ON A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY SPACINGS

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Summary

Let $X$ be a non-negative random variable with probability distribution function $F$. Suppose $X_{i,n}$ ($i=1, \ldots, n$) is the $i$th smallest order statistic in a random sample of size $n$ from $F$. A necessary and sufficient condition for $F$ to be exponential is given which involves the identical distribution of the random variables $(n-i)(X_{i+1,n}-X_{i,n})$ and $(n-j)(X_{j+1,n}-X_{j,n})$ for some $i$, $j$ and $n$, ($1 \leq i < j < n$).

1. Introduction

Let $X$ be a random variable (rv) whose probability density function $f$ is given, for some $\theta > 0$, by

$$f_\theta(x) = \begin{cases} \theta^{-1} \exp(-x/\theta), & x > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size $n$ from a population with density $f$ and let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$, be the associated order statistics. We shall define the standardized spacings as

$$D_{r,n} = (n-r)(X_{r+1,n}-X_{r,n}), \quad 1 \leq r < n, \text{ with } D_{0,n} = nX_{1,n} \text{ and } D_{n,n} = 0.$$ 

Kotz [7] and Galambos [5] discussed various characterizations of the exponential distribution. Puri and Rubin [8] proved that if $X_1$ and $X_2$ are independent copies of an rv $X$ with density $f$, then $X$ and $D_{1,2}$ have the same distribution if and only if $f$ is as given in (1.1). Ahsanullah [2], [3] gave characterization of the exponential distribution by assuming respectively the identical distributions of $D_{i,n}$, $D_{0,n}$ and $D_{i,n}$, $X$. 

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In this paper we will give a characterization of the exponential distribution by considering identical distributions of $D_{i,n}$ and $D_{j,n}$ with some $i, j$ and $n$, $(1 \leq i < j < n)$.

2. Notation and results

Let $F$ be the distribution function of a non-negative rv $X$ with the density $f$ and $\bar{F}=1-F$ and with hazard rate $H(x)$ as $H(x)=f(x)\cdot (\bar{F}(x))^{-1}$, for $x \geq 0$, and $\bar{F}(x)>0$. We will call $F$ has increasing hazard rate (IHR) if $H(x) \leq H(x+y)$, $x, y \geq 0$ and $F$ has decreasing hazard rate (DHR), if $H(x) \geq H(x+y)$, $x, y \geq 0$. We will say that $F$ belongs to class C if $F$ is either IHR or DHR.

**Theorem.** Let $X$ be a non-negative rv having an absolutely continuous (with respect to Lebesgue measure) distribution function $F$ that is strictly increasing on $[0, \infty)$. Then the following properties are equivalent:

(a) $X$ has an exponential distribution with density as given in (1.1).

(b) For some $i, j$ and $n$, $1 \leq i < j < n$, the statistics $D_{i,n}$ and $D_{j,n}$ are identically distributed and $F$ belongs to class C.

**Proof.** It is known (see, Galambos [5]) that (a)$\Rightarrow$(b), so we prove only that (b)$\Rightarrow$(a). From the conditional joint density of $X_{i,n}$ and $X_{j+1,n}$ given $X_{i,n}=x$, which is given e.g. by Govindarajulu [6], it follows that the conditional density of $D_{j,n}$ given $X_{i,n}=x$, is

\[
(2.1) \quad f_{D_{j,n}}(d|X_{i,n}=x)=K \int_0^\infty ((\bar{F}(x)-\bar{F}(x+s))(\bar{F}(x))^{-1})^{j-i-1} \\
\times ((\bar{F}(x+s+d(n-j)^{-1}))(\bar{F}(x))^{-1})^{n-j-1} \\
\times (f(x+s)(\bar{F}(x))^{-1})(f(x+s+d(n-j)^{-1})) \\
\times (\bar{F}(x))^{-1})ds,
\]

where $K=(n-i)!(j-i-1)!(n-j)!^{-1}$, and $1 \leq i < j < n$. Integrating (2.1) with respect to $d$ from $d$ to $\infty$, we get

\[
(2.2) \quad \bar{F}_{D_{j,n}}(d|X_{i,n}=x)=K \int_0^\infty ((\bar{F}(x)-\bar{F}(x+s))(\bar{F}(x))^{-1})^{j-i-1} \\
\times ((\bar{F}(x+s+d(n-j)^{-1}))(\bar{F}(x))^{-1})^{n-j} \\
\times (f(x+s)(\bar{F}(x))^{-1})ds.
\]

Again we know (see, e.g. Galambos [5], p. 82) that the conditional distribution of $D_{i,n}$ satisfies the following relation

\[
(2.3) \quad \bar{F}_{D_{i,n}}(d|X_{i,n}=x)=((\bar{F}(x+d(n-i)^{-1}))(\bar{F}(x))^{-1})^{n-i}, \quad 1 \leq i < n.
\]
Since \( F \) belongs to class \( C \) and the distributions of \( D_{i,n} \) and \( D_{i,n} \) are identical, so also their conditional distributions given \( X_{i,n} = x \). Writing
\[
K^{-1} = \int_0^\infty (\bar{F}(x+s))(\bar{F}(x))^{-i}((\bar{F}(x) - \bar{F}(x+s))(\bar{F}(x))^{-i})^{-1}
\times f(x+s)(\bar{F}(x))^{-i}ds,
\]
we get on simplification from (2.2) and (2.3),
\[
0 = \int_0^\infty (\bar{F}(x+s)(\bar{F}(x))^{-i}((\bar{F}(x) - \bar{F}(x+s))(\bar{F}(x))^{-i})^{-1}
\times f(x+s)(\bar{F}(x+s))^{-i}G(x, s, d)ds,
\tag{2.4}
\]
for all \( d \) and any given \( x \), and
\[
G(x, s, d) = ((\bar{F}(x+d(n-i)^{-1}))(\bar{F}(x))^{-i} - ((\bar{F}(x+s+d(n-j)^{-1}))
\times (\bar{F}(x+s))^{-i}h(x+s))^{-i}j.
\tag{2.5}
\]
Differentiating \( G(x, s, d) \) with respect to \( s \), we obtain,
\[
\frac{\partial}{\partial s} G(x, s, d) = ((\bar{F}(x+s+d(n-j)^{-1}))(\bar{F}(x+s))^{-i}j
\times (h(x+s+d(n-j)^{-1}) - h(x+s)).
\tag{2.6}
\]
(i) If \( F \) is IHR, then \( G(x, s, d) \) is increasing in \( s \) for fixed \( x \) and \( d \). Thus for (2.4) to be true, we must have \( G(x, 0, d) \leq G(x, s, d) \leq 0 \). If \( F \) has IHR, then we know (see, e.g. Barlow and Prochan [4]) that log \( \bar{F}(x) \) is concave, hence using Jensen's inequality, we have
\[
\log \bar{F}(x+d(n-i)^{-1}) \geq ((j-i)(n-i)^{-1}) \log \bar{F}(x)
+ ((n-j)(n-i)^{-1}) \log \bar{F}(x+d(n-j)^{-1})
\]
i.e.
\[
(\bar{F}(x+d(n-i)^{-1}))^{n-i} \geq (\bar{F}(x))^{i-1}((\bar{F}(x+d(n-j)^{-1}))^{n-j}.
\]
Which shows that \( G(x, 0, d) \geq 0 \). Hence if (2.4) is true, we must have \( G(x, 0, d) = 0 \), for all \( d \) and any given \( x \).

(ii) If \( F \) has DHR, similarly we get \( G(x, 0, d) = 0 \), for all \( d \) and any given \( x \). Substituting \( x = 0 \), we have \( G(0, 0, d) = 0 \) for all \( d \), i.e.,
\[
(\bar{F}(d(n-i)^{-1}))^{n-i} = (\bar{F}(d(n-j)^{-1}))^{n-j}, \text{ for all } d \geq 0, \text{ and some } i, j \text{ and } \frac{n}{2} (1 \leq i < j < n).
\tag{2.7}
\]
Taking \( \phi(d) = -\log \bar{F}(d) \) and \( z = d(n-i)^{-1} \), we get
\[
\phi(z) = ((n-j)(n-i)^{-1}) \phi(z(n-i)(n-j)^{-1}), \text{ for all } z \geq 0 \text{ and some } i, j \text{ and } n \text{ with } 1 \leq i < j < n.
\tag{2.8}
\]
The non null solution of (2.8), (see Aczél [1], p. 31) is

\[ \phi(x) = cx, \text{ where } c \text{ is a constant, and so } F(x) = 1 - e^{-x}. \]

Using the boundary conditions \( F(0) = 0 \), and \( F(\infty) = 1 \), we get

\[ F(x) = 1 - e^{-\theta x}, \text{ where } \theta > 0. \]

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References


