \log \psi(b_1)\ , \qquad x > 0\ , \tag{16}$$

where for $x > 0$

$$\zeta(x) = \log \psi(-\rho_2 + x) = \log \psi(b_1/(1-a_1) + x)\ .$$

The general solution of the functional equation (16) is

$$\zeta(x) = -c \log x + \lambda(x) \tag{17}$$

where $c = -(\log \psi(b_1))/\log a_1$ and where λ is any function satisfying $\lambda(x) = \lambda(a_1 x)$ for $x > 0$. We therefore obtain

$$\psi(x) = \eta(\rho_2 + x)/(\rho_2 + x)^c\ , \qquad x > -\rho_2\ , \tag{18}$$

where $\eta(x)$ satisfies

$$\eta(x) = \eta(a_1 x)\ , \qquad x > 0\ . \tag{19}$$

It follows from (14) and Fatou's lemma that

$$\int_{(-\infty,\infty)} \exp(-\rho_2 t) F(dt) \leqq \liminf_{x \downarrow -\rho_2} \psi(x)\ . \tag{20}$$

The function $\psi(x)$ is continuous for all $x > -\rho_2$ and hence so is $\eta(x)$. The functional equation (19) implies

$$0 < \sup_{x > 0} \eta(x) = \sup_{a_1 < x \leqq 1} \eta(x) < \infty \tag{21}$$

and hence if c were strictly negative it would follow from (18) and (21) that $\liminf_{x \downarrow -\rho_2} \psi(x) = 0$ contradicting (20). For $c \geqq 0$ we have

$$\limsup_{x \to \infty} \int_{(0,\infty)} \exp(xt) F(dt) \leqq \limsup_{x \to \infty} \psi(x) < \infty$$

by (14), (18) and (21). This can only be the case if $F(\infty)=F(0)$ and we may therefore write

$$\psi(x)=\int_{(-\infty,0]}\exp(xt)F(dt)\ .$$

Suppose now that $c=0$. Then

$$F(0)-F(0-)=\lim_{x\to\infty}\int_{(-\infty,0]}\exp(xt)F(dt)=\lim_{x\to\infty}\psi(x)=\lim_{x\to\infty}\eta(x)$$

by (18) with $c=0$. However, it follows from (19) that $\lim_{x\to\infty}\eta(x)$ exists if and only if $\eta(x)\equiv$constant in which case $\psi(x)\equiv$constant. This implies that F corresponds to the unit mass situated at the origin which in turn implies $\rho_2=\infty$, a contradiction. We have therefore shown that $c>0$ and that $F(0)=F(0-)=1$.

We may therefore write

$$\psi(x)=\int_{(-\infty,0)}\exp(xt)F(dt)\ . \tag{22}$$

For $t<0$ we define

$$G(t)=-\int_{(t,0)}\exp(-\rho_2 s)F(ds)\ . \tag{23}$$

It follows that

$$\psi(x)=\int_{(-\infty,0)}\exp(xt)F(dt)=\int_{(-\infty,0)}\exp((x+\rho_2)t)G(dt) \tag{24}$$

for all $x>-\rho_2$.

On integrating (24) by parts we obtain

$$\psi(x)=(\rho_2+x)^{-c}\int_{(-\infty,0)}|t|^c g(t/(\rho_2+x))e^t dt$$

where

$$g(t)=-|t|^{-c}G(t)\ ,\qquad t<0\ , \tag{25}$$

and the constant $c>0$ is as in (18). From (18) we conclude that

$$\eta(x)=\int_{(-\infty,0)}|t|^c g(t/x)e^t dt\ ,\qquad x>0\ ,$$

and the functional equation (19) implies

$$\int_{(-\infty,0)}|t|^c g(t)e^{xt}dt=\int_{(-\infty,0)}|t|^c g(t/a_1)e^{xt}dt$$

for all $x>0$.

It follows from the uniqueness of the Laplace transform that $g(t) = g(t/a_1)$ for almost all t. However, as $G(t)$ is right continuous because of (23) we may in fact conclude that $g(t)=g(t/a_1)$ for all $t<0$ or

$$g(t)=g(a_1 t)\ , \qquad t<0\ . \tag{26}$$

On using (23) ond (25) we obtain

$$F(t)=1-\rho_2\int_{(t,0)}|s|^c g(s)\exp(\rho_2 s)ds-e^{\rho_2 t}|t|^c g(t) \tag{27}$$

for all $t<0$ which is the general form of F given in (a).

We have already shown in (26) that (i) of (a) holds. The fact that (ii) holds follows from (23) and (25) as (23) implies that G is non-decreasing (F non-decreasing). Finally (iii) of (a) follows from the requirement $\lim_{t\to-\infty} F(t)=0$.

Conversely it is easily seen that any function F of the form described in (a) is a distribution function whose corresponding characteristic function φ satisfies the functional equation $\varphi(z)=\varphi(a_1 z-ib_1)/\varphi(-ib_1)$.

5. The infinitely divisible cases

5.1. LEMMA 2. *Let*

$$L_1=\inf_{m\geq 1}\min_{(j_1,\cdots,j_m)}(b_{j_1}a_{j_2}\cdots a_{j_m}+b_{j_2}a_{j_3}\cdots a_{j_m}+\cdots+b_{j_m})>-\infty$$

and

$$L_2=\sup_{m\geq 1}\max_{(j_1,\cdots,j_m)}(b_{j_1}a_{j_2}\cdots a_{j_m}+b_{j_2}a_{j_3}\cdots a_{j_m}+\cdots+b_{j_m})<\infty$$

where the minimum and maximum extend over all m-vectors $(j_1,\cdots,j_m)$ *with* $1\leqq j_k\leqq n$, $1\leqq k\leqq m$. *Suppose that* $\int_{(-\infty,\infty)}\exp(xL_1)P_0(dx)$ *and* $\int_{(-\infty,\infty)}\exp(xL_2)P_0(dx)$ *are both finite. Then* φ *is an infinitely divisible characteristic function.*

PROOF. The proof of this lemma is elementary and is omitted.

LEMMA 3.
(a) *If* $\rho_1<\infty$ *then* $L_2=\rho_1$.
(b) *If* $\rho_2<\infty$ *then* $L_1=-\rho_2$.

PROOF. The lemma follows from Theorem 1 and the corollary.

5.2. The cases which give rise to infinitely divisible families are those for which $\rho_1=\infty$ or $p_1\geqq 2$ together with $\rho_2=\infty$ or $p_2\geqq 2$. We require the following lemma.

LEMMA 4.

(a) *If* $\rho_1=\infty$ *or* $p_1\geqq 2$ *then* $\int_{(-\infty,\infty)}\exp(L_2x)P_0(dx)<\infty$.

(b) *If* $\rho_2=\infty$ *or* $p_2\geqq 2$ *then* $\int_{(-\infty,\infty)}\exp(L_1x)P_0(dx)<\infty$.

PROOF. We prove only case (a). If $\rho_1=\infty$ we have $-\rho_2<L_2<\rho_1$ and hence from (4) and (5) we conclude that (a) of the lemma holds in this case. We may therefore suppose that $0<\rho_1<\infty$ and that $p_1\geqq 2$. Let $\psi(x)$ be the moment generating function of X as defined by (14) where F is the distribution function of X. It is clear that $\psi(x)>0$ for $-\rho_2<x<\rho_1$ and that ψ satisfies the functional equation

$$\psi(x)=\prod_{j=1}^{n}\psi(a_jx+b_j)/\psi(b_j)\;, \qquad -\rho_2<x<\rho_1\,. \tag{28}$$

Let $\eta(x)=(d^2/dx^2)\log\psi(\rho_1-x)$ for $0<x<\rho_1+\rho_2$ and note that $\eta(x)>0$. Then the one can show that $\sum_{j=1}^{p_1}a_j^2<1$ and

$$\begin{aligned}\eta(x)&=\eta_0(x)+\sum_{s=1}^{\infty}\left(\sum_{j_1=p_1+1}^{n}\sum_{j_2=1}^{p_1}\cdots\sum_{j_s=1}^{p_1}a_{j_1}^2\cdots a_{j_s}^2\eta(a_{j_1}\cdots a_{j_s}x+c_{j_1})\right)\\&=\eta_0(x)+R_0(x)\end{aligned} \tag{29}$$

where $c_j=(1-a_j)\rho_1-b_j>0$, $p_1+1\leqq j\leqq n$, $\sup_{0<x<\rho_1+\rho_2-\varepsilon}R_0(x)<\infty$ for all ε satisfying $0<\varepsilon<\rho_1+\rho_2$, and $\eta_0(x)=\lim_{m\to\infty}\sum_{j_1=1}^{p_1}\cdots\sum_{j_m=1}^{p_1}a_{j_1}^2\cdots a_{j_m}^2\eta(a_{j_1}\cdots a_{j_m}x)$ is continuous and satisfies the functional equation

$$\eta_0(x)=\sum_{j=1}^{p_1}a_j^2\eta_0(a_jx)\;, \qquad x>0\,. \tag{30}$$

The general non-negative right-continuous solution of the functional equation (30) is given by Theorem 1 and Lemma 1 (including the footnote) of [2] and is as follows:

$$\eta_0(x)=x^{-\beta_1}\Delta(x) \tag{31}$$

where β_1 is such that $\sum_{j=1}^{p_1}a_j^{2-\beta_1}=1$ and where $\Delta(x)\equiv$constant if $(a_1,\cdots,a_{p_1})\in\mathfrak{A}(0,p_1)$ and $\Delta(\alpha_1x)=\Delta(x)$ if $(a_1,\cdots,a_{p_1})\in\mathfrak{A}(\alpha_1,p_1)$. In either case $\Delta(x)$ is bounded and as $p_2\geqq 2$ it follows from $p_1\geqq 2$ and $\sum_{j=1}^{p_1}a_j^2<1$ that $0<\beta_1<2$. We obtain

$$\eta(x)<K_1x^{-\beta_1}+K_2\;, \qquad 0<x<\rho_1 \tag{32}$$

and hence

$$\psi(x)<\exp(K_3(\rho_1-x)^{2-\beta_1}+K_4)\ , \qquad 0<x<\rho_1 .$$

As $\beta_1<2$ this implies $\lim_{x\uparrow\rho_1}\psi(x)<\infty$ and hence from Fatou's lemma we conclude

$$\int_{(-\infty,\infty)}\exp(\rho_1 t)F(dt)<\infty .$$

As $L_2=\rho_1$ by Lemma 3 part (a) of the lemma is proved.

LEMMA 5. *The characteristic function φ is infinitely divisible in all of the following cases*: $(2:\infty:\infty)$, $(2:\infty:\rho_2, p_2, \mathfrak{A}(\alpha_2, p_2))$, $(2:\rho_1, p_1, \mathfrak{A}(\alpha_1, p_1):\infty)$, $(2:\rho_1, p_1, \mathfrak{A}(\alpha_1, p_1):\rho_2, p_2, \mathfrak{A}(\alpha_2, p_2))$.

PROOF. This is an immediate consequence of Lemmas 2 and 4.

5.3. According to the Kolmogorov representation for analytic infinitely divisible characteristic functions we may write with the usual notation

$$\varphi(z)=\exp\left(i\mu z-\frac{1}{2}\sigma^2z^2+\int_{(-\infty,0)}(\exp(izu)-1-izu)\frac{M(du)}{u^2}\right.$$
$$\left.+\int_{(0,\infty)}(\exp(izu)-1-izu)\frac{N(du)}{u^2}\right) . \tag{33}$$

THEOREM 4.

(a) *If* $\min(\rho_1, \rho_2)<\infty$ *then* $\sum_1^n a_j^2<1$.

(b) *The general form of M is as follows*:

(i) *If* $\rho_2=\infty$ *then* $M(u)\equiv 0$

(ii) *If* $\rho_2<\infty$ *then* $M(u)=\sum_{s=0}^{\infty}\int_{(-\infty,u]}\exp(\rho_2 v)K_s(dv)$, $-\infty<u<0$,

where (1) $K_0(u)=-|u|^{\alpha}\Gamma_0(u)$, (2) α, $0<\alpha<2$, *is such that* $\sum_{j=1}^n a_j^{2-\alpha}=1$, (3) $\Gamma_0(u)\equiv$ *constant* >0 *if* $(a_{n-p_2+1},\cdots,a_n)\in\mathfrak{A}(0,p_2)$, (4) $\Gamma_0(\alpha_2 u)=\Gamma_0(u)>0$ *for all* $u<0$ *if* $(a_{n-p_2+1},\cdots,a_n)\in\mathfrak{A}(\alpha_2,p_2)$ *with* $\alpha_2>0$, (5) $K_0(u)$ *is non-decreasing and right-continuous*, $-\infty<u<0$, *and* (6) *the* $K_s(u)$ *are defined recursively by*

$$K_{s+1}(u)=\sum_{t=1}^{\infty}\left(\sum_{j_1=1}^{n-p_2}\sum_{j_2=n-p_2+1}^{n}\cdots\sum_{j_t=n-p_2+1}^{n}a_{j_1}^2\cdots a_{j_t}^2\right)$$
$$\cdot\int_{[u/(a_{j_1}\cdots a_{j_s}),0)}\exp(v((1-a_{j_1})\rho_2+b_{j_1}))K_s(dv) .$$

(c) *The general form of $N(u)$ is as follows*:

(i) *If* $\rho_1=\infty$ *then* $N(u)\equiv 0$.

(ii) *If* $\rho_1<\infty$ *then*

$$N(u)=\sum_{s=0}^{\infty}\int_{(0,u]}\exp(-\rho_1 v)L_s(dv)\ , \qquad 0<u<\infty\ ,$$

where (1) $L_0(u)=u^{\alpha}\Delta_0(u)$, $u>0$, (2) α, $0<\alpha<2$, *is such that* $\sum_{j=1}^{n} a_j^{2-\alpha}=1$, (3) $\Delta_0(u)\equiv$ *constant* >0 *if* $(a_1,\cdots,a_{p_1})\in\mathfrak{A}(0,p_1)$, (4) $\Delta_0(a_1 u)=\Delta_0(u)>0$ *for all* $u>0$ *if* $(a_1,\cdots,a_{p_1})\in\mathfrak{A}(\alpha_1,p_1)$ *with* $\alpha_1>0$, (5) $L_0(u)$ *is non-decreasing and right-continuous and* (6) *the* $L_s(u)$ *are defined recursively by*

$$L_{s+1}(u)=\sum_{t=1}^{\infty}\left(\sum_{j_1=p_1+1}^{n}\sum_{j_2=1}^{p_1}\cdots\sum_{j_t=1}^{p_1}a_{j_1}^2\cdots a_{j_t}^2\right)\cdot\int_{(0,u/(a_{j_1}\cdots a_{j_s})]}\exp(-(\rho_1(1-a_{j_1})-b_{j_1})v)L_s(dv)\ .$$

The proof of part (a) is simple. The proof of part (c) and hence also of part (b) is similar to the proof of Theorem 6 below. The functions θ_s'' and θ'' are replaced by the Laplace transforms $\tilde{L}_s$ and $\tilde{L}$ of certain Lebesgue-Stieltjes functions, thus for example, $\tilde{L}$ denotes the Laplace transform of the Lebesgue-Stieltjes function

$$L(\nu)=\int_{(0,\nu]}\exp(\rho_1 u)N(du)\ .$$

We omit the details.

The results of this section can be summarized as follows:

THEOREM 5. *For the cases* $(2\colon \rho_1, p_1, \mathfrak{A}(\alpha_1,p_1):\infty)$, $(2\colon \infty:\rho_2,p_2,\mathfrak{A}(\alpha_2,p_2))$ *and* $(2\colon \rho_1,p_1,\mathfrak{A}(\alpha_1,p_1):\rho_2,p_2,\mathfrak{A}(\alpha_2,p_2))$ *the following hold*:

(a) $\sum_{j=1}^{n} a_j^2<1$,

(b) *the characteristic function* φ *is infinitely divisible with* $\sigma^2=0$ *and Levy functions M and N as described in Theorem* 4.

6. The cases $(2:\rho_1:\infty)$ and $(2:\infty:\rho_2)$

THEOREM 6.

(a) *The general form of* $\varphi(z)$ *for the case* $(2\colon \rho_1:\infty)$ *is given by*

$$\varphi(z)=\exp(i\mu z)\prod_{s=0}^{\infty}(\varphi_s(z)\exp(-i\mu_s z))$$

where

(i) $\varphi_0(z)$ *is the general solution for the case* $(1\colon \rho_1:\infty)$,

(ii) *the* $\varphi_s(z)$ *are defined recursively by*

$$\varphi_{s+1}(z)=\prod_{t=0}^{\infty}\prod_{j=2}^{n}\left(\frac{\varphi_s(a_j(a_1^t z-i\rho_1(1-a_1^t))-ib_j)}{\varphi_s(-ia_j\rho_1(1-a_1^t)-ib_j)}\right)$$

(iii) μ_s *and* μ *are respectively the means of the distributions associated with* φ_s *and* φ.

(iv) $\sum_{j=1}^{n} a_j^2 < 1$.

(b) *The general form of* $\varphi(z)$ *for the case* $(2:\ \infty : \rho_2)$ *is as in* (a) *above where however* $\varphi_0(z)$ *is now the general solution for the case* $(1:\ \infty : \rho_2)$ *and where* a_1 *and* a_n *and* ρ_1 *and* $-\rho_2$ *are interchanged.*

PROOF. We restrict ourselves to a proof of part (a) of the theorem. Again denoting the moment-generating function of the distribution associated with φ by $\phi(x)$ we have

$$\phi(x)=\frac{\phi(a_1x+b_1)}{\phi(b_1)}\prod_{j=2}^{n}\frac{\phi(a_jx+b_j)}{\phi(b_j)}\,, \qquad x<\rho_1\,. \tag{34}$$

On iterating the first term only m times and on remembering $\rho_1=b_1/(1-a_1)$ we obtain

$$\phi(x)=\frac{\phi(a_1^m x+\rho_1(1-a_1^m))}{\phi(\rho_1(1-a_1^m))}\prod_{s=0}^{m-1}\prod_{j=2}^{n}\frac{\phi(a_j(a_1^s x+\rho_1(1-a_1^s))+b_j)}{\phi(a_j\rho_1(1-a_1^s)+b_j)}\,. \tag{35}$$

As $0<a_1<1$ the mean value theorem implies

$$\lim_{m\to\infty}\prod_{s=0}^{m-1}\prod_{j=2}^{n}\left(\frac{\phi(a_j(a_1^s x+\rho_1(1-a_1^s))+b_j)}{\phi(a_j\rho_1(1-a_1^s)+b_j)}\right)=\chi_0(x) \tag{36}$$

exists for all $x<\varphi_1$. From (35) it follows at once that

$$\lim_{m\to\infty}\frac{\phi(a_1^m x+\rho_1(1-a_1^m))}{\phi(\rho_1(1-a_1^m))}=\phi_0(x) \tag{37}$$

exists and from the results given in [1] we may conclude that $\phi_0(x)$ is the moment generating function of a probability distribution and is defined for all $x<\rho_1$. Furthermore, it is easily seen that $\phi_0(x)$ satisfies the functional equation

$$\phi_0(x)=\phi_0(a_1x+b_1)/\phi_0(b_1)\,. \tag{38}$$

We may therefore write

$$\phi(x)=\phi_0(x)\chi_0(x)\,.$$

In general we obtain

$$\phi(x)=\prod_{s=0}^{m}\phi_s(x)\chi_m(x) \tag{39}$$

where χ_m and ϕ_s, $0\leqq s\leqq m$, are moment generating functions and are defined recursively by

(40) $$\psi_{s+1}(x)=\prod_{t=0}^{\infty}\prod_{j=2}^{n}\left(\frac{\psi_s(a_j(a_1^t x+\rho_1(1-a_1^t))+b_j)}{\psi_s(a_j\rho_1(1-a_1^t)+b_j)}\right), \qquad x<\rho_1 ,$$

and

(41) $$\chi_{s+1}(x)=\prod_{t=0}^{\infty}\prod_{j=2}^{n}\left(\frac{\chi_s(a_j(a_1^t s+\rho_1(1-a_1^t))+b_j)}{\chi_s(a_j\rho_1(1-a_1^t)+b_j)}\right), \qquad x<\rho_1 .$$

On writing $\theta_s(x)=\log\psi_s(s)$, $0\leqq s<\infty$, we conclude that

(42) $$\theta''_{s+1}(x)=\sum_{t=0}^{\infty} a_1^2 \sum_{j=2}^{n} a_j^2\theta''_s(a_j^t x+\rho_1(1-a_1^t))+b_j)$$

and that

(43) $$\theta''_0(x)=\lim_{m\to\infty} a_1^{2m}\theta''(a_1^m+\rho_1(1-a_1^m))$$

where $\theta(x)=\log\psi(x)$.

With $\theta(x)$ as defined above (34) implies

$$\theta''(x)=a_1^2\theta''(a_1x+b_1)+\sum_{j=2}^{n} a_j^2\theta''(a_jx+b_j) .$$

We obtain after some calculation

(44) $$\theta''(x)=\sum_{s=0}^{m} g_s(x)+R_s(x)$$

where

(45) $$g_0(x)=\lim_{m\to\infty} a_1^{2m}\theta''(a_1^m x+\rho_1(1-a_1^m)) ,$$

(46) $$R_0(x)=\sum_{t=0}^{\infty} a_1^{2t} \sum_{j=2}^{n} a_j^2\theta''(a_j^t x+\rho_1(1-a_1^t))+b_j)$$

and where the g_s and R_s are defined recursively by

(47) $$g_{s+1}(x)=\sum_{t=0}^{\infty} a_1^{2t} \sum_{j=2}^{n} a_j^2 R_s(a_j(a_1^t x+\rho_1(1-a_1^t))+b_j)$$

and

(48) $$R_{s+1}(x)=\sum_{t=0}^{\infty} a_1^{2t} \sum_{j=2}^{n} a_j^2 R_s(a_j(a_1^t x+\rho_1(1-a_1^t))+b_j) .$$

It follows at once from (42), (43), (44) and (47) that $g_s(x)=\theta''_s(x)$ for all $x<\rho_1$ and all $s>0$. As $\theta''(x)>0$ for all $x<\rho_1$ it follows that $\theta''_s(x)>0$ and $R_s(x)>0$ for all $x<\rho_1$ and all $s>0$. From this and (44) we conclude that $\sum_{s=0}^{\infty}\theta''_s(x)$ converges for all $x<\rho_1$ and that $\lim_{s\to\infty} R_s(x)=R(x)$ exists for all $x<\rho_1$. We may therefore write

$$\theta''(x)=\sum_{s=0}^{\infty}\theta_s''(x)+R(x)\ . \tag{49}$$

It may be shown that $R(x)\equiv 0$ if $\sum_{j=1}^{n} a_j^2<1$ and that $R(x)\equiv$constant if $\sum_{j=1}^{n} a_j^2=1$. However, in the latter case we would have $\psi(x)=\exp(\lambda_0+\lambda_1 x+\lambda_2 x^2)$ which would contradict the assumption that $\rho_1<\infty$. We therefore conclude that $\sum_1^n a_j^2<1$ and that

$$\theta''(x)=\sum_{s=0}^{\infty}\theta_s''(x)\ , \qquad -\infty<x<\rho_1\ . \tag{50}$$

Since term by term integration is permissible we obtain

$$\psi(x)=\exp(\mu x)\prod_{s=0}^{\infty}(\psi_s(x)\exp(-\mu_s x))$$

where $\mu_s(\mu)$ denotes the mean of the distribution associated with $\psi_s(\psi)$. On translating these results back into terms of φ we obtain part (a) of the theorem.

7. The cases with $\max(\rho_1, \rho_2)<\infty$ and $\min(p_1, p_2)=1$

The remaining cases are covered by our final theorem.

THEOREM 7.

(a) *The general form of* $\varphi(z)$ *for the case* $(2\colon \rho_1 : \rho_2)$ *is given by*

$$\varphi(z)=\varphi_1(z)\varphi_2(z)$$

where $\varphi_1(z)$ *and* $\varphi_2(z)$ *are the general forms of* $\varphi(z)$ *for the cases* $(2\colon \rho_1 : \infty)$ *and* $(2\colon \infty : \rho_2)$ *respectively.*

(b) *The general form of* $\varphi(z)$ *for the case* $(2\colon \rho_1 : \rho_2, p_2, \mathfrak{A}(\alpha_2, p_2))$ *is given by*

$$\varphi(z)=\varphi_1(z)\varphi_2(z)$$

where $\varphi_1(z)$ *and* $\varphi_2(z)$ *are the general forms of* $\varphi(z)$ *for the cases* $(2\colon \rho_1 : \infty)$ *and* $(2\colon \infty : \rho_2, p_2, \mathfrak{A}(\alpha_2, p_2))$ *respectively.*

(c) *The general form of* $\varphi(z)$ *for the case* $(2\colon \rho_1, p_1, \mathfrak{A}(\alpha_1, p_1) : \rho_2)$ *is given by*

$$\varphi(z)=\varphi_1(z)\varphi_2(z)$$

where $\varphi_1(z)$ *and* $\varphi_2(z)$ *are the general forms of* $\varphi(z)$ *for the cases* $(2\colon \rho_1, p_1, \mathfrak{A}(\alpha_1, p_1) : \infty)$ *and* $(2\colon \infty : \rho_2)$ *respectively.*

(d) *In all cases we have* $\sum_{j=1}^{n} a_j^2<1$.

PROOF. We restrict ourselves to proofs of parts (a) and (d) of the theorem.

Arguing as in the proof of Theorem 6 we obtain

$$\phi(x)=\chi_0(x)\eta_0(x)\ , \qquad -\rho_2<x<\rho_1\ ,$$

where $\eta_0(x)$ is a moment generating function which satisfies the functional equation (38) for $-\rho_2<x<\rho_1$. Obviously $\eta_0(x)$ may be extended to a moment generating function which is defined for all $x<\rho_1$ and which satisfies the functional equation (38).

In general we obtain as in the proof of Theorem 6

$$\phi(x)=\chi_m(x)\prod_{s=0}^{m}\eta_s(x)\ , \qquad -\rho_2<x<\rho_1\ , \tag{51}$$

where the moment generating function $\eta_0(x)$ is a general solution of (38) and is defined for all $x<\rho_1$. The moment generating functions χ_s are recursively defined by (36) and (41) and the moment generating functions η_s are defined recursively by

$$\eta_{s+1}(x)=\prod_{t=0}^{\infty}\prod_{j=2}^{n}\left(\frac{\eta_s(a_j(a_1^t x+\rho_1(1-a_1^t))+b_j)}{\eta_s(a_j\rho_1(1-a_1^t)+b_j)}\right) \tag{52}$$

and are defined for all $x<\rho_1$. The infinite product in (36) converges for all x satisfying

$$-\rho_2<x<\rho_1+\min_{2\leq j\leq n}((\rho_1(1-a_j)-b_j)/a_j)$$

and it follows by induction that $\chi_s(x)$ is defined at least for all x satisfying

$$-\rho_2<x<\rho_1+c\sum_{j=1}^{s+1}a^{-j}\ . \tag{53}$$

Similarly the $\eta_s(x)$, $s\geqq 1$, are defined at least for all x satisfying $-\infty<x<\rho_1+c\sum_{j=1}^{s}a^{-j}$ respectively.

From (51) we have

$$\chi_m(x)=\eta_{m+1}(x)\chi_{m+1}(x)\ , \qquad -\rho_2<x<\rho_1 \tag{54}$$

and hence

$$\chi_m(x)=\eta_{m+1}(x)\chi_{m+1}(x)\ , \qquad -\rho_2<x<\rho_1+c\sum_{j=1}^{m}a^{-j}\ . \tag{55}$$

On setting $\theta(x)=\log\phi(x)$, $\theta_s(x)=\log\eta_s(x)$, and $\lambda_s(x)=\log\chi_s(x)$, $s>0$, we obtain from (55)

$$\lambda_m''(x)=\theta_{m+1}''(x)+\lambda_{m+1}''(x)\ , \qquad -\rho_2<x<\rho_1+c\sum_{j=1}^{m}a^{-j}\ . \tag{56}$$

The functions η_s and $\chi_s(x)$, $s>0$, are all moment generating functions and hence the $\lambda_s''(x)$ and $\theta_s''(x)$ are non-negative. This combined with (56) implies that $\lambda_m''(x) \geqq \lambda_{m+1}''(x) \geqq 0$ for all x satisfying $-\rho_2 < x < \rho_1 + c \sum_{j=1}^{m} a^{-j}$ so that

$$\lambda''(x) = \lim_{m\to\infty} \lambda_m''(x) \tag{57}$$

exists for all $x > -\rho_2$.

From (51) we obtain

$$\theta''(x) = \sum_{s=0}^{m} \theta_s''(x) + \lambda_m''(x) , \qquad -\rho_2 < x < \rho_1 ,$$

and hence

$$\theta''(x) = \sum_{s=0}^{\infty} \theta_s''(x) + \lambda''(x) , \qquad -\rho_2 < x < \rho_1 . \tag{58}$$

Again as in the proof of Theorem 6 the convergence of $\sum_{s=0}^{\infty} \theta_s''(x)$ implies $\sum_{j=1}^{n} a_j^2 < 1$ which proves (d) of the theorem.

For all K sufficiently large such that $-K \leqq -a_j K + b_j$, $j=1, \cdots, n$ we have

$$\sup_{-K<x<\rho_1} \theta_s''(x) < \left(\left(\sum_{j=2}^{n} a_j^2 \right) \Big/ (1-a_j^2) \right)^s \sup_{-K<x<\rho_1 - c} \theta_0''(x)$$

and as $\sup_{-K<x<\rho_1-c} \theta_0''(x) < \infty$ we may deduce that $\sum_{s=0}^{\infty} \theta_s''(x) < \infty$ for all $x < \rho_1$.

This implies that the infinite product

$$\prod_{s=0}^{\infty} (\eta_s(x) \exp(-\mu_s x)) \tag{59}$$

converges for all x satisfying $-\infty < x < \rho_1$. Part (a) of Theorem 6 implies the existence of a μ such that if

$$\psi_1(x) = \exp(\mu x) \prod_{s=0}^{\infty} (\eta_s(x) \exp(-\mu_s x)) \tag{60}$$

that $\psi_1(x)$ satisfies the functional equation

$$\psi_1(x) = \prod_{j=1}^{n} \frac{\psi_1(a_j x + b_j)}{\psi_1(b_j)} , \qquad -\infty < x < \rho_1 . \tag{61}$$

Similarly the fact that the $\lambda_m''(x)$ are monotone decreasing and that $\lambda''(x)$ as given by (58) exists for all $x > -\rho_2$ implies that there exists a sequence $(\gamma_m)_1^\infty$ such that

(62) $$\chi(x)=\lim_{m\to\infty}\chi_m(x)\exp(-\gamma_m x)$$

exists for all $x>-\rho_2$. As the χ_m are moment generating functions this implies that $\chi(x)$ is also a moment generating function. On combining this with (51) and (59) we deduce that

$$\begin{aligned}\phi(x)&=\exp(-\gamma x)\chi(x)\prod_{s=0}^{\infty}(\chi_s(x)\exp(-\mu_s x))\\&=\exp(-\delta x)\chi(x)\phi_1(x)\,,\qquad -\rho_2<x<\rho_1\,.\end{aligned}$$

As $\phi_1(x)$ and $\phi(x)$ satisfy the functional equation (61) for all x satisfying $-\rho_2<x<\rho_1$, it follows that

$$\phi_2(x)=\exp(-\delta x)\chi(x)$$

also satisfies (61) for $-\rho_2<x<\rho_1$.

However, as $\phi_2(x)$ is a moment generating function which is defined for all $x>-\rho_2$ it follows via Lemma 1 that $\phi_2(x)$ satisfies (61) for all $x>-\rho_2$. We have therefore shown that

$$\phi(x)=\phi_1(x)\phi_2(x)\,,\qquad -\rho_2<x<\rho_1\,,$$

where $\phi_1(x)$ and $\phi_2(x)$ are moment generating functions defined for all $x<\rho_1$ and $x>-\rho_2$ respectively and which satisfy the functional equation

$$\phi_i(x)=\prod_{j=1}^{n}\frac{\phi_i(a_jx+b_j)}{\phi_i(b_j)}\,,\qquad i=1,2\,,$$

for $x<\rho_1$ and $x>-\rho_2$ respectively. On translating these results into terms of the respective characteristic functions we obtain part (a) of the theorem.

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