Ann. Inst. Statist. Math.
30 (1978), Part B, 199-209

COMPARISON OF TWO TYPES OF MULTIDIMENSIONAL SCALING METHODS

—MINIMUM DIMENSION ANALYSIS MDA-OR AND MDA-UO—

CHIKIO HAYASHI AND FUMI HAYASHI

(Received Apr. 19, 1977; revised Dec. 10, 1977)

1. Introduction

Minimum dimension analysis, the abbreviation of which is shown as MDA, is a kind of well known multidimensional scaling methods. MDA has two types, MDA-OR (case of order class belonging) and MDA-UO (case of unordered class belonging), which are closely related to the ideas shown in [1], [2], [3], [4], [5] and have been published in [6], [7]. The relevant interesting paper of ALSCAL on flexible idea has been published by Takane, Young and de Leeuw [8]. In the present paper the difference between the two will be described with examples and the practical example of MDA-UO will be shown too. Theoretically speaking, the distance is used in MDA-OR and generalized variance is used in MDA-UO, where the distance corresponds to the generalized variance in unidimensional case. The relations R's between the two elements are represented in the form of rank ordered class belonging in MDA-OR and are given in the form of unordered i.e. nominal class belonging in MDA-UO. The idea runs as below: rank ordered class belonging→unidimensional treatment→distance representation and nominal class belonging→multidimensional treatment→generalized variance representation. Thus, naturally, the Euclidean distance* is used in MDA-OR for heuristic understanding and generalized variance is used in MDA-UO as the tools of methods of quantification of the elements with the construction of the minimum dimensional space.

For the convenience of reading, the ideas of MDA-OR and MDA-UO are briefly mentioned as followings (the details are in [6] and [7]).

MDA-OR:

We give a numerical value x_i to i-element ($i=1, 2, \cdots, N$). Here consider the S-dimensional Euclidean distance $m_{ij}=d_{ij}^2=\sum_{s}^{S}(x_{is}-x_{js})^2$

* Any distance function may be taken, however, it is too sophisticated for intuitive understanding.

though x's are unknown, we define $\delta_{ij}(g)$

$$\delta_{ij}(g)=\begin{cases} 1, & \text{if } R_{ij}, \text{ the relation between } i\text{-element and } j\text{-element, belongs to the } g\text{th class} \\ 0, & \text{otherwise} \end{cases}$$

$$i, j=1, 2, \cdots, N, \quad g=1, 2, \cdots, G.$$

Here the gth class means the gth dissimilarity, where the 1st means the lowest dissimilarity class and G means the highest dissimilarity class.

We have $\sum_g^G \delta_{ij}(g)=1$, G being the number of class which shows the class of the highest dissimilarity and $\sum_g \sum_i \sum_j \delta_{ij}(g)=T$, T being the total number of the pair which is equal to $N(N-1)$ if there is no missing pair. It is our purpose to make d_{ij}^2 to correspond to R_{ij}, i.e., to find out the space including N elements and their spacings and their configuration in the minimum dimensional space in order that the relations of d's may imply those of R's. This idea is along the line of so-called multidimensional scaling method by Shepard, Kruskal, Guttman, Carroll, Young, Takane, de Leeuw and etc.

It is noticeable that d^2's are used with validity as those corresponding to unidimensional rank order (scalar-to-order correspondence, both being unidimensional) and correlation ratio is adopted as the discrimination measure of rank ordered groups.

MDA-UO:

In this case, a different method will be presented. Pair (i, j) belongs to the gth class or not. The classes have not any rank ordered property but only the meaning of grouping. That is to say, the ordering among the classes is not found but only nominal classification exists.

So, without operating any direct correspondence of Euclidean distance with the classification, the idea of maximization of discrimination power among the classes, i.e. of effective clustering, is adopted in this case. Then, generalized variance as the tool of such a pattern recognition without any rank order may be reasonably used in the essential clustering.

2. Comparison by artificial data—1

The comparison between MDA-OR and MDA-UO was shown using the actual data in [7]. Here, using simple artificial data which have a clear and easy structure to interpret, the data analysis by two methods will be shown as below.

Table 1 Relational matrix R_{ij}

	1	2	3	4
1	*	A	A	B
2		*	C	B
3			*	C
4				*

* means no definition.
A, B and C mean nominal class.

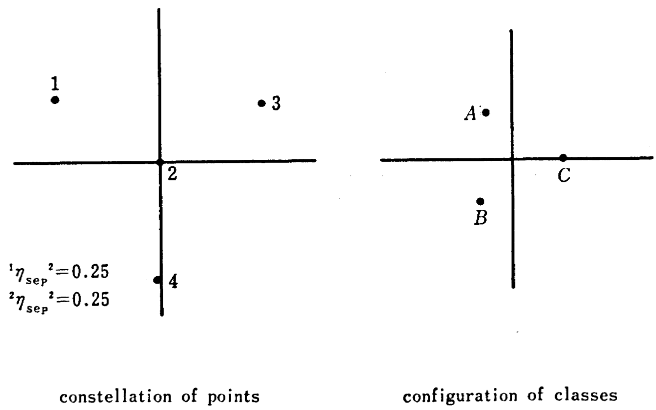


Fig. 1

Relational data matrix: four elements, symmetric
The result by MDA-UO is given Fig. 1 with $^1\eta_{sep}^2=0.25$, $^2\eta_{sep}^2=0.25$. The relations between the elements are specified into two kinds according to distance and shown in Table 2 in which S means short distance and L means long distance. This is quite different from Table 1 and dose not mean the relationships by classification (A, B, C). If MDA-OR is used, we have 6 cases. A, B and C are to be determined in the sense of "rank ordered groups belonging". The 6 cases are shown in Table 3. We have dif-

Table 2

	1	2	3	4
1	0	S	L	L
2		0	S	S
3			0	L
4				0

Table 3

Case α	A	B	C
1	S	M	L
2	M	L	S
3	L	S	M
4	L	M	S
5	M	S	L
6	S	L	M

S means small dissimilarity,
M means medium dissimilarity
and L means large dissimilarity,
i.e. S corresponds to short distance,
M to medium distance
and L to long distance

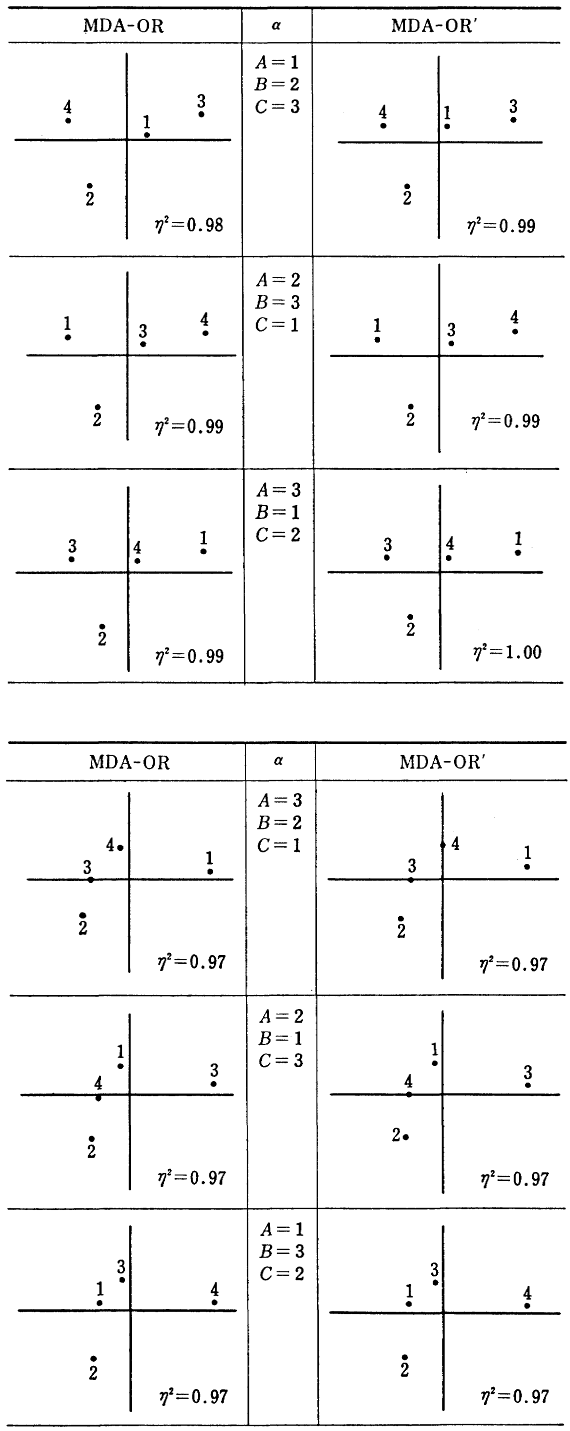
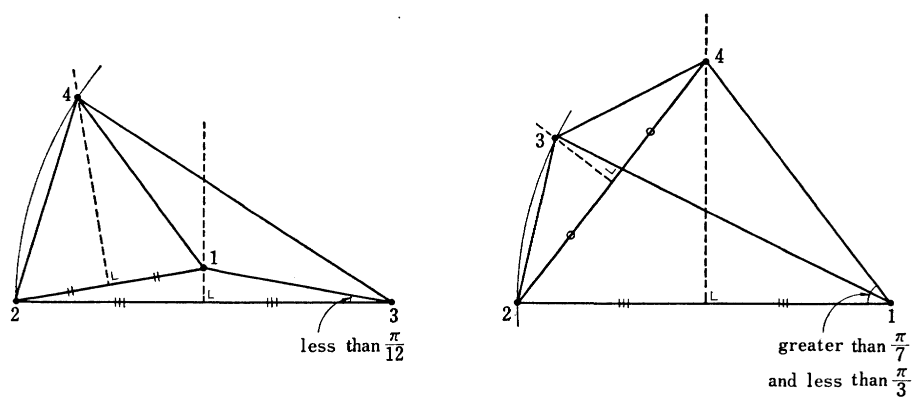

MDA-OR
α
MDA-OR'
A=1
B=2
C=3
η²=0.98
η²=0.99
A=2
B=3
C=1
η²=0.99
η²=0.99
A=3
B=1
C=2
η²=0.99
η²=1.00
MDA-OR
α
MDA-OR'
A=3
B=2
C=1
η²=0.97
η²=0.97
A=2
B=1
C=3
η²=0.97
η²=0.97
A=1
B=3
C=2
η²=0.97
η²=0.97

Fig. 2

ferent results according to 6 cases which are shown in Fig. 2 which are classified into two kinds (1, 2, 3) cases and (4, 5, 6) cases in Table 3. In the calculation, as the initial values, $S=1$, $M=2$, $L=3$ are taken. The results by MDA-OR are quite different from that by MDA-UO. If MDA-OR is used in nominal classification, the selection of a case in Table 3 is indispensable. It must be remarked that this selection implies an additional condition which is essentially unnecessary in the nominal classification and, as it were, gives a pain to the lily. So, it is impossible to pick out a valid one among the results since the results are different according to the case selected. The obtained η^2's are regarded to be equal though η^2's apparently vary from 0.97 to 1.00 since they are under the calculation error by MDA-OR computer algorithm. Really, we can rigorously draw two types of the configuration as Fig. 3.



Corresponds to case 1,2,3
(The figure shows case 1)

Corresponds to case 4,5,6
(The figure shows case 4)

Fig. 3

This existence is easily proved by the idea of elementary geometry. However, the solution is not unique. The solution by MDA-OR is determined by that algorithm and considered to be one realization of the rigorous solutions mentioned above which are not unique. Suppose that the conditions of rank ordered group in MDA-OR are omitted which we call MDA-OR′ as shown in Section 2 of [7]. The results depend upon the adoption of initial values. If those values corresponding to α in Table 3 (see initial values in Fig. 2, $\alpha=1, 2, \cdots, 6$) are used as the initial values, the result gives the same configuration with that of the case α by MDA-OR as it says. The results are shown too in Fig. 2. This means that the formal MDA-OR calculation (MDA-OR′) technique omitting the conditions of the rank ordered gives the different results according to the selection of initial values which is to be arbitrary and the same results with those by MDA-OR when the same initial values are used according to cases in Table 3.

We may conclude that
(1) MDA-UO method is useful in the case of nominal classification while MDA-OR is preferable in the case of rank ordered classification (see Section 3 in [7]),
(2) MDA-OR method is not desirable in the case of nominal classification because the unnecessary rank order conditions (see Table 1) for the very problem to be solved, that are essentially to be free, must be added to nominal classes according to the selection of which the different results are obtained,
(3) formal MDA-OR calculation technique omitting the rank order conditions, i.e. MDA-OR′ is also improper because the adoption of the initial values, which are essentially to be arbitrary and must not influence on the matter of fact, determines the different results and
(4) the idea of use of Euclidean distance even in multidimensional space does not lead to the valid numerical representation of the elements in case of "nominal unordered class belonging" because of the reason mentioned above.

3. Comparison by artificial data—2

In this case, the modificated data of simple example mentioned in Section 2 are used. The data are shown in Table 4. $1l, 1m, 1n$ in this matrix correspond to the element 1 in the former matrix (Table 1). $2l, 2m, 2n$ in this matrix correspond to the element 2 in the former matrix, $3l, 3m, 3n$ in this to the element 3 in the former matrix and $4l, 4m, 4n$ in this to the element 4 in the former matrix. In this case,

Table 4

		1			2			3			4		
		l	*m*	*n*	*l*	*m*	*n*	*l*	*m*	*n*	*l*	*m*	*n*
1	*l*		*D*	*D*	*A*	*A*	*A*	*A*	*A*	*A*	*B*	*B*	*B*
	m			*D*	*A*	*A*	*B*	*A*	*A*	*A*	*B*	*C*	*B*
	n				*A*	*A*	*A*	*A*	*B*	*A*	*B*	*B*	*B*
2	*l*					*D*	*D*	*C*	*C*	*C*	*B*	*B*	*B*
	m						*D*	*C*	*C*	*C*	*B*	*B*	*A*
	n							*C*	*C*	*C*	*B*	*B*	*B*
3	*l*								*D*	*D*	*C*	*C*	*C*
	m									*D*	*C*	*C*	*C*
	n										*C*	*C*	*B*
4	*l*											*D*	*D*
	m												*D*
	n												

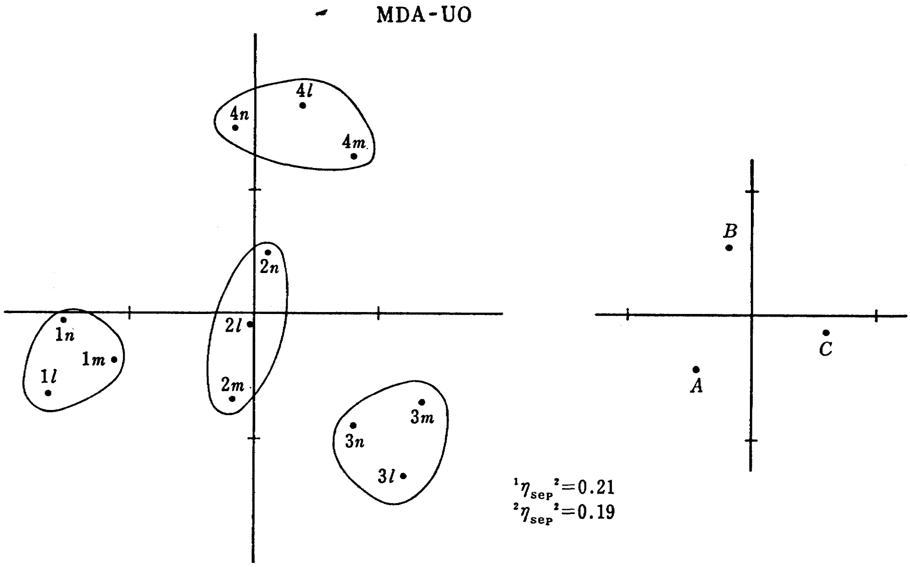

MDA-UO
4l
4n
4m
2n
1n
1m
1l
2l
2m
3m
3n
3l
B
C
A
$^1\eta_{sep}{}^2=0.21$
$^2\eta_{sep}{}^2=0.19$

Fig. 4

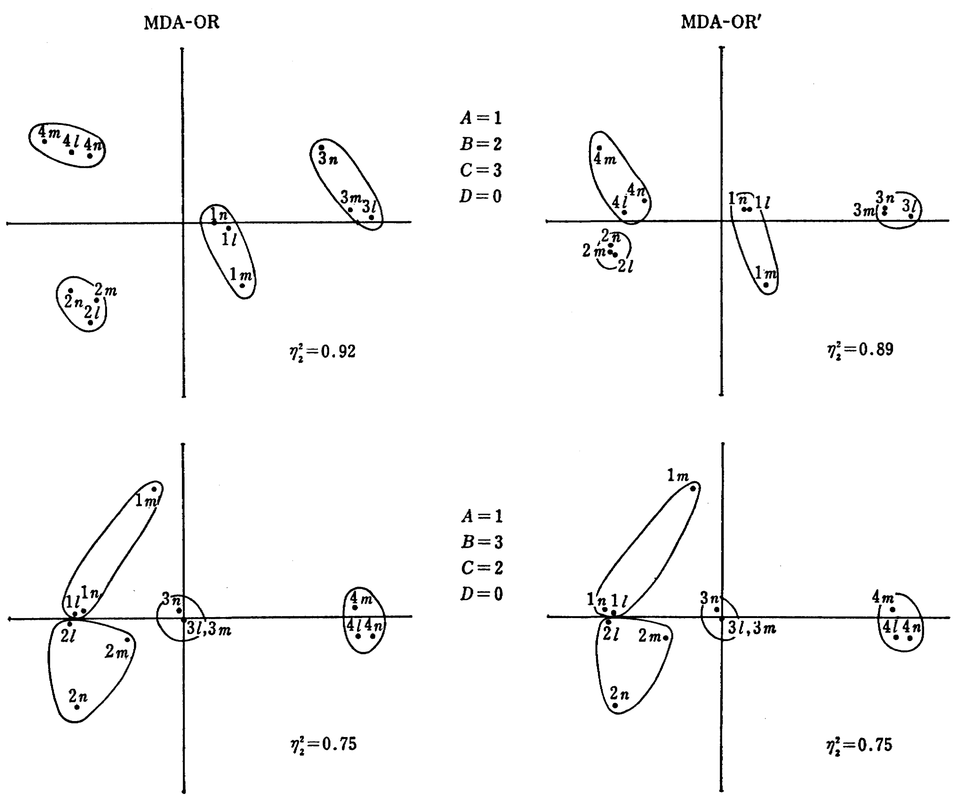

MDA-OR
MDA-OR′
A=1
B=2
C=3
D=0
$\eta_2^2=0.92$
$\eta_2^2=0.89$
A=1
B=3
C=2
D=0
$\eta_2^2=0.75$
$\eta_2^2=0.75$

Fig. 5

four classes A, B, C and D are used. The analysis by MDA-UO is shown in Fig. 4. In this case, D is regarded as no datum, i.e. $*$ in Table 1, because this corresponds to the former case. It gives the similar configuration to that in Fig. 1. As in Table 3, two cases are used in Table 4 which correspond to the cases to give different results. The results by MDA-OR in cases 1 and 6 are shown in the left side of Fig. 5. In this case, four classes A, B, C and D are used and the initial value of D is taken as 0 in both cases, because D corresponds to the smaller dissimilarity class than S. The different configurations are obtained in different cases. Those by the formal technique MDA-OR′ omitting the conditions of rank order are given in the right side of Fig. 5 which are quite the same with those by MDA-OR according to the adoption of initial values. The interpretation of these comparisons leads to the same conclusion as in Section 2.

4. MDA-UO analysis by actual data

The results for some kinds of artificial data are shown in [7]. Here, the psychological data[1] for color harmony are used which were given by research group[2] of color space (the chief is A. Motoaki, Prof. of Psychology in Waseda University). Thirty-three colors are selected from Munsell Color chart. $528=\binom{33}{2}$ kinds of color combination (the size of material for survey is: 160 mm × 120 mm) were presented to about 100 students of a women's university in the faculty of literature except department of psychology. The responses are shown as followings;

(1) Harmony—Disharmony scale is divided into five categories (++, +, ±, −, − −)

(2) Like—Dislike scale is divided into five categories (++, +, ±, −, − −)

(3) Consolidated—Diffuse scale is divided into five categories (++, +, ±, −, − −)

(4) The question, "How strongly does the combination set off each other to advantage?" is given. response categories (+++++, ++++, +++, ++, +, 0)

The responses in the questions for 4 questions are considered in

Harmony	(++, +),	Disharmony	(−, − −)	in (1)
Like	(++, +),	Dislike	(−, − −)	in (2)
Consolidated	(++, +),	Diffuse	(−, − −)	in (3)
Setting off each other		(+++++, ++++)		in (4).

[1] The details are published in Studies of Color, Japan Color Research Institute, vol. 21, no. 1-2, 1974.

[2] We should like to acknowledge the considerable assistance of the member of the group.

Table 5

Color number		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
1	vR	0	6	4	4	4	4	4	4	4	2	2	6	6	2	4	4	4	6	6	2	4	2	4	6	6	6	2	3	7	4	7	1	1
2	vrO	6	0	3	3	4	4	2	2	4	6	2	6	2	1	7	2	4	6	6	3	3	2	2	4	2	3	5	7	7	4	7	5	7
3	vyO	6	3	0	3	3	7	7	7	7	4	6	4	2	3	3	4	4	4	6	3	3	2	4	4	7	5	5	7	7	7	7	5	7
4	vY	4	3	3	0	1	1	7	7	7	7	4	6	2	3	3	1	4	4	2	3	3	1	1	4	7	5	1	7	7	7	5	1	7
5	vYG	4	4	3	1	0	1	3	4	4	4	4	4	2	2	3	3	2	4	2	2	3	3	4	2	4	3	1	1	1	4	3	1	7
6	vG	4	4	7	7	3	0	1	4	4	4	2	6	4	4	1	3	4	4	2	6	3	3	2	2	4	4	2	1	1	4	5	3	1
7	vBG	4	4	7	7	3	3	0	6	6	4	4	6	2	4	1	3	1	4	6	2	4	3	6	6	2	4	2	3	1	2	7	2	7
8	vgB	4	2	4	7	4	4	2	0	6	2	6	4	4	4	7	3	1	4	4	4	4	2	6	2	6	2	2	2	3	4	7	1	7
9	vB	4	4	7	7	4	4	6	6	0	2	4	6	6	4	4	4	3	6	6	2	4	2	3	6	6	4	4	2	3	4	7	1	1
10	vV	4	4	7	7	4	4	4	2	2	0	3	6	2	4	7	4	3	1	2	6	6	4	6	3	2	4	4	2	3	3	1	3	1
11	vP	2	6	6	4	4	2	4	6	2	3	0	6	6	4	2	6	6	3	6	2	6	6	2	3	6	2	4	4	2	3	7	6	6
12	vRP	6	6	6	6	4	4	6	6	6	6	6	0	6	6	4	4	4	6	6	6	4	2	2	6	6	6	2	6	6	6	7	6	6
13	ltR	6	6	2	6	2	4	4	6	6	4	6	6	0	6	2	6	6	6	3	6	2	6	2	3	3	3	3	2	4	3	1	3	7
14	ltyO	6	1	3	3	4	4	2	4	4	4	6	6	6	0	3	6	6	6	1	3	3	2	4	4	3	1	1	7	7	4	1	1	7
15	ltY	7	7	3	3	3	1	1	7	4	4	6	2	2	3	0	1	1	6	3	1	3	3	1	1	7	1	1	1	1	7	1	3	7
16	ltG	4	4	4	3	3	3	3	3	4	4	6	6	6	6	3	0	3	2	6	2	3	3	1	4	4	1	1	1	3	4	3	3	7
17	ltB	4	4	4	4	2	4	6	1	3	3	6	2	6	6	3	3	0	3	6	6	6	3	3	3	4	5	1	3	1	3	1	3	7
18	ltP	2	6	2	4	4	4	4	4	6	3	3	6	6	6	6	2	3	0	6	6	6	2	3	3	6	2	2	4	3	3	3	3	7
19	dR	6	6	2	2	6	2	6	4	6	6	6	6	3	3	3	2	6	6	0	6	6	6	6	6	3	6	2	6	6	6	3	6	6
20	dyO	2	3	3	3	2	4	2	4	4	4	2	6	2	3	1	6	6	6	6	0	3	6	2	2	3	3	3	3	2	2	1	2	1
21	dY	4	3	3	3	3	3	4	4	4	2	4	6	6	3	3	3	6	6	3	3	0	3	2	6	3	3	1	3	3	6	1	3	1
22	dG	4	2	2	3	3	3	3	6	2	4	6	2	6	6	3	3	3	2	6	6	3	0	3	6	6	3	3	3	3	2	1	3	1
23	dB	4	2	4	7	4	2	6	6	3	6	6	6	6	6	1	1	3	3	6	6	2	3	0	6	6	6	2	2	3	2	1	3	1
24	dP	6	6	4	4	4	4	2	2	6	3	3	6	3	4	1	6	3	3	6	6	6	6	6	0	6	6	6	4	6	3	1	1	6
25	dkR	6	2	7	7	4	4	2	6	6	2	6	6	3	3	7	4	4	6	3	3	3	6	6	6	0	6	6	2	6	6	7	6	6
26	dkyO	2	3	3	1	5	4	6	2	2	4	2	6	3	1	3	1	5	6	6	3	3	3	6	6	6	0	3	1	6	6	7	3	3
27	dkY	4	5	5	5	3	3	2	6	2	2	4	2	5	1	5	3	1	2	2	3	3	3	2	6	6	3	0	3	6	6	7	3	3
28	dkG	2	7	7	7	1	3	3	6	2	2	6	6	6	7	5	3	3	4	6	3	3	3	6	6	6	1	3	0	6	6	7	3	3
29	dkB	7	7	7	7	1	1	3	3	3	3	6	6	4	7	1	3	3	6	6	6	3	3	3	6	6	6	6	6	0	6	1	3	6
30	dkP	4	4	7	7	4	4	2	4	4	3	3	6	3	4	7	4	7	3	6	6	6	2	2	3	6	6	6	2	6	0	7	6	6
31	W	7	7	7	1	1	5	1	7	7	1	7	7	1	3	1	3	5	3	7	3	1	1	1	1	7	7	1	1	1	7	0	1	1
32	mGy	3	5	5	7	5	4	2	1	4	3	6	6	3	3	3	3	3	3	6	2	3	3	3	1	6	3	6	3	1	6	1	0	3
33	Bk	7	7	7	7	7	7	7	1	5	1	6	6	7	7	7	7	5	7	6	3	1	1	1	6	6	3	3	1	6	6	1	3	0

1: Harmony, 2: Disharmony, 3: Like, 4: Dislike, 5: Consolidated,
6: Diffuse, 7: Setting off each other

Thus each color combination has representation as a numerical vector represented by response percentage for seven characteristics mentioned above. The character which gives the maximum value of vector component is regarded as the label characteristic of the color combination. This idea may be fruitful as a breakthrough to reveal the underlying feature by representing the characteristic of color combination in an exaggerated description. So, the labels of 528 color combinations are determined as the followings. Nominal classification A, B, C, D, E, F and G correspond to Harmony, Disharmony, Like, Dislike, Consolidated, Diffuse and Setting off each other. Thus, MDA-UO is applicable.

The results are shown in Fig. 6. The sign given to point is Munsell color code. The constellation of colors (Fig. 6-1) and the configuration of classes (Fig. 6-2) reveal interesting interpretation of color-characteristics for young Japanese girls.

The motivation of preparing this paper was stimulated through the discussions with Prof. F. W. Young, and Dr. Y. Takane, University of North Calorina Chapell Hill and Dr. J. B. Kruskal, Bell Telephone Labo-

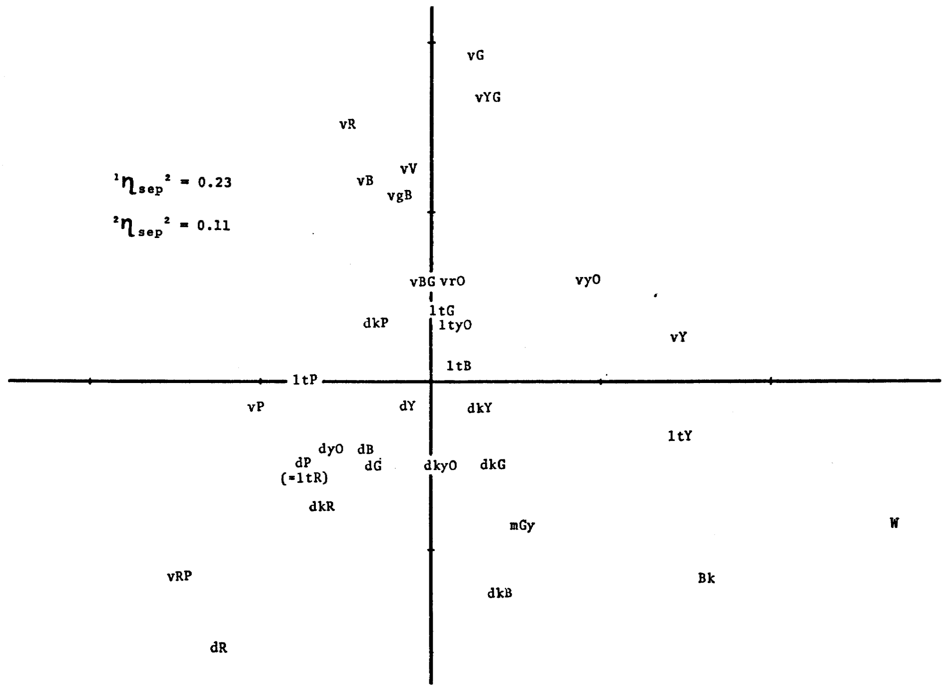

vG
vYG
vR
vV
vB
vgB
$^1\eta_{sep}^2 = 0.23$
$^2\eta_{sep}^2 = 0.11$
vBG vrO
vyO
ltG
dkP
ltyO
vY
ltB
ltP
vP
dY
dkY
ltY
dyO
dB
dP
(=ltR)
dG
dkyO
dkG
dkR
mGy
W
vRP
Bk
dkB
dR

Fig. 6-1

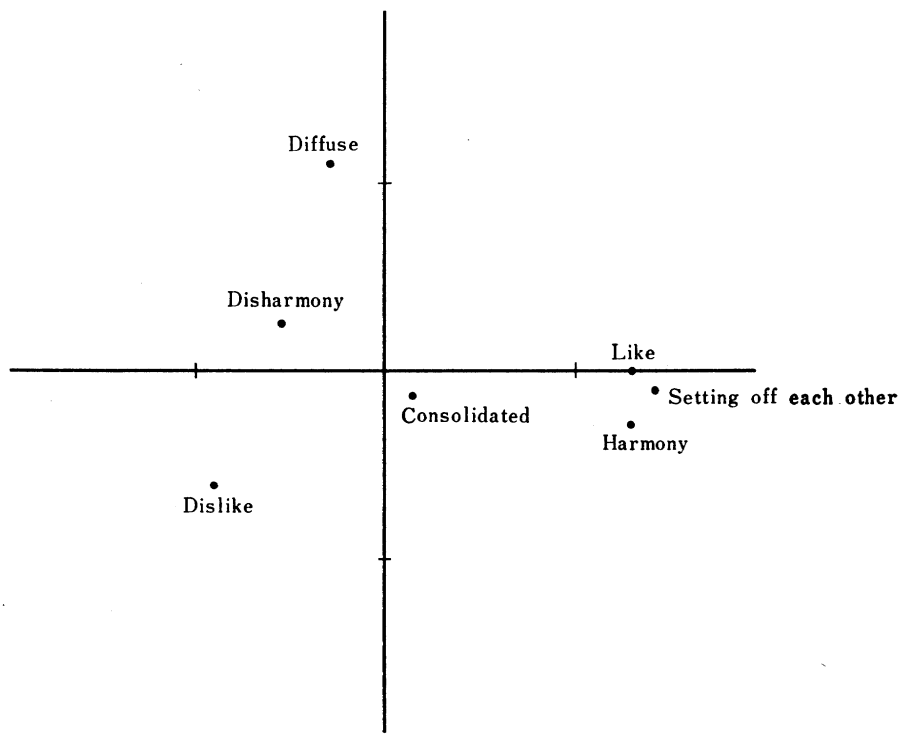

Diffuse
Disharmony
Like
Setting off each other
Consolidated
Harmony
Dislike

Fig. 6-2

ratory. We would like to express our appreciations to them and also our thanks for the referees' helpful comments.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] Hayashi, C. (1952). On the prediction of phenomena from qualitative data and the quantification of qualitative data from the mathematico-statistical point of view, *Ann. Inst. Statist. Math.*, **2**, 93-96.

[2] Hayashi, C. (1964, 1967). Multidimensional quantification of the data obtained by the method of paired comparison, and its Note, *Ann. Inst. Statist. Math.*, **16**, 231-245; **19**, 363-365.

[3] Hayashi, C. (1969). Distance and dimension from the statistical point of view (in Japanese), *Marketing Research* (Marketing Center Co.), **1**, 26-34.

[4] Hayashi, C., Higuti, I. and Komazawa, T. (1970). *Information processing and statistical mathematics* (in Japanese), Sangyo Tosho press, 273-289.

[5] Hayashi, C. (1972). Two dimensional quantification based on the measure of dissimilarity among three elements, *Ann. Inst. Statist. Math.*, **24**, 251-257.

[6] Hayashi, C. (1974). Minimum dimension analysis MDA, *Behaviormetrika*, **1**, 1-24.

[7] Hayashi, C. (1976). Minimum dimension analysis, MDA-OR and MDA-UO, *Essays in Probability and Statistics* (ed. by S. Ikeda and others), Shinko Tsusho Co. Ltd., 395-412.
This paper was originally published at the U.S.-Japan Seminar on "Theory, Methods and Applications of Multidimensionally Scaling and Related Topics", held at the Univ. of Calif. San Diego in August 20-24, 1975.

[8] Takane, Y., Young, F. W. and de Leeuw, J. (1977). Nonmetric individual differences multidimensional scaling: An alternating least squares method with optimal scaling features, *Psychometrika*, **1**, 7-68.