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ON A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY SPACINGS

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Summary

Let X be a non-negative random variable with probability distribution function F. Suppose $X_{i,n}$ $(i=1,\cdots,n)$ is the ith smallest order statistics in a random sample of size n from F. A necessary and sufficient condition for F to be exponential is given which involves the identical distribution of the random variables $(n-i)(X_{i+1,n}-X_{i,n})$ and $(n-j)(X_{j+1,n}-X_{j,n})$ for some i, j and n, $(1\leqq i<j<n)$.

1. Introduction

Let X be a random variable (rv) whose probability density function f is given, for some $\theta>0$, by

$$(1.1)\qquad f_\theta(x)=\begin{cases}\theta^{-1}\exp(-x/\theta)\,, & x>0\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

Suppose $X_1, X_2,\cdots, X_n$ is a random sample of size n from a population with density f and let $X_{1,n}<X_{2,n}<\cdots<X_{n,n}$, be the associated order statistics. We shall define the standardized spacings as

$$D_{r,n}=(n-r)(X_{r+1,n}-X_{r,n})\,,\qquad 1\leqq r<n,\ \text{with}\ D_{0,n}=nX_{1,n}\ \text{and}\ D_{n,n}=0\,.$$

Kotz [7] and Galambos [5] discussed various characterizations of the exponential distribution. Puri and Rubin [8] proved that if X_1 and X_2 are independent copies of an rv X with density f, then X and $D_{1,2}$ have the same distribution if and only if f is as given in (1.1). Ahsanullah [2], [3] gave characterization of the exponential distribution by assuming respectively the identical distributions of $D_{i,n}$, $D_{0,n}$ and $D_{i,n}$, X.

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In this paper we will give a characterization of the exponential distribution by considering identical distributions of $D_{j,n}$ and $D_{i,n}$ with some i, j and n, $(1 \leqq i < j < n)$.

2. Notation and results

Let F be the distribution function of a non-negative rv X with the density f and $\bar{F}=1-F$ and with hazard rate $H(x)$ as $H(x)=f(x)\cdot(\bar{F}(x))^{-1}$, for $x \geqq 0$, and $\bar{F}(x)>0$. We will call F has increasing hazard rate (IHR) if $H(x) \leqq H(x+y)$, $x, y \geqq 0$ and F has decreasing hazard rate (DHR), if $H(x) \geqq H(x+y)$, $x, y \geqq 0$. We will say that F belongs to class C if F is either IHR or DHR.

THEOREM. *Let X be a non-negative rv having an absolutely continuous (with respect to Lebesgue measure) distribution function F that is strictly increasing on $[0, \infty)$. Then the following properties are equivalent:*

(a) *X has an exponential distribution with density as given in* (1.1).

(b) *For some i, j and n, $1 \leqq i < j < n$, the statistics $D_{i,n}$ and $D_{j,n}$ are identically distributed and F belongs to class C.*

PROOF. It is known (see, Galambos [5]) that (a)$\Rightarrow$(b), so we prove only that (b)$\Rightarrow$(a). From the conditional joint density of $X_{j,n}$ and $X_{j+1,n}$ given $X_{i,n}=x$, which is given e.g. by Govindarajulu [6], it follows that the conditional density of $D_{j,n}$ given $X_{i,n}=x$, is

$$(2.1) \qquad \begin{aligned} f_{D_{j,n}}(d \mid X_{i,n}=x) = K \int_0^\infty & ((\bar{F}(x)-\bar{F}(x+s))(\bar{F}(x))^{-1})^{j-i-1} \\ & \times((\bar{F}(x+s+d(n-j)^{-1}))(\bar{F}(x))^{-1})^{n-j-1} \\ & \times(f(x+s)(\bar{F}(x))^{-1})(f(x+s+d(n-j)^{-1}) \\ & \times(\bar{F}(x))^{-1})ds \, , \end{aligned}$$

where $K=(n-i)!((j-i-1)!(n-j)!)^{-1}$, and $1 \leqq i < j < n$. Integrating (2.1) with respect to d from d to ∞, we get

$$(2.2) \qquad \begin{aligned} \bar{F}_{D_{j,n}}(d \mid X_{i,n}=x) = K \int_0^\infty & ((\bar{F}(x)-\bar{F}(x+s))(\bar{F}(x))^{-1})^{j-i-1} \\ & \times((\bar{F}(x+s+d(n-j)^{-1}))(\bar{F}(x))^{-1})^{n-j} \\ & \times(f(x+s)(\bar{F}(x))^{-1})ds \, . \end{aligned}$$

Again we know (see, e.g. Galambos [5], p. 82) that the conditional distribution of $D_{i,n}$ satisfies the following relation

$$(2.3) \qquad \bar{F}_{D_{i,n}}(d \mid X_{i,n}=x) = ((\bar{F}(x+d(n-i)^{-1}))(\bar{F}(x))^{-1})^{n-i}\, , \qquad 1 \leqq i < n \, .$$

Since F belongs to class C and the distributions of $D_{j,n}$ and $D_{i,n}$ are identical, so also their conditional distributions given $X_{i,n}=x$. Writing

$$K^{-1}=\int_0^\infty ((\bar{F}(x+s))(\bar{F}(x))^{-1})^{n-j}((\bar{F}(x)-\bar{F}(x+s))(\bar{F}(x))^{-1})^{j-i-1} \times f(x+s)(\bar{F}(x))^{-1}ds ,$$

we get on simplification from (2.2) and (2.3),

$$0=\int_0^\infty (\bar{F}(x+s)(\bar{F}(x))^{-1})^{n-j}((\bar{F}(x)-\bar{F}(x+s))(\bar{F}(x))^{-1})^{j-i-1} \times f(x+s)(\bar{F}(x+s))^{-1}G(x, s, d)ds , \tag{2.4}$$

for all d and any given x, and

$$G(x, s, d)=((\bar{F}(x+d(n-i)^{-1}))(\bar{F}(x))^{-1})^{n-i}-((\bar{F}(x+s+d(n-j)^{-1})) \times(\bar{F}(x+s))^{-1})^{n-j} . \tag{2.5}$$

Differentiating $G(x, s, d)$ with respect to s, we obtain,

$$\frac{\partial}{\partial s}G(x, s, d)=((\bar{F}(x+s+d(n-j)^{-1}))(\bar{F}(x+s))^{-1})^{n-j} \times(h(x+s+d(n-j)^{-1})-h(x+s)) . \tag{2.6}$$

(i) If F is IHR, then $G(x, s, d)$ is increasing in s for fixed x and d. Thus for (2.4) to be true, we must have $G(x, 0, d)\leqq G(x, s, d)\leqq 0$. If F has IHR, then we know (see, e.g. Barlow and Prochan [4]) that $\log \bar{F}(x)$ is concave, hence using Jensen's inequality, we have

$$\log \bar{F}(x+d(n-i)^{-1})\geqq((j-i)(n-i)^{-1})\log \bar{F}(x) +((n-j)(n-i)^{-1})\log \bar{F}(x+d(n-j)^{-1})$$

i.e.

$$(\bar{F}(x+d(n-i)^{-1}))^{n-i}\geqq(\bar{F}(x))^{j-i}(\bar{F}(x+d(n-j)^{-1}))^{n-j} .$$

Which shows that $G(x, 0, d)\geqq 0$. Hence if (2.4) is true, we must have $G(x, 0, d)=0$, for all d and any given x.

(ii) If F has DHR, similarly we get $G(x, 0, d)=0$, for all d and any given x. Substituting $x=0$, we have $G(0, 0, d)=0$ for all d, i.e.,

(2.7) $(\bar{F}(d(n-i)^{-1}))^{n-i}=(\bar{F}(d(n-j)^{-1}))^{n-j}$, for all $d\geqq 0$, and some i, j and n $(1\leqq i<j<n)$.

Taking $\phi(d)=-\log \bar{F}(d)$ and $z=d(n-i)^{-1}$, we get

(2.8) $\phi(z)=((n-j)(n-i)^{-1})\phi(z((n-i)(n-j)^{-1}))$, for all $z\geqq 0$ and some i, j and n with $1\leqq i<j<n$.

The non null solution of (2.8), (see Aczél [1], p. 31) is

(2.9) $\psi(z)=cz$, where c is a constant, and so $F(x)=1-e^{cx}$.

Using the boundary conditions $F(0)=0$, and $F(\infty)=1$, we get

(2.10) $F(x)=1-e^{-\theta x}$, where $\theta>0$.

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REFERENCES

[1] Aczél, J. (1966). *Lectures on Functional Equations and Their Applications*, Academic Press, New York.
[2] Ahsanullah, M. (1976). On a characterization of the exponential distribution by order statistics, *J. Appl. Prob.*, **13**, 818-822.
[3] Ahsanullah, M. (1977). A characteristic property of the exponential distribution, *Ann. Statist.*, **5**, 580-582.
[4] Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing Probability Models*, Holt, Rinehart and Winston Inc., New York.
[5] Galambos, J. (1975). Characterization of probability distributions by properties of order statistics I, *Statistical Distribution in Scientific Work*, **3**, 71-88 (G. P. Patil et al., eds.), Reidel, Dordrecht.
[6] Govindarajulu, Z. (1966). Characterization of the exponential and power distributions, *Skand. Aktuarietidskr.*, **10**, 132-136.
[7] Kotz, S. (1974). Characterizations of statistical distribution, a supplement to recent survey, *Rev. Int. Statist. Inst.*, **42**, 39-65.
[8] Puri, P. S. and Rubin, H. (1970). A characterization based on absolute difference of two i.i.d. random variables, *Ann. Math. Statist.*, **41**, 2113-2122.