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BETWEENNESS FOR REAL VECTORS AND LINES, I
BASIC GENERALITIES

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1. Introduction

In a recent paper [3] one of the authors showed in detail how the present notion of flanking is built up from a certain elementary notion of betweenness for lines through the origin in a real vector space. A concise analytical formulation of this betweenness notion was obtained, and this formulation was carried over formally into a complex unitary space and shown to be equally well a betweenness notion for 1-dimensional linear manifolds there. The build-up procedure then went on to translate statements about 1-dimensional linear manifolds into statements about 1-dimensional projections, after which, with the employment of the pairwise spectral analysis of projections, it pieced together from the betweenness of 1-dimensional projections a "betweenness" for higher-dimensional projections in an n-dimensional space. This "betweenness" (not *strict* betweenness, according to definition) was called *flanking*, and its suitable analytic characterization, of a form not tied to finite dimensionality or discreteness, was derived and was seen to be exactly the general definition of flanking that had been given in [2]. Now, that build-up of a betweenness-like relation for projections in a general complex unitary space displays in bold relief the very simple notion of betweenness for real lines that it proceeds from. If one considers that there may be many other interesting (and perhaps, also, some uninteresting) notions of betweenness in real space from which one might proceed similarly (or with certain modifications), then one must suspect that the present notion of flanking is only the first in a long line of interesting and distinctly different flanking notions still to be brought out. And indeed it appears that this is the situation. The general betweenness concept has a wealth of particular realizations for vectors and for 1-dimensional linear manifolds in a real space, and from these it is possible to obtain, in various ways, a corresponding wealth of partic-

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ular betweenness notions for 1-dimensional linear manifolds in a complex space; from each of those one can then go on to build another flanking notion. It is our intention to carry out this program and ultimately arrive at some of the possible new flanking notions. In an initial series of three or more papers we will examine real betweenness in a fair amount of detail and generate some particular examples that are not commonly encountered. This present paper is the first in this initial series, and in it we will start with a perfectly general study of betweenness, subsequently specializing to a real unitary space for the elaboration of significant examples. Continuing development of theoretical generalities on betweenness will proceed hand-in-hand with the discussion of these examples. The second article in this series will continue the general study by examining the relatedness of betweenness notions for real vectors and those for real lines. The third article will look still further at the examples earlier discussed and at new ones.

We anticipate that our general notion of betweenness will find increasing application in fields such as probability and statistics—in the more standard lines of investigation in these fields as well as in the study of flanking. We are seeing, in these times, a number of instances of "enlargement" of routine older mathematical notions in the effort to achieve better understanding of real phenomena. Some examples of this are generalized functions, finitely additive probability measures, and fuzzy sets. In the same way, the general betweenness notion constitutes an enlargement of, for example, the constantly arising linear point betweenness in a vector space—so that the notion of convex hull is broadened to the general notion of span function (see Section 3). It seems to us, indeed—in the light of such observations—that the formalization and purposeful study of the notion of betweenness is overdue.

We begin our study in Section 2 here with the formal definition of a betweenness relation in a space $\mathscr{X}$, and we then discuss the characterization of such a relation by a *spread function*. These functions are defined on a domain of sets each containing not more than two points; therefore, in Section 3 we investigate the extension of a spread function to a *span function*, this latter having as domain the class of all subsets of $\mathscr{X}$. We are able to characterize, for a given spread function τ, both the inclusionally minimal extension of τ to a span function—which we designate by g_τ—and the inclusionally maximal extension of τ to a span function—which we label g^τ. We also give in Section 3, in Theorem 3.3, the correspondence between span functions and betweenness relations; this is facilitated by the notion of a *self-core span function*. In Section 4 we observe that the notion of span function has been studied in the algebraic literature under the name of *closure operator*; we briefly review the results of interest to us here.

These results then enable us to give, in Theorem 5.1 of Section 5, an effective representation of betweenness relations in terms of binary relations. We go on in Section 5 to illustrate this mode of representation in mainly familiar examples. In Section 6 we introduce, via the Theorem 5.1 representation mode, a new class of betweenness relations for vectors, and we devote this section to their study.

Section 7 gives a brief survey of some of the past literature that has gone under the name "betweenness". We have tried to make it clear that our definition, in Section 2, is far more specific than that which has governed past investigations. We have accordingly applied the new term *betwixtness* to the older investigations and have developed Section 7 with the point of view of showing that certain betwixtness relations are betweenness relations and that others are not.

It will be useful to list a few items of notation and terminology here. We will use $\mathscr{X}$ generally to denote the space in which we have a betweenness relation. $\mathscr{A}$ will denote the collection of all subsets of $\mathscr{X}$, $\tilde{\mathscr{A}}$ the collection of all at-most-two-point sets, and $\tilde{\mathscr{A}}^{\cdot}$ the class of all one- and two-point sets. We will abbreviate the phrase "at-most-two-point set" by "a.m.t.p. set". When we specialize our discussion to betweennesses in a (finite-dimensional) real unitary space, we shall use the symbol $\mathscr{K}$ in place of $\mathscr{X}$. The symbol $\mathcal{O}_{\mathscr{K}}$ will denote the unit sphere in $\mathscr{K}$. And $\mathscr{K}^p$ will designate the space of 1-dimensional linear manifolds in $\mathscr{K}$.

The letter B will generically denote a betweenness relation. The Japanese hiragana *hi*, ひ, will be used with various affixes to label specific cases of betweenness. R will denote a binary relation, in general between a point of the space $\mathscr{X}$ and a point of a space Ω. Other unusual symbols that will be used are the hiragana *to*, と, and the Hebrew *beth*, $\beth$. We follow our customary practice of representing the null element of a vector space by θ. The sign $\Rightarrow$ will denote implication, and $\Leftrightarrow$ will denote equivalence.

2. Definition of betweenness, and its characterization by a spread function

Let $\mathscr{X}$ be any space, with points denoted by x, y, z, etc. For a relation B on ordered point-triplets of $\mathscr{X}$, we write $B(x, y; z)$ if B holds for the ordered triplet $\langle x, y, z\rangle$, and otherwise we write $B^c(x, y; z)$. Then the fully general definition of betweenness is as follows:

DEFINITION 2.1. The relation B, on ordered point-triplets of the space $\mathscr{X}$, is said to be a *betweenness relation* if it has the following properties:

$$(2.1)\quad \begin{cases} \text{(i)} & \text{if } B(x, y; z), \text{ then } B(y, x; z); \\ \text{(ii)} & \text{for any } x \text{ and } y,\ B(x, y; x) \text{ and } B(x, y; y); \\ \text{(iii)} & \text{if } B(x, y; z_1),\ B(x, y; z_2) \text{ and } B(z_1, z_2; w), \text{ then } B(x, y; w)\,. \end{cases}$$

The statement $B(x, y; z)$ is equivalently made by saying "z is between x and y". The property (i) is to the effect that B is symmetric in the first two elements of the point-triplet. Property (ii) says that, for any two points x and y, each of x and y is between x and y. And property (iii) asserts this: if both z_1 and z_2 are between x and y, and w is a point that is between z_1 and z_2, then w is also between x and y. This last property we call the *hereditary property* of the betweenness relation.

Many obvious examples of betweenness relations spring to mind immediately. Since our interest will be in betweenness for vectors and for lines through the origin, we shall confine ourselves always to examples in terms of these elements. Let $\mathcal{K}$ be a real unitary space. We have a betweenness relation in $\mathcal{K}$ if we define $B(x, y; z)$, for points x, y, z of $\mathcal{K}$, to mean

$$(2.2)\qquad z = ax + by\,, \qquad a \geqq 0,\ b \geqq 0\,.$$

This elementary example serves, furthermore, to illustrate a certain aspect of the general definition. One might have considered including in the definition the property that for any point x, the only point between x and x is x itself. But the example of (2.2) in $\mathcal{K}$, which we *do* want to call a betweenness relation, does not have this property. In this case, for any vector $x \neq \theta$, every vector in the non-negative ray determined by x is between x and x. A more forceful reason for not including the just-mentioned condition in our definition is that the general developments we are going to elaborate in this paper would become more cumbersome.

It may be noted that any betweenness relation can be minimally modified precisely so as to satisfy the condition in question. Specifically, if B is any particular betweenness relation, then the relation B', defined in the following manner, is likewise a betweenness relation:

$$(2.3)\qquad \begin{aligned} &B'(x, y; z) \Leftrightarrow B(x, y; z)\,, \qquad \text{if } x \neq y \\ &B'(x, x; z) \Leftrightarrow z = x\,. \end{aligned}$$

This assertion is readily proved using Definition 2.1 directly.

If $\mathcal{O}_{\mathcal{K}}$ denotes the unit sphere in $\mathcal{K}$, we get a betweenness relation in $\mathcal{O}_{\mathcal{K}}$ by again using the defining relation (2.2). In this case we see that it happens incidentally to be true that, for any $x \in \mathcal{O}_{\mathcal{K}}$, x is the only element between x and x.

Of particular interest also is the fact that on any subset of $\mathcal{O}_{\mathcal{K}}$ as well (2.2) defines a betweenness relation; for example, on any unit hemisphere, either open or closed, or on a unit hemisphere open except for one point on the boundary (which is a case that will be of interest later).

Let $\mathcal{K}^p$ denote the set of all 1-dimensional linear manifolds in $\mathcal{K}$, that is, the set of all lines through the origin. A simple example of a betweenness relation in this set is as follows: an element N is between the non-orthogonal elements L and M if N is in the linear manifold spanned by L and M and furthermore falls in the (closed) acute angle formed by L and M; if L and M are orthogonal then every N in the linear manifold spanned by L and M is between L and M. This betweenness notion is precisely the one from which the present flanking notion is built up (see [3]).

Returning now to the general betweenness relation, it is rather natural to consider, for any points x and y, the set of all points each of which is between x and y. We shall call this set the *spread* of the set $\{x, y\}$, and denote it by $\tau(\{x, y\})$. This definition applies whether $x \neq y$ or $x = y$. By (ii) and (iii) of (2.1) we see that

(2.4) $$\tau(\{x\}) \subseteq \tau(\{x, y\}) \qquad \text{for all } x,\ y\,;$$

and therefore we may state:

(2.5) $$\tau(A) = \{z \in \mathcal{X} \mid B(x, y; z),\ x \in A,\ y \in A\} \qquad \text{for every one- or two-point set } A \subseteq \mathcal{X}\,.$$

Property (i) of (2.1) is, of course, already incorporated into our definition of spread in that we have asserted that the spread depends only on the set $\{x, y\}$, and not on the ordered pair $\langle x, y\rangle$. Property (ii) of (2.1) translates immediately into

(2.6) $$x \in \tau(\{x, y\}) \qquad \text{for all } x,\ y\,.$$

And property (iii) is expressed by

(2.7) $$[x \in \tau(A),\ y \in \tau(A)] \Rightarrow [\tau(\{x, y\}) \subseteq \tau(A)] \qquad \text{for every one- or two-point set } A\,.$$

These two statements are next seen to be most concisely expressed as follows:

for all one- and two-point sets A and B,

(2.8) $$A \subseteq \tau(A)$$

and

$$A \subseteq \tau(B) \Rightarrow \tau(A) \subseteq \tau(B) . \tag{2.9}$$

Thus, we have the result that a betweenness relation defines a function τ on the class of all one- and two-point subsets of $\mathcal{X}$ which satisfies (2.8) and (2.9). It is seen without difficulty that, conversely, any such function τ defines a betweenness relation B through the identity

$$B(x, y; z) \equiv [z \in \tau(\{x, y\})] . \tag{2.10}$$

This identity equally well defines the τ formed from a given B, and so we have a 1-1 correspondence between the class of betweenness relations on $\mathcal{X}$ and the class of functions τ described.

For subsequent purposes it is useful to take another step at this juncture, namely, to go over to functions τ whose domain is not merely the class of one- and two-point sets, but the class of all at-most-two-point sets. In other words, we want to include the empty set $\emptyset$ in the domain. This does not introduce any difficulties. Notice that if τ is defined on one- and two-point sets, and we formally adjoin the statement $\tau(\emptyset) = \emptyset$, then (2.8) and (2.9) continue to hold. And notice that the same is true if, instead, we adjoin the statement

$$\tau(\emptyset) = \bigcap_{\substack{A \text{ a one- or} \\ \text{two-point set}}} \tau(A) . \tag{2.11}$$

The right-hand side of (2.11) may be non-empty, so that in general there are here two possibilities for extending the domain of τ to include $\emptyset$ and still have (2.8) and (2.9) holding. It is interesting to observe now that these are the only possibilities. To see this, suppose τ is defined on all a.m.t.p. (at-most-two-point) sets, and let us set

$$C_0 \overset{\text{def.}}{=} \bigcap_{A \text{ a one-point set}} \tau(A) . \tag{2.12}$$

By (2.4) we see that this set C_0 is identical with the right-hand side of (2.11). Suppose C_0 is not empty, and let $x \in C_0$. By (2.12) we have $C_0 \subseteq \tau(\{x\})$. On the other hand, again by (2.12), $x \in \tau(\{y\})$ for every y, and therefore, by (2.9), $\tau(\{x\}) \subseteq \tau(\{y\})$ for every y, whence $\tau(\{x\}) \subseteq C_0$. Hence, we have

$$\tau(\{x\}) = C_0 \qquad \text{for every } x \in C_0 \neq \emptyset . \tag{2.13}$$

Now, suppose $\tau(\emptyset) \neq \emptyset$. From (2.9) we obtain that

$$\tau(\emptyset) \subseteq C_0 . \tag{2.14}$$

Equally well from (2.9) we have that

$$\tau(\{x\}) \subseteq \tau(\emptyset) \qquad \text{for every } x \in \tau(\emptyset) . \tag{2.15}$$

From these last three statements it follows that $\tau(\emptyset)=C_0$, and this establishes our assertion.

The result of these deliberations is clearly this: a function τ on the class of one- and two-point sets, which satisfies (2.8) and (2.9), can be extended in one of at most two ways to include $\emptyset$ in its domain and still satisfy (2.8) and (2.9) on its entire domain; and any τ defined on all a.m.t.p. sets, and satisfying (2.8) and (2.9) on this domain, has a unique restriction to one- and two-point sets and so defines a unique betweenness relation. We could, of course, make a conventional restriction to only functions τ such that $\tau(\emptyset)=\emptyset$, and then we could affirm a 1-1 correspondence between the class of such functions on all a.m.t.p. sets and the class of betweenness relations. However, we have no particular need to do this. We do, however, have an advantage of simplicity in our developments in having $\emptyset$ in the domain of our functions τ. We shall, therefore, make the following definition:

DEFINITION 2.2. A function τ, with domain the class of all a.m.t.p. subsets of $\mathcal{X}$, and range-space the class of all subsets of $\mathcal{X}$, and satisfying the two conditions

$$A\subseteq\tau(A)\,, \tag{2.16}$$

$$A\subseteq\tau(B)\Rightarrow\tau(A)\subseteq\tau(B)\,, \tag{2.17}$$

for all A and B in the said domain, will be called a *spread function*. For any A in the domain, $\tau(A)$ is called the *spread of* A.

Spread functions are, then, equivalent to betweenness relations, although not in 1-1 correspondence with them; the equivalence is expressed by (2.10). A full exact statement of the correspondence will be incorporated into Theorem 3.3 in the next section.

It is quite easy to see what the spread functions are that correspond to the particular betweenness notions we have presented as examples above. In all of those examples the set C_0 (see (2.12)) is $\emptyset$, so that the spread function in question is unique. There is a very simple instance of a betweenness relation that has two distinct spread functions associated with it; namely, the case of $B(x, y; z)$ for all x, y, z in $\mathcal{X}$. In this case, $\tau(A)=\mathcal{X}$ for all one- and two-point sets A, and therefore $\tau(\emptyset)$ may be either $\emptyset$ or $\mathcal{X}$.

3. Extensions of a spread function: span functions

We were led in a natural way from betweenness relations to the consideration of spread functions. Now another motivation comes to urge us further along in the same direction. When we look at the defining conditions for a spread function, namely (2.16) and (2.17), we see that they are in no way *innately* tied to a.m.t.p. sets; they could very well apply to the characterization of a function on a larger domain. Specifically, we are driven to ask, regarding any particular spread function, if it is not in fact just the restriction to a.m.t.p. sets of a function defined on perhaps the domain of *all* subsets of $\mathcal{X}$, and satisfying relations of the form (2.16) and (2.17). An approach to this question suggests itself immediately. A given spread function τ has a unique associated betweenness relation, say B_τ, and this betweenness relation in turn defines τ for one- and two-point sets A, by (2.5). But the right-hand side of (2.5) is not restricted, for meaningfulness, to just one- and two-point sets. It is therefore suggested that we examine the function $\tau^{(1)}$ defined by

$$(3.1)\qquad \begin{cases} \tau^{(1)}(A) \overset{\text{def.}}{=} \{z \in \mathcal{X} \mid B_\tau(x, y; z),\ x \in A,\ y \in A\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all non-empty } A \subseteq \mathcal{X}\,, \\ \tau^{(1)}(\emptyset) \overset{\text{def.}}{=} \tau(\emptyset)\,. \end{cases}$$

Let us introduce the symbol $\mathcal{A}$ to denote the collection of all subsets of $\mathcal{X}$, and $\tilde{\mathcal{A}}$ to denote the collection of all a.m.t.p. subsets of $\mathcal{X}$. We note then with no difficulty that (3.1) can be re-expressed as follows:

$$(3.2)\qquad \tau^{(1)}(A) = \bigcup_{\substack{C \in \tilde{\mathcal{A}} \\ C \subseteq A}} \tau(C)\,, \qquad A \in \mathcal{A}\,.$$

And this form of expression of $\tau^{(1)}$ moreover unburdens us of the detour through B_τ.

It is immediately clear from (3.2) that $A \subseteq \tau^{(1)}(A)$ for every set A; that is, $\tau^{(1)}$ fulfills our desire as regards a more general context for (2.16). But this is not true for (2.17). The relation

$$(3.3)\qquad A \subseteq \tau^{(1)}(B) \Rightarrow \tau^{(1)}(A) \subseteq \tau^{(1)}(B)$$

will hold unrestrictedly for A if B is an element of $\tilde{\mathcal{A}}$, or is the spread of an element of $\tilde{\mathcal{A}}$. We see this from the form of the right-hand side of (3.2). But from that form we see also that (3.3) will not hold for just any set B. What happens in general is that A, satisfying $A \subseteq \tau^{(1)}(B)$, may contain a point x' of $\tau(C')$ and a point x'' of $\tau(C'')$, where C' and C'' are distinct $\tilde{\mathcal{A}}$-subsets of B; and then $\tau(\{x', x''\})$ may contain points that are not contained in any $\tau(C)$ for an $\tilde{\mathcal{A}}$-subset, C, of B. This can occur in the case of a 3-point set B. Thus, the function $\tau^{(1)}$ does not fulfill all our hopes. However, what we have just seen suggests that we may overcome the difficulty by iterating the function $\tau^{(1)}$, that is, by considering the function

$$\tau^{(2)}(A)=\tau^{(1)}(\tau^{(1)}(A)) , \qquad A \in \mathcal{A} . \tag{3.4}$$

But it is soon seen that this function presents the very same kind of difficulty. Nevertheless, since iteration persists, in this situation, as an evident way of attempting to "catch up" with the difficulty involved, we take the clear hint that our quest may be answered by considering the indefinite iteration of $\tau^{(1)}$. We therefore define all the functions

$$\tau^{(n)}(A)\overset{\text{def.}}{=}\tau^{(1)}(\tau^{(n-1)}(A)) , \qquad A \in \mathcal{A},\ n=2, 3,\cdots , \tag{3.5}$$

as well as $\tau^{(1)}$ and $\tau^{(0)}\equiv$the identity function. And noting that, since $A\subseteq\tau^{(1)}(A)$ for all $A \in \mathcal{A}$, we have

$$A\equiv\tau^{(0)}(A)\subseteq\tau^{(1)}(A)\subseteq\tau^{(2)}(A)\subseteq\cdots , \tag{3.6}$$

we then define the function g_τ on $\mathcal{A}$ by:

$$g_\tau(A)\overset{\text{def.}}{=}\lim_{n\to\infty}\tau^{(n)}(A)\equiv\bigcup_{n=1}^{\infty}\tau^{(n)}(A) . \tag{3.7}$$

We observe that we have the usual property:

$$\tau^{(m+n)}(A)=\tau^{(m)}(\tau^{(n)}(A)) , \quad A \in \mathcal{A} , \qquad m,\ n \text{ non-negative integers} . \tag{3.8}$$

From (3.6) we see immediately that

$$A\subseteq g_\tau(A) , \qquad A \in \mathcal{A} , \tag{3.9}$$

so that the (2.16)-type relationship is preserved by g_τ. To see that the same is true for the (2.17)-type relationship, let us first obtain some of the properties of the function g_τ. One of these properties comes directly from (3.8) by letting $m\to\infty$ while keeping n fixed: we get

$$g_\tau(\tau^{(n)}(A))=g_\tau(A) \qquad \text{for all } n=0, 1, 2,\cdots ;\ A \in \mathcal{A} . \tag{3.10}$$

From (3.2) we have

$$A\subseteq B\Rightarrow\tau^{(1)}(A)\subseteq\tau^{(1)}(B) . \tag{3.11}$$

Let $\langle A_n,\ n=1, 2,\cdots\rangle$ be a monotone non-decreasing sequence of sets, with limit A. Then $\langle\tau^{(1)}(A_n),\ n=1, 2,\cdots\rangle$ is likewise monotone, non-decreasing. Since every $A_n\subseteq A$, we have

$$\lim_{n\to\infty}\tau^{(1)}(A_n)\subseteq\tau^{(1)}(A) . \tag{3.12}$$

Conversely, consider a point $x \in \tau^{(1)}(A)$. Then $x \in \tau(C)$ for some a.m.t.p. subset C of A. This C is a subset of some $A_{n'}$; and therefore $x \in \tau^{(1)}(A_{n'})$. Consequently x is an element of the set on the left-hand side of (3.12). We have thus established the reverse inclusion to (3.12), and thereby the equality of the two members of (3.12). It follows that we have proved the statement

$$A_n \uparrow A \Rightarrow \tau^{(1)}(A_n) \uparrow \tau^{(1)}(A) . \tag{3.13}$$

We may apply this result as follows:

$$\tau^{(2)}(A_n) = \tau^{(1)}(\tau^{(1)}(A_n)) \uparrow \tau^{(1)}(\tau^{(1)}(A)) ; \tag{3.14}$$

thus we have

$$A_n \uparrow A \Rightarrow \tau^{(2)}(A_n) \uparrow \tau^{(2)}(A) . \tag{3.15}$$

Repeating the argument, we get in general

$$A_n \uparrow A \Rightarrow \tau^{(m)}(A_n) \uparrow \tau^{(m)}(A) \qquad \text{for all } m=0, 1, 2, \cdots . \tag{3.16}$$

With this result we can now let $n \to \infty$ in (3.8) and obtain the identity

$$\tau^{(m)}(g_\tau(A)) = g_\tau(A) \qquad \text{for all } m=0, 1, 2, \cdots ;\ A \in \mathcal{A} . \tag{3.17}$$

We may next let $m \to \infty$ in this last relation; we get:

$$g_\tau(g_\tau(A)) = g_\tau(A) , \qquad A \in \mathcal{A} . \tag{3.18}$$

Iterating (3.11) repeatedly, we find the result that

$$A \subseteq B \Rightarrow g_\tau(A) \subseteq g_\tau(B) . \tag{3.19}$$

And now it is these last two results with which we can show that a (2.17)-type implication holds for g_τ. Indeed, if $A \subseteq g_\tau(B)$, then $g_\tau(A) \subseteq g_\tau(g_\tau(B))$ by (3.19), and the desired result comes out by applying the identity (3.18). We do therefore have

$$A \subseteq g_\tau(B) \Rightarrow g_\tau(A) \subseteq g_\tau(B) , \qquad A, B \in \mathcal{A} . \tag{3.20}$$

Before going on we may observe that the discussion just completed gives us also the result

$$A_n \uparrow A \Rightarrow g_\tau(A_n) \uparrow g_\tau(A) . \tag{3.21}$$

To see this, note first that with $A_n \uparrow A$, (3.19) gives the monotone, non-decreasing character of the $g_\tau(A_n)$. Now apply (3.16) as follows:

$$\begin{aligned} A_n \uparrow A &\Rightarrow \bigcup_{n=1}^{\infty} \tau^{(m)}(A_n) = \tau^{(m)}(A) \qquad \text{for all } m \\ &\Rightarrow \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \tau^{(m)}(A_n) = g_\tau(A) \\ &\Leftrightarrow \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \tau^{(m)}(A_n) = g_\tau(A) \end{aligned} \tag{3.22}$$

$$\Leftrightarrow \bigcup_{n=1}^{\infty} g_\tau(A_n)=g_\tau(A)$$
$$\Leftrightarrow g_\tau(A_n) \uparrow g_\tau(A) .$$

Thus, (3.21) is established.

If A is an a.m.t.p. set, we see by (3.2) that $\tau^{(1)}(A)=\tau(A)$ and $\tau^{(1)}(\tau(A)) =\tau(A)$. (We alluded to this fact earlier, in our discussion of (3.3).) By iteration we have, therefore,

$$\tau^{(n)}(A)=\tau(A) , \qquad n=1, 2,\cdots ; \ A \in \tilde{\mathcal{A}} . \tag{3.23}$$

It follows immediately that

$$g_\tau(A)=\tau(A) , \qquad A \in \tilde{\mathcal{A}} , \tag{3.24}$$

so that indeed—as we were expecting to be the case—g_τ is an extension of τ.

Now let us derive one more property of g_τ before we summarize these present results. Suppose f is another extension of τ to the domain $\mathcal{A}$, and that it has the two crucial properties:

$$A\subseteq f(A) , \tag{3.25}$$

$$A\subseteq f(B)\Rightarrow f(A)\subseteq f(B) . \tag{3.26}$$

Let B be any set, and suppose A_0 is an a.m.t.p. subset of B. Then $A_0\subseteq B\subseteq f(B)$. Therefore $f(A_0)\subseteq f(B)$. But, f being an extension of τ, we have $f(A_0)=\tau(A_0)$; and therefore we have $\tau(A_0)\subseteq f(B)$. This holds for every a.m.t.p. subset of B, and it therefore follows that $\tau^{(1)}(B)\subseteq f(B)$. Similar argumentation shows, as a next step, that $\tau^{(2)}(B)\subseteq f(B)$. And so on. We thus get the result that $g_\tau(B)\subseteq f(B)$. We have hereby shown that g_τ is, among all extensions of τ to $\mathcal{A}$, the inclusionally minimal one. On this account we shall call g_τ the *core extension of* τ *to* $\mathcal{A}$.

Preparatory to bringing our results together in the form of a theorem, we make a definition:

DEFINITION 3.1. A function f on $\mathcal{A}$ to $\mathcal{A}$ which has the properties (3.25) and (3.26) will be called a *span function*. For any $A \in \mathcal{A}$, $f(A)$ will be called the *span of* A.

We may now state

THEOREM 3.1. *A spread function τ has an extension to a span function on $\mathcal{A}$. Among all such extensions there is one, g_τ—necessarily unique—which is inclusionally minimal; that is, for any span function f that extends τ, $g_\tau(A)\subseteq f(A)$ for all $A \in \mathcal{A}$. The function g_τ is called the core extension of τ, and it is given explicitly in terms of τ by* (3.7), (3.5) *and* (3.2).

In the course of our deliberations we saw that g_τ has the property of idempotency, that is, the property (3.18). The question arises, of course, whether this is tied to the fact that g_τ is in particular a core extension. It is not. Consider any span function f. For any $B \in \mathcal{A}$, if we put $A=f(B)$ in (3.25), we get $f(B) \subseteq f(f(B))$. On the other hand, if we make the same substitution in (3.26), we get $f(f(B)) \subseteq f(B)$. Thus, we have equality, and so the conclusion that every span function is idempotent.

It is also true that every span function is isotone, that is,

$$A \subseteq B \Rightarrow f(A) \subseteq f(B) . \tag{3.27}$$

Indeed, if $A \subseteq B$, then by (3.25) $A \subseteq f(B)$, and therefore (3.26) provides the implication stated in (3.27).

The two properties of idempotency and isotony together conversely imply (3.26): if $A \subseteq f(B)$ then $f(A) \subseteq f(f(B)) = f(B)$.

We gather these facts together into a lemma:

LEMMA 3.1. *Every span function is both idempotent and isotone. Conversely, if a function f on $\mathcal{A}$ to $\mathcal{A}$ has the property* (3.25) *and is also idempotent and isotone, then it is a span function.*

A very familiar example shows that in general there will be more than one extension of a given spread function to a span function on $\mathcal{A}$. Take $\mathcal{X}$ to be an infinite-dimensional Banach space, and let $\tau(A)$ denote the set of all linear combinations of elements in the a.m.t.p. set A. This τ clearly corresponds to the betweenness notion that defines z to be between x and y if z is a linear combination of x and y. Notice that in this case there are two distinct possibilities for $\tau(\emptyset)$; namely, $\emptyset$ and the set consisting of just the zero-element of the Banach space. The core extension, g_τ, of τ has the value, for any particular non-empty $A \in \mathcal{A}$, which is the linear manifold spanned by A. If, for each non-empty $A \in \mathcal{A}$, we define $f(A)$ to be the *closed* linear manifold spanned by A, and define $f(\emptyset)=\tau(\emptyset)$, then we see that f also is an extension of τ to a span function on $\mathcal{A}$, and that f is not identical with g_τ.

There is no great problem about knowing whether or not a particular spread function has a unique extension to a span function. For, in fact, it is possible to state quite explicitly the inclusionally *maximal* extension of any given spread function, τ, to a span function on $\mathcal{A}$. If we denote this latter by g^τ, then of course the comparison of g_τ with g^τ provides the answer to the uniqueness question. The following theorem gives the formula for this g^τ.

THEOREM 3.2. *Among all the extensions of a given spread function*

τ to a span function on $\mathcal{A}$, there is one, g^τ—necessarily unique—which is inclusionally maximal; that is, for any span function f that extends τ, $f(A) \subseteq g^\tau(A)$ for all $A \in \mathcal{A}$. The function g^τ is given explicitly by

$$g^\tau(A) = \begin{cases} \displaystyle\bigcap_{\substack{B \in \tilde{\mathcal{A}} \\ A \subseteq \tau(B)}} \tau(B), & \textit{if there exists some } B \in \tilde{\mathcal{A}} \textit{ such that } A \subseteq \tau(B), \\ \mathcal{X}, & \textit{otherwise}. \end{cases} \tag{3.28}$$

Let us first see that g^τ is a span function. It is clear immediately from (3.28) that $A \subseteq g^\tau(A)$ for every $A \in \mathcal{A}$, so that g^τ has the first of the two defining properties of a span function. To prove that it has the second as well, suppose $A \subseteq g^\tau(C)$ for two particular sets, A and C, of $\mathcal{A}$. If $g^\tau(C) = \mathcal{X}$ then without further argument we have the desired result that $g^\tau(A) \subseteq g^\tau(C)$. If, on the other hand, $g^\tau(C) \neq \mathcal{X}$, then, by (3.28),

$$g^\tau(C) = \bigcap_{\substack{B \in \tilde{\mathcal{A}} \\ C \subseteq \tau(B)}} \tau(B). \tag{3.29}$$

For each set $B \in \tilde{\mathcal{A}}$ that figures in the intersection on the right-hand side of this equality, we have $A \subseteq g^\tau(C) \subseteq \tau(B)$, so that such a set B is also among those that define $g^\tau(A)$ according to (3.28). Thus, we have

$$g^\tau(A) = \bigcap_{\substack{B \in \tilde{\mathcal{A}} \\ A \subseteq \tau(B)}} \tau(B) \subseteq \bigcap_{\substack{B \in \tilde{\mathcal{A}} \\ C \subseteq \tau(B)}} \tau(B) = g^\tau(C). \tag{3.30}$$

The relation $g^\tau(A) \subseteq g^\tau(C)$ is therefore in all cases a consequence of the inclusion $A \subseteq g^\tau(C)$. This is the assertion that g^τ has the second defining property of a span function, and so it is established that g^τ is indeed a span function.

That g^τ is an extension of τ is proved as follows. For any particular $A \in \tilde{\mathcal{A}}$ we have $A \subseteq \tau(A)$ (see (2.8)), and if B is any other set in $\tilde{\mathcal{A}}$ such that $A \subseteq \tau(B)$, then $\tau(A) \subseteq \tau(B)$ (see (2.9)). Therefore,

$$g^\tau(A) = \bigcap_{\substack{B \in \tilde{\mathcal{A}} \\ A \subseteq \tau(B)}} \tau(B) = \tau(A), \qquad A \in \tilde{\mathcal{A}}. \tag{3.31}$$

Now finally to see that g^τ is maximal, let f be any particular extension of τ. For a set $A \in \mathcal{A}$, if there does not exist $B \in \tilde{\mathcal{A}}$ such that $A \subseteq \tau(B)$, then $g^\tau(A) = \mathcal{X} \supseteq f(A)$. If there does exist such a B then we have, utilizing Lemma 3.1,

$$f(A) \subseteq f(\tau(B)) = f(f(B)) = f(B) = \tau(B); \tag{3.32}$$

and therefore

$$f(A) \subseteq \bigcap_{\substack{B \in \tilde{\mathcal{A}} \\ A \subseteq \tau(B)}} \tau(B) = g^{\tau}(A) . \tag{3.33}$$

This shows the asserted maximality, and hence the theorem is completely proved.

Clearly, the restriction to $\tilde{\mathcal{A}}$ of any span function f on $\mathcal{A}$ is a particular spread function; the core extension of that spread function will be called the *core* (*span function*) of f. The core of any core span function is obviously that function itself. A span function that is its own core will be called a *self-core span function.* It is evident that the class of spread functions is in 1-1 correspondence with the class of self-core span functions, and that through this correspondence we get a correspondence between self-core span functions and betweenness relations. We state all of this precisely in the following theorem, which details the findings of Section 2 as well:

THEOREM 3.3. *Let $\tilde{\mathcal{A}}^{\cdot}$ denote the class of all one-point and two-point subsets of $\mathcal{X}$. Then, the class of all betweenness relations is in a* 1-1 *correspondence with the class of all restrictions of spread functions to $\tilde{\mathcal{A}}^{\cdot}$. The correspondence is explicitly given by* (2.10).

If the values of a spread function τ over the domain $\tilde{\mathcal{A}}^{\cdot}$ have an intersection C which is not the null set, then there are exactly two spread functions having the same restriction to $\tilde{\mathcal{A}}^{\cdot}$ as τ: for one of these the value at $\emptyset$ is $\emptyset$; for the other the value at $\emptyset$ is C. If $C=\emptyset$, then $\tau(\emptyset)=\emptyset$ and τ is the unique spread function that takes on its particular values over $\tilde{\mathcal{A}}^{\cdot}$.

There is a 1-1 *correspondence between the class of all spread functions and the class of all self-core span functions: this is the correspondence which associates with each spread function τ its core extension g_{τ}, or, conversely, which associates with each self-core span function its restriction to $\tilde{\mathcal{A}}$. From this correspondence there derives in the obvious way a* 1-1 *correspondence between the class of all restrictions of spread functions to $\tilde{\mathcal{A}}^{\cdot}$ and the class of all restrictions of self-core span functions to $\mathcal{A}^{\cdot}$—where $\mathcal{A}^{\cdot}$ is the class of all non-empty subsets of $\mathcal{X}$.*

It follows that there is a 1-1 *correspondence between the class of all restrictions of self-core span functions to $\mathcal{A}^{\cdot}$ and the class of all betweenness relations, this correspondence being that which associates with a given betweenness relation, B, the restriction to $\mathcal{A}^{\cdot}$ of the core extension of a spread function τ related to B through* (2.10).

Consider two span functions, h and f, satisfying the following condition:

$$(3.34) \qquad h(A) \subseteq f(A) , \qquad A \in \mathcal{A} .$$

It follows immediately by (3.26) that $f(h(A)) \subseteq f(A)$. By (3.25) (as it pertains to h) we have $A \subseteq h(A)$, and therefore, by the isotone property of f, $f(A) \subseteq f(h(A))$. Hence we have the equality: $f(h(A)) = f(A)$.

By (3.25) once again, $f(A) \subseteq h(f(A))$. Conversely, by (3.34) and the idempotency of f, we have $h(f(A)) \subseteq f(f(A)) = f(A)$. Thus, $h(f(A)) = f(A)$. And so we have shown that from (3.34) it follows that

$$(3.35) \qquad h(f(A)) = f(h(A)) = f(A) , \qquad A \in \mathcal{A} .$$

If we suppose that (3.35) holds for two span functions h and f, then notice that, for any A, it follows from $A \subseteq f(A)$ that $h(A) \subseteq h(f(A)) = f(A)$; that is, (3.34) is satisfied. We have thus arrived at the following result:

LEMMA 3.2. *For two span functions, h and f, the conditions* (3.34) *and* (3.35) *are equivalent.*

This lemma enables us to state, for example, the following alternative characterizations of the core extension and the maximal extension of a given spread function:

LEMMA 3.3. *The core extension, g_τ, of a given spread function τ is that span function extension of τ such that for any span function extension, f, of τ, we have*

$$(3.36) \qquad g_\tau(f(A)) = f(g_\tau(A)) = f(A) , \qquad A \in \mathcal{A} .$$

And the maximal extension, g^τ, of τ is that span function extension of τ such that for any span function extension, f, of τ, we have

$$(3.37) \qquad g^\tau(f(A)) = f(g^\tau(A)) = g^\tau(A) , \qquad A \in \mathcal{A} .$$

4. The Birkhoff-Ore-Everett structure of span functions

Our persistence in pushing the examination of general betweenness relations all the way to the discovery of their connection with span functions now turns out to be far more than just an interesting algebraic exercise. For, the fact is that span functions are well-known in the field of algebra, and there are general results concerning them that are enlightening for our purposes. Garrett Birkhoff [4], Oystein Ore [11] and C. J. Everett [6] are the principal authors associated with these developments. Span functions, defined on partially ordered sets in general—not only on lattices of subsets, as in our case—are studied by algebraists under the name *closure operators*. (In our geometric context the term "span function" remains the preferable one.) The immediately interesting facts concerning these operators are the following. Birkhoff had considered binary relations between the elements of a space $\mathscr{X}$ and those of a space Ω, and from such a relation had induced two mappings, one from $\mathscr{A}$ to $\mathscr{B}$ and one from $\mathscr{B}$ to $\mathscr{A}$—where $\mathscr{A}$ and $\mathscr{B}$ are the classes of all subsets of $\mathscr{X}$ and of Ω, respectively. Let $P_{\mathscr{A}}$ and $P_{\mathscr{B}}$ denote these mappings, for a given binary relation. Birkhoff showed that a certain triplet of conditions is satisfied by this pair of mappings; and he showed also that $P_{\mathscr{B}}P_{\mathscr{A}}$ is a closure operator in $\mathscr{A}$ and, similarly, that $P_{\mathscr{A}}P_{\mathscr{B}}$ is a closure operator in $\mathscr{B}$ (see Section 5 in Chapter IV of Birkhoff [4]). Ore then fixed upon the mentioned triplet of conditions, positing them as satisfied for mappings, $P_{\mathscr{A}}$ and $P_{\mathscr{B}}$, more generally between two partially ordered sets $\mathscr{A}$ and $\mathscr{B}$; and he showed that this was all that was needed for the result that $P_{\mathscr{B}}P_{\mathscr{A}}$ and $P_{\mathscr{A}}P_{\mathscr{B}}$ are closure operators in $\mathscr{A}$ and $\mathscr{B}$, respectively. A pair of mappings, $P_{\mathscr{A}}$, $P_{\mathscr{B}}$, fulfilling the three conditions in question, Ore called a *Galois connection*, or *Galois correspondence*, between the partially ordered sets $\mathscr{A}$ and $\mathscr{B}$. The work of Everett followed and supplied complete converses. Everett showed that every closure operator in a partially ordered set $\mathscr{A}$ results in the above manner from a Galois connection between $\mathscr{A}$ and another partially ordered set $\mathscr{B}$. And in the case where $\mathscr{A}$ is the class of all subsets of a space $\mathscr{X}$, he showed that every closure operator in $\mathscr{A}$ results from a Galois connection which in turn derives, in the manner of Birkhoff, from a binary relation between the elements of $\mathscr{X}$ and those of another space Ω. It is this last result that will tell us much more about betweenness relations.

Let us now describe exactly the notions that have just been introduced. And we shall, in doing so, remain within the context that is of interest to us, namely, $\mathscr{A}$ the class of all subsets of our space $\mathscr{X}$. We shall also now leave behind the term "closure operator" and henceforth speak of span functions.

DEFINITION 4.1. There is said to exist a (binary) relation R between the points of $\mathscr{X}$ and the points of another space Ω if, for every pair of points, $\langle x, \omega\rangle$, $x \in \mathscr{X}$, $\omega \in \Omega$, it is specified either that x is in the relation R to ω or that x is not in the relation R to ω. If x is in the relation R to ω, we write $R(x;\omega)$; otherwise, we write $R^c(x;\omega)$.

DEFINITION 4.2. Let R be a relation between $\mathscr{X}$ and another space Ω. Let $\mathscr{A}$ and $\mathscr{B}$ be the classes of all subsets of $\mathscr{X}$ and of Ω, respectively. We define mappings $P^R_{\mathscr{A}}$: $\mathscr{A}\rightarrow\mathscr{B}$ and $P^R_{\mathscr{B}}$: $\mathscr{B}\rightarrow\mathscr{A}$ as follows:

$$P^R_{\mathscr{A}}(A)=\{\omega\in\Omega \mid R(x;\omega) \text{ for all } x\in A\}\ , \tag{4.1}$$

$$P^R_{\mathcal{B}}(B) = \{x \in \mathcal{X} \mid R(x, \omega) \text{ for all } \omega \in B\} . \tag{4.2}$$

$P^R_{\mathcal{A}}(A)$ is called the *polar of A under R*, and $P^R_{\mathcal{B}}(B)$ is called the *polar of B under R*.

The very apt term "polar" is suggested by Birkhoff. These two polar mappings are the ones presented by Birkhoff, and he showed, as mentioned above, that $P^R_{\mathcal{B}}P^R_{\mathcal{A}}$ is a span function on $\mathcal{A}$ and that $P^R_{\mathcal{A}}P^R_{\mathcal{B}}$ is a span function on $\mathcal{B}$.

DEFINITION 4.3. Let $\mathcal{A}$ and $\mathcal{B}$ denote, again, the classes of all subsets of the spaces $\mathcal{X}$ and Ω, respectively. Let $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ be mappings, on $\mathcal{A}$ to $\mathcal{B}$ and on $\mathcal{B}$ to $\mathcal{A}$, respectively, satisfying the following conditions:

$$A_1 \supseteq A_2 \Rightarrow P_{\mathcal{A}}(A_1) \subseteq P_{\mathcal{A}}(A_2) , \tag{4.3}$$

$$B_1 \supseteq B_2 \Rightarrow P_{\mathcal{B}}(B_1) \subseteq P_{\mathcal{B}}(B_2) , \tag{4.4}$$

$$A \subseteq P_{\mathcal{B}}P_{\mathcal{A}}(A) \quad \text{and} \quad B \subseteq P_{\mathcal{A}}P_{\mathcal{B}}\,(B) \qquad \text{for all } A \in \mathcal{A},\ B \in \mathcal{B} . \tag{4.5}$$

Then the pair of mappings, $P_{\mathcal{A}}$, $P_{\mathcal{B}}$, is said to define a *Galois connection* between $\mathcal{A}$ and $\mathcal{B}$.

For any relation R between $\mathcal{X}$ and Ω, the pair, $P^R_{\mathcal{A}}$, $P^R_{\mathcal{B}}$, defines a Galois connection between $\mathcal{A}$ and $\mathcal{B}$ (Birkhoff). For any Galois connection, $P_{\mathcal{A}}$, $P_{\mathcal{B}}$, the mapping $P_{\mathcal{B}}P_{\mathcal{A}}$ is a span function on $\mathcal{A}$, and the mapping $P_{\mathcal{A}}P_{\mathcal{B}}$ is a span function on $\mathcal{B}$ (Ore). The result of Everett that is of interest to us may now be stated as follows:

THEOREM 4.1 (Everett). *Let f be any particular span function on $\mathcal{A}$, the class of all subsets of the space $\mathcal{X}$. Then there is a space Ω, and a relation R between $\mathcal{X}$ and Ω, such that $f = P^R_{\mathcal{B}}P^R_{\mathcal{A}}$, where $\mathcal{B}$ is the class of all subsets of Ω and $P^R_{\mathcal{A}}$ and $P^R_{\mathcal{B}}$ are defined by* (4.1) *and* (4.2).

The reader will find discussions of closure operators and Galois correspondences also in Cohn [5] and in Grätzer [8].

5. The structure of betweenness; explicit examples

We can immediately turn the facts revealed in the preceding section into a statement of structure for betweenness relations. We prove the following theorem.

THEOREM 5.1. *B is a betweenness relation in the space $\mathcal{X}$ if and only if there is a space Ω and a relation R between $\mathcal{X}$ and Ω such that*

$B(x, y; z)$ if and only if $R(z, \omega)$ for every $\omega \in \Omega$ such that $R(x, \omega)$ and $R(y, \omega)$.

PROOF. Suppose there is a space Ω and a relation R such that a certain relation B is related to R in the manner indicated. We shall show that B is then a betweenness relation. Let $\mathcal{A}$ and $\mathcal{B}$ denote, as usual, the classes of all subsets of $\mathcal{X}$ and Ω, respectively. Let us set

$$B_{x,y} = \{\omega \in \Omega \mid R(x, \omega) \text{ and } R(y, \omega)\} . \tag{5.1}$$

Then notice that the relation between B and R is that $B(x, y; z)$ if and only if

$$z \in \{w \in \mathcal{X} \mid R(w, \omega) \text{ for all } \omega \in B_{x,y}\} . \tag{5.2}$$

In terms of the mappings $P^R_{\mathcal{A}}$ and $P^R_{\mathcal{B}}$ defined by (4.1) and (4.2), we see that

$$B_{x,y} = P^R_{\mathcal{A}}(\{x, y\}) , \tag{5.3}$$

and that (5.2) is the statement

$$z \in P^R_{\mathcal{B}}(B_{x,y}) . \tag{5.4}$$

Therefore, we have the assertion that $B(x, y; z)$ if and only if

$$z \in P^R_{\mathcal{B}} P^R_{\mathcal{A}}(\{x, y\}) . \tag{5.5}$$

By Birkhoff's result, $P^R_{\mathcal{B}} P^R_{\mathcal{A}}$ is a span function on $\mathcal{A}$, and if $\tau^{\cdot}$ denotes the restriction of this span function to $\tilde{\mathcal{A}}^{\cdot}$ (the class of all one- and two-point subsets of $\mathcal{X}$) then (5.5) states that $z \in \tau^{\cdot}(\{x, y\})$. But, being a restriction of a span function, $\tau^{\cdot}$ is a spread function on $\tilde{\mathcal{A}}^{\cdot}$, that is, it has the properties (2.8) and (2.9). Therefore, the equivalence of $B(x, y; z)$ and (5.5) is an identity of the form (2.10). And this establishes the result that B is a betweenness relation.

Now, conversely, let us suppose that B is given as a betweenness relation in $\mathcal{X}$. Then, by Theorem 3.3, there is a span function f on $\mathcal{A}$ such that

$$B(x, y; z) \equiv [z \in f(\{x, y\})] . \tag{5.6}$$

By Theorem 4.1 there exist Ω and R such that $f = P^R_{\mathcal{B}} P^R_{\mathcal{A}}$. Thus, the right-hand side of (5.6) is of the form

$$z \in P^R_{\mathcal{B}} P^R_{\mathcal{A}}(\{x, y\}) . \tag{5.7}$$

The definitions of $P^R_{\mathcal{A}}$ and $P^R_{\mathcal{B}}$ tell us that (5.7) is precisely the statement "$R(z, \omega)$ for every $\omega \in \Omega$ such that $R(x, \omega)$ and $R(y, \omega)$". And therefore the equivalence (5.6) establishes exactly the R-structure asserted by our theorem for the betweenness relation B.

The proof of Theorem 5.1 is therefore complete.

It is, of course, true that we could have given a direct proof of this theorem, thereby avoiding entirely all of our discussion of spread functions, span functions, and the results in the preceding section. That, however, would have been exceedingly undesirable. We are looking forward to rather extensive studies emanating from the notion of betweenness. Those studies may, in ways as yet unanticipated, be benefitted by facts regarding betweenness relations that relate to such associated notions as span functions, Galois connections, etc. It was, therefore, very much in order for us to bring all of this material to light.

When a space Ω and a relation R are specified, the Theorem 5.1 structure determines B completely. On the other hand, a given B may be generated by many different pairs Ω, R. (Examples of this will be encountered in the ensuing discussions.) However, there is always a relation R of a certain particular form that is available to provide a representation of a given B. We shall prove this now, after first defining the form in question.

DEFINITION 5.1. A relation R will be said to be of (or, to have) *membership form in* $\mathscr{X}$ if it is a relation between $\mathscr{X}$ and a class, Ω, of subsets of $\mathscr{X}$, and specifically

$$R(x, \omega)\equiv[x \in \omega] , \qquad x \in \mathscr{X},\ \omega \in \Omega . \tag{5.8}$$

The theorem to be proved is thus the following:

THEOREM 5.2. *Every betweenness relation B in a space $\mathscr{X}$ has a Theorem* 5.1 *representation in which R is a relation of membership form in $\mathscr{X}$.*

To prove this theorem let a given betweenness relation B have an associated spread function τ. We take

$$\Omega=\{\tau(A)\,|\,A \in \tilde{\mathscr{A}}\} , \tag{5.9}$$

and let R then be the relation specified by (5.8). Let us denote by B' the betweenness relation determined by this R. We want to show that B' is precisely B. According to Theorem 5.1, z is B'-between x and y if and only if z is an element of every $\omega \in \Omega$ to which both x and y belong. Thus,

$$B'(x, y; z)\equiv[z \in \cap(\tau(A);\ A \in \tilde{\mathscr{A}},\ \{x, y\}\subseteq\tau(A))] . \tag{5.10}$$

By (2.17), every $\tau(A)$ in the intersection here has $\tau(\{x, y\})$ as a subset. Furthermore, by (2.16), $\tau(\{x, y\})$ is itself one of the pertinent $\tau(A)$'s. Therefore, the intersection is exactly $\tau(\{x, y\})$, and so (5.10) becomes

$$B'(x, y; z) \equiv [z \in \tau(\{x, y\})] . \tag{5.11}$$

By Theorem 3.3, the right-hand side of this equality determines uniquely the betweenness relation that determines τ on $\tilde{\mathcal{A}}^{\cdot}$, that is, B. Hence $B' \equiv B$, and the theorem is proved.

It will be noticed that in place of (5.9) we could have taken

$$\Omega = \{f(A) \mid A \in \mathcal{A}\} , \tag{5.12}$$

where f is any particular span-function extension of τ. This Ω is seen to be just the *closure system* corresponding to f according to Theorem 1.1, p. 43 in Cohn [5].

Small differences among betweenness relations are often most penetratingly understood or presented in terms of generating relations of membership form. For example, in (2.3) we formed a betweenness relation B' from a betweenness relation B. If R gives a Theorem 5.1 representation of B, and R is of membership form in $\mathcal{X}$ based on a class Ω of subsets of $\mathcal{X}$ (i.e., (5.8) holds), then, in fact, B' is the betweenness relation generated by the relation R', of membership form in $\mathcal{X}$, based on the class of subsets $\Omega' = \Omega \cup \{\{x\} \mid x \in \mathcal{X}\}$; that is the singletons in $\mathcal{X}$ are adjoined to Ω. We shall cite other comparisons of this kind that come to the fore in our discussions below.

It is instructive now to examine in detail certain examples of the structure bespoken by Theorem 5.1. Let us first consider the instances of betweenness exemplified in Section 2. For ease of reference we shall give these definite names.

DEFINITION 5.2. In a real unitary space $\mathcal{K}$ the relation

$$z = ax + by , \qquad a \geqq 0,\ b \geqq 0 , \tag{5.13}$$

of a point z to the pair of points x and y, is a betweenness relation. We call this *ひ$^v_0$-betweenness in $\mathcal{K}$*. If (5.13) holds we say that *z is ひ$^v_0$-between x and y*; and we abbreviate this statement with the symbolic expression ひ$^v_0(x, y; z)$.

If C is any subset of $\mathcal{K}$, (5.13) defines a betweenness relation for triplets of elements of C; we call this *ひ$^v_0$-betweenness in C*, and again use terminology and notation as above.

(The character ひ is the Japanese hiragana *hi*. The superscript v is meant to indicate that the betweenness relation is one that is defined for vectors—as distinguished from the ひ$_0$-betweenness that was defined in [3] and which pertains to lines, or more precisely, to 1-dimensional linear manifolds. This latter notion of betweenness we present here in our next definition (see further below).)

Let $\mathcal{K}$ be a real unitary space, with inner product denoted, as usual, by $(\cdot, \cdot)$. It is a matter of a few elementary calculations—which we shall not detail here—to show that, in $\mathcal{K}$, z is $\mathcal{V}_0^v$-between x and y if and only if

$$\left.\begin{array}{l}(x, \omega)\geqq 0\\ (y, \omega)\geqq 0\end{array}\right\}\Rightarrow (z, \omega)\geqq 0 \,. \tag{5.14}$$

This fact is immediately seen to give us a realization of Theorem 5.1 for $\mathcal{V}_0^v$-betweenness in $\mathcal{K}$, in which Ω as well as $\mathcal{X}$ is the space $\mathcal{K}$ and the relation R is given by

$$R(x, y)\equiv[(x, y)\geqq 0] \,. \tag{5.15}$$

Under this binary relation R—a fully symmetric relation in this case—the polar of a set $A\subseteq\mathcal{K}$ is

$$\{y\in\mathcal{K}\,|\,(x, y)\geqq 0 \text{ for all } x\in A\} \,. \tag{5.16}$$

This particular definition of a polar set has already done much service in the literature; for example, see Weyl [16] and Gale [7], as well as the publication [1].

If C is any particular subset of $\mathcal{K}$ we get a Theorem 5.1 representation of $\mathcal{V}_0^v$-betweenness in C by again taking $\Omega=\mathcal{K}$ and taking the relation R between $x\in C$ and $y\in\mathcal{K}$ to be once again (5.15).

Notice that the relation (5.15) can be read as follows: x lies in the positive closed half-space determined by the (non-null) vector y. Thus, R is equivalent to the relation of membership form, (5.8), with $\mathcal{X}=\mathcal{K}$ and $\Omega=$the collection of all closed half-spaces in $\mathcal{K}$. Accordingly, we can say further that the set of vectors which are $\mathcal{V}_0^v$-between two given vectors is just the intersection of all closed half-spaces each of which contains the two given vectors. By making small modifications in the collection Ω we obtain betweennesses that differ only slightly from $\mathcal{V}_0^v$-betweenness. For example, if the sets of Ω are taken to be not the closed half-spaces, but the open half-spaces each augmented by the null vector, then the resulting betweenness relation is described by (5.13) when x and y do not fulfill the condition

$$0\neq(x, y)=-\|x\|\cdot\|y\| \,, \tag{5.17}$$

whereas when x and y do fulfill this condition then every point in $\mathcal{K}$ is between x and y. One gets still another, slightly different betweenness if one takes Ω to be the collection of all open half-spaces. And so on.

In Section 2 we cited also, as another example, the betweenness for 1-dimensional linear manifolds in $\mathcal{K}$ that was shown in [3] to be at the root of the present flanking notion. We have described this notion hereabove in purely geometrical terms, as the "in-the-acute-angle-of" notion of betweenness. In [3] we showed, however, that there is a very concise analytical characterization of this notion. We use this characterization now in stating the definition of real $\mho_0$-betweenness.

DEFINITION 5.3. Let L, M and N denote 1-dimensional linear manifolds in the real unitary space $\mathcal{K}$. We say that *N is $\mho_0$-between L and M* if, for any particular unit vectors $x \in L$, $y \in M$ and $z \in N$, the following four conditions are satisfied:

$$(5.18) \qquad \begin{cases} \text{i)} & z \text{ is a linear combination of } x \text{ and } y, \\ \text{ii)} & |(z, x)| \geqq |(x, y)|, \\ \text{iii)} & |(y, z)| \geqq |(x, y)|, \\ \text{iv)} & (z, x)(x, y)(y, z) \geqq 0. \end{cases}$$

(This characterization was given, in [3], the name "$\mho_0$-betweenness" explicitly only in the case of a complex unitary space. We are now extending the use of this name to the real case as well.)

We shall now give a Theorem 5.1 representation for $\mho_0$-betweenness in $\mathcal{K}$. And it will be noted that this discussion effectively gives another proof—other than the one given in [3]—that (5.18) is the analytical characterization of the geometrical notion of "in-the-acute-angle-of" betweenness. We begin with a definition:

DEFINITION 5.4. A collection, Γ, of 1-dimensional linear manifolds in $\mathcal{K}$ will be called a *bicone* (*of* 1-*dimensional linear manifolds*) if the following holds: for any L and M in Γ, if x and y are unit vectors in L and M, respectively, such that $(x, y) \geqq 0$, and if z is a unit vector which is a non-negative linear combination of x and y, then the linear manifold spanned by $\{z\}$ is in Γ.

(Our use of the term "bicone" here is consistent with the definition given in [2].) Let the set of all these bicones in $\mathcal{K}$ be denoted by $\beth$. (This last character is the Hebrew letter *beth*.) Recalling that we have already introduced (in Section 2) the notation $\mathcal{K}^p$ for the set of all 1-dimensional linear manifolds in $\mathcal{K}$, we now define a relation R between $\mathcal{K}^p$ and $\beth$, as follows: for $L \in \mathcal{K}^p$ and $\Gamma \in \beth$,

$$(5.19) \qquad R(L, \Gamma) \equiv [L \in \Gamma],$$

We now assert that this membership-form relation R gives a Theorem 5.1 structure for $\mho_0$-betweenness. We proceed to prove this.

Let B denote the betweenness relation that does have the Theorem 5.1 relationship to the R defined by (5.19) (—so that our objective is to prove that $B=\mho_0$). Thus, $B(L, M; N)$ if and only if $R(N, \Gamma)$ for every Γ such that $R(L, \Gamma)$ and $R(M, \Gamma)$. Restating this, we have: $B(L, M; N)$ if and only if N belongs to every Γ to which both L and M belong. Now, it is readily seen, according to the Definition 5.4, that intersections of bicones are bicones. Hence, we have a smallest bicone containing any given set of elements of $\mathcal{K}^p$, designated, as usual, *the bicone generated by* the given set of elements. In particular, then, in our present case of twe elements, L and M, an element N belongs to every Γ to which both L and M belong if and only if N belongs to the bicone generated by L and M. Consequently: $B(L, M; N)$ if and only if N belongs to the bicone generated by L and M.

Consider, for a given pair L and M, the following subset of $\mathcal{K}^p$:

$$(5.20)\quad \Gamma'=\{N'\in\mathcal{K}^p \mid \text{there exist } x\in L,\ y\in M,\ z\in N';\ a\geqq 0,\ b\geqq 0 \text{ with } \|x\|=\|y\|=\|z\|=1;\ (x, y)\geqq 0;\ z=ax+by\}\ .$$

Two facts are easily established concerning Γ': first, that it is a bicone, and second, that it is a subset of every bicone that contains L and M. It follows that Γ' is the bicone generated by L and M. And therefore we may state: $B(L, M; N)$ if and only if $N\in\Gamma'$. Accordingly, to prove that $B=\mho_0$, it now remains only to show that the defining conditions for Γ' in (5.20) are equivalent to the conditions (5.18).

Let us suppose first that (5.18) is satisfied. That is, for *any* particular unit vectors $x\in L$, $y\in M$ and $z\in N$, the conditions (5.18) hold. If $(x, y)=0$, then, according to i), z is a linear combination of x and y—say $\alpha x+\beta y$; rewrite it as $(e_1\alpha)(e_1x)+(e_2\beta)(e_2y)$, where $e_1=\pm 1$ and $e_2=\pm 1$, and $e_1\alpha\geqq 0$ and $e_2\beta\geqq 0$. Then we have that the defining conditions of Γ' are satisfied for $e_1x\in L$, $e_2y\in M$, $z\in N$, $a=e_1\alpha$, $b=e_2\beta$. Thus, $N\in\Gamma'$.

If the x, y and z fulfilling (5.18) are such that $(x, y)\neq 0$, we first take $y'=e_1y$, where $e_1=\pm 1$ and $(x, y')>0$. By ii) and iii) we have $(z, x)\neq 0$ and $(y', z)\neq 0$. We next choose $z'=e_2z$, where $e_2=\pm 1$ and $(y', z')>0$. It then follows from iv) that also $(z', x)>0$. Having done all this, we can now drop the primes and consider that we have unit vectors $x\in L$, $y\in M$ and $z\in N$ such that z is a linear combination of x and y, and such that

$$(5.21)\qquad (z, x)\geqq(x, y)>0\ ,\qquad (y, z)\geqq(x, y)>0\ .$$

If $(x, y)=1$, then these inequalities imply that $z=y=x$, and we see that the defining conditions of Γ' are satisfied with, for example, $a=1$, $b=0$. And so we have $N\in\Gamma'$.

Suppose now that $(x, y)<1$. We write

$$z=ax+by \ , \tag{5.22}$$

and derive from this the two equalities

$$(z, x)=a+b(x, y) \ , \qquad (y, z)=a(x, y)+b \ . \tag{5.23}$$

Solving for a and b, we have

$$a=\frac{(z, x)-(x, y)(y, z)}{1-(x, y)^2} \ , \qquad b=\frac{(y, z)-(z, x)(x, y)}{1-(x, y)^2} \ . \tag{5.24}$$

Because of (5.21) we see by these expressions that a and b are non-negative. This being the case we once again find the defining conditions of Γ' satisfied, and so again we have the conclusion that $N \in \Gamma'$.

We have now completed the demonstration that the satisfaction of (5.18) implies that $N \in \Gamma'$. To prove the converse, let the unit vectors $x \in L$, $y \in M$, $z \in N$ fulfill the defining conditions of Γ'. If $(x, y)=0$ we see immediately that (5.18) is fully satisfied. Let us suppose, then, that $(x, y)>0$. If $b=0$ then $a=1$ and $z=x$, and in this case it is again immediate that (5.18) is satisfied. Similarly if $a=0$, (5.18) is an immediate consequence. We may suppose, then, that both a and b are positive. We have (5.22) holding and again may derive (5.23) from it. From (5.23) we obtain immediately the fact that $(z, x)>0$ and $(y, z)>0$. We therefore have

$$(z, x)(x, y)(y, z)>0 \ . \tag{5.25}$$

It follows also from (5.23) that

$$a+b(x, y)\leqq 1 \ , \qquad a(x, y)+b\leqq 1 \tag{5.26}$$

or

$$\frac{1-a}{b}\geqq(x, y) \ , \qquad \frac{1-b}{a}\geqq(x, y) \ . \tag{5.27}$$

From the fact that $\|z\|=1$ we have

$$a(z, x)+b(y, z)=1 \ . \tag{5.28}$$

From this we deduce the two inequalities

$$a+b(y, z)\geqq 1 \ , \qquad a(z, x)+b\geqq 1 \ , \tag{5.29}$$

or

$$(y, z)\geqq\frac{1-a}{b} \ , \qquad (z, x)\geqq\frac{1-b}{a} \ . \tag{5.30}$$

Combining these with (5.27), we have

$$(y, z) \geqq (x, y) , \qquad (z, x) \geqq (x, y) . \tag{5.31}$$

We may summarize now, pointing out that for our present unit vectors $x \in L$, $y \in M$, $z \in N$, we have (5.22), (5.25) and (5.31) holding. But under these circumstances it is then true that (5.18) holds for *any* unit vectors $x \in L$, $y \in M$, $z \in N$. This observation completes the proof that if $N \in \Gamma'$ then (5.18) holds.

We have hereby shown fully that B and $\mathcal{V}_0$ are the same relation. And consequently we have established our assertion that the relation (5.19) gives a Theorem 5.1 representation of $\mathcal{V}_0$-betweenness. It will be noticed by (5.20) that Γ' is quite evidently the collection of 1-dimensional linear manifolds "in the acute angle between" L and M; and therefore, in proving the equivalence of the defining conditions of Γ' and the characterization (5.18), we have, as we foretold earlier, given a proof that (5.18) is indeed the analytical formulation of "in-the-acute-angle-of" betweenness.

6. Further examples

In all the examples thus far discussed we have illustrated Theorem 5.1 by taking a known betweenness relation and finding an associated R-relation. We shall now give a class of examples in which we proceed in the reverse direction. Let ν be a fixed number in the interval $(0, 1]$, and define the relation R between $\mathcal{K}$ and $\mathcal{K}$ as follows:

$$R(x, y) \equiv [(x, y) \geqq \nu \|x\| \cdot \|y\|] , \qquad x \in \mathcal{K},\ y \in \mathcal{K} . \tag{6.1}$$

Our motivation is clearly to generalize (5.15) and thereby to generalize $\mathcal{V}_0^v$-betweenness. Let us immediately give this new betweenness relation a name:

DEFINITION 6.1. The betweenness relation in $\mathcal{K}$ that derives according to Theorem 5.1 from the relation R defined by (6.1) will be called *$\mathcal{V}_\nu^v$-betweenness in $\mathcal{K}$*.

Now, we want to obtain a characterization of $\mathcal{V}_\nu^v$-betweenness in $\mathcal{K}$ corresponding to the characterization of $\mathcal{V}_0^v$-betweenness in Definition 5.2. The better way to this, however, is an indirect one, because the relation R of (6.1) is overly cumbersome to work with, involving, as it does, the norms of the elements x and y. To explain the procedure we have in mind, we shall first detail certain generalities.

We have already had, in Definition 5.2, an application of the following general definition:

DEFINITION 6.2. Let B be a betweenness relation in a space $\mathcal{X}$. If C is any particular subset of $\mathcal{X}$, the restricted application of B to point triplets $\langle x, y, z\rangle$ belonging only to C is a betweenness relation in C; it is called the *restriction of B to C*, or *B-betweenness in C*.

That the restricted application of B to C is, as asserted here, a betweenness relation in C is readily verified directly by Definition 2.1. By contrast, there is no subtlety at all to the fact that if R is a binary relation between $\mathcal{X}$ and Ω, then its restricted application to subsets $C\subseteq\mathcal{X}$ and $\Omega'\subseteq\Omega$ is a binary relation between C and Ω'; we shall call it the *restriction of R to $C\times\Omega'$*, or *the relation R in $C\times\Omega'$*. With these definitions of restrictions now laid down, the following lemma answers a natural question.

LEMMA 6.1. *Let B be a betweenness relation in $\mathcal{X}$; and let R be a relation between $\mathcal{X}$ and a space Ω, which gives a Theorem* 5.1 *representation of B. If C is any particular subset of $\mathcal{X}$, then the restriction of R to $C\times\Omega$ gives a Theorem* 5.1 *representation of B-betweenness in C.*

This result is due basically to the fact that the Theorem 5.1 relationship between B and R is a point-wise relationship. The detailed proof of the lemma is clear and we shall omit it here.

Lemma 6.1 deals with the contraction of a betweenness relation. We next give a result that enables us to extend, and more generally to transfer, a betweenness relation.

LEMMA 6.2. *Let t be a function on the space $\mathcal{Y}$ to the space $\mathcal{X}$, and let B be a betweenness relation in $\mathcal{X}$. Then the relation B_t on triplets of points $\langle x', y', z'\rangle$ in $\mathcal{Y}$, defined by*

$$B_t(x', y'; z')\equiv B(t(x'), t(y'); t(z')) , \tag{6.2}$$

is a betweenness relation in $\mathcal{Y}$.

This lemma is proved immediately by appealing to Definition 2.1. We can now go further, and again tie this procedure on B to a corresponding procedure on an associated relation R.

LEMMA 6.3. *Let $\mathcal{X}$, $\mathcal{Y}$, t and B be as in the preceding lemma. Let R be a relation between $\mathcal{X}$ and Ω that gives a Theorem* 5.1 *representation of B. Let R_t be the relation between $\mathcal{Y}$ and Ω defined by*

$$R_t(x', \omega)\equiv R(t(x'), \omega) , \qquad x'\in\mathcal{Y},\ \omega\in\Omega . \tag{6.3}$$

Then R_t gives a Theorem 5.1 *representation of the betweenness relation B_t.*

The proof of this statement is immediate from the definitions.

Let us return now to our discussion of $\mathcal{V}_{\nu}^{v}$-betweenness. Recalling that $\mathcal{O}_{\mathcal{K}}$ denotes the unit sphere in $\mathcal{K}$, we note first of all the following fact: *for given elements* x, y, z *in* $\mathcal{K}$, $(z, \omega)\geqq\nu\|z\|\cdot\|\omega\|$ *for every* $\omega\in\mathcal{K}$ *such that* $(x, \omega)\geqq\nu\|x\|\cdot\|\omega\|$ *and* $(y, \omega)\geqq\nu\|y\|\cdot\|\omega\|$ *if and only if* $(z, \omega)\geqq\nu\|z\|$ *for all* $\omega\in\mathcal{O}_{\mathcal{K}}$ *such that* $(x, \omega)\geqq\nu\|x\|$ *and* $(y, \omega)\geqq\nu\|y\|$. This statement is easily seen to hold true, and it tells us that also the relation

$$R(x, y)\equiv[(x, y)\geqq\nu\|x\|]\ , \qquad x\in\mathcal{K},\ y\in\mathcal{O}_{\mathcal{K}} \tag{6.4}$$

gives a Theorem 5.1 representation of $\mathcal{V}_{\nu}^{v}$-betweenness in $\mathcal{K}$. Now, by Lemma 6.1, the restriction of R of (6.4) to $\mathcal{O}_{\mathcal{K}}\times\mathcal{O}_{\mathcal{K}}$ will give us a Theorem 5.1 representation of $\mathcal{V}_{\nu}^{v}$-betweenness in $\mathcal{O}_{\mathcal{K}}$. Thus, the restriction of $\mathcal{V}_{\nu}^{v}$-betweenness to $\mathcal{O}_{\mathcal{K}}$ can be studied through the relation

$$R(x, y)\equiv[(x, y)\geqq\nu]\ , \qquad x\in\mathcal{O}_{\mathcal{K}},\ y\in\mathcal{O}_{\mathcal{K}}\ . \tag{6.5}$$

This is markedly advantageous over the study of either the relation (6.1) or the relation (6.4). But furthermore, the study of (6.5) will in fact accomplish the study of $\mathcal{V}_{\nu}^{v}$-betweenness over all of $\mathcal{K}$. We see this as follows. Let t be the function on $\mathcal{K}-\{\theta\}$ to $\mathcal{O}_{\mathcal{K}}$ defined by

$$t(x)=\frac{x}{\|x\|}\ . \tag{6.6}$$

Then the Lemma 6.2 extension, under this t, of $\mathcal{V}_{\nu}^{v}$-betweenness in $\mathcal{O}_{\mathcal{K}}$ to $\mathcal{K}-\{\theta\}$ has, by Lemma 6.3, a Theorem 5.1 representation in terms of the relation

$$\begin{aligned} R(x, y)&\equiv\left[\left(\frac{x}{\|x\|}, y\right)\geqq\nu\right] \\ &\equiv[(x, y)\geqq\nu\|x\|]\ , \qquad x\in\mathcal{K}-\{\theta\},\ y\in\mathcal{O}_{\mathcal{K}}\ . \end{aligned} \tag{6.7}$$

Through (6.4) we see that this relation generates the restriction of $\mathcal{V}_{\nu}^{v}$-betweenness to $\mathcal{K}-\{\theta\}$. Hence, by Lemma 6.3, the Lemma 6.2 extension, under (6.6), of $\mathcal{V}_{\nu}^{v}$-betweenness in $\mathcal{O}_{\mathcal{K}}$ is $\mathcal{V}_{\nu}^{v}$-betweenness in $\mathcal{K}-\{\theta\}$. Finally, it is seen without too much difficulty (see the discussion further below) by (6.4) that the $\mathcal{V}_{\nu}^{v}$-betweenness relations involving the element θ are:

$$\begin{cases} \qquad\qquad \mathcal{V}_{\nu}^{v}(x, y; \theta) & \text{for all } x,\ y\in\mathcal{K}\ ; \\ \qquad\qquad \mathcal{V}_{\nu}^{v}(\theta, \theta; z) & \text{only for } z=\theta\ ; \\ \text{for } x\neq\theta,\quad \mathcal{V}_{\nu}^{v}(x, \theta; z) & \text{if and only if } z=\alpha x,\ \alpha\geqq 0\ . \end{cases} \tag{6.8}$$

And these statements can be adjoined to a characterization of $\mathcal{V}_{\nu}^{v}$-betweenness in $\mathcal{K}-\{\theta\}$ to give a complete characterization of $\mathcal{V}_{\nu}^{v}$-betweenness in all of $\mathcal{K}$. In other words, we have shown the following: if some analytic characterization is found for $\mho_\nu^v$-betweenness in $\mathcal{O}_{\mathcal{K}}$, then the transformation (6.6) will convert that to a characterization of $\mho_\nu^v$-betweenness in $\mathcal{K}-\{\theta\}$, and finally the explicit adjunction to this (if necessary) of the statements (6.8) will provide a characterization of $\mho_\nu^v$-betweenness over all of $\mathcal{K}$. This is precisely the procedure we had in mind when we spoke, following Definition 6.1, of using an indirect way of arriving at a characterization of $\mho_\nu^v$-betweenness in $\mathcal{K}$. Thus, we shall undertake the inquiry into $\mho_\nu^v$-betweenness by studying first its restriction to $\mathcal{O}_{\mathcal{K}}$.

Before we do that let us say a word about (6.8). The first two assertions there are established very easily, and we omit their derivations here. To establish the third one, however, we require a somewhat more detailed argument. We must show that $(z, \omega)\geqq\nu\|z\|$ *for every* $\omega\in\mathcal{O}_{\mathcal{K}}$ *such that* $(x, \omega)\geqq\nu\|x\|$ *if and only if* $z=\alpha x$ *for some* $\alpha\geqq 0$. The sufficiency of this last condition is clear. We proceed to prove its necessity. If $z=\theta$ then the condition is satisfied (with $\alpha=0$). If $z\neq\theta$ let us suppose that z is not of the form αx for some $\alpha>0$. We can then present z in the form

$$z=\alpha x+y\,,\quad y\perp x\qquad \text{where } y\neq\theta \text{ if } \alpha\geqq 0\,. \tag{6.9}$$

We are now to show that for this z there is an $\omega\in\mathcal{O}_{\mathcal{K}}$ such that $(x, \omega)\geqq\nu\|x\|$ while $(z, \omega)<\nu\|z\|$. In fact, let us take the following ω_0:

$$\omega_0=\frac{|\alpha|x-\delta y}{\||\alpha|x-\delta y\|}\,, \tag{6.10}$$

where δ is, in the case of $y\neq\theta$, the unique positive number such that $(x, \omega_0)=\nu\|x\|$, that is,

$$\frac{|\alpha|\cdot\|x\|}{\||\alpha|x-\delta y\|}=\nu\,. \tag{6.11}$$

If $y=\theta$ and (therefore necessarily) $\alpha<0$, then ω_0 is simply $x/\|x\|$ and we see immediately that $(x, \omega_0)\equiv\|x\|\geqq\nu\|x\|$ while $(z, \omega_0)\equiv\alpha\|x\|<0<\nu\|z\|$. In the case of $y\neq\theta$ we proceed to examine

$$\begin{aligned}\frac{(z, \omega_0)}{\|z\|}&=\frac{(\alpha x+y, |\alpha|x-\delta y)}{\||\alpha|x-\delta y\|\sqrt{\alpha^2\|x\|^2+\|y\|^2}}\\ &=\frac{\alpha\|x\|-\delta\|y\|^2/(|\alpha|\cdot\|x\|)}{\||\alpha|x-\delta y\|\sqrt{1+\|y\|^2/(\alpha^2\|x\|^2)}}\,,\qquad \text{by (6.9)}\\ &=\nu\frac{\operatorname{sgn}\alpha-\delta(\|y\|/(\alpha\|x\|))^2}{\sqrt{1+(\|y\|/(\alpha\|x\|))^2}}\,,\qquad \text{by (6.11)}\,.\end{aligned} \tag{6.12}$$

We see that this last expression is indeed $<\nu$, and therefore our demonstration is complete; that is, the third assertion of (6.8), as well as the first two, is established.

We can now start into our study of ひ_ν^v-betweenness in $\mathcal{O}_{\mathcal{K}}$. We want to find a direct characterization of it. To this end we state a first theorem:

THEOREM 6.1. *Let x and y be elements of $\mathcal{O}_{\mathcal{K}}$, and let と denote the inner product (x, y). As postulated above, $\nu>0$.*

If the condition

$$\nu>\sqrt{\frac{1+\text{と}}{2}} \tag{6.13}$$

holds, then every $z\in\mathcal{O}_{\mathcal{K}}$ is ひ_ν^v-between x and y.

If, on the other hand, we have the inequality

$$\nu\leqq\sqrt{\frac{1+\text{と}}{2}} \tag{6.14}$$

holding, then the following statements describe the ひ_ν^v-betweenness in $\mathcal{O}_{\mathcal{K}}$:

(1) *if $\dim\mathcal{K}\leqq 2$, then $z\in\mathcal{O}_{\mathcal{K}}$ is ひ_ν^v-between x and y if and only if it is a non-negative linear combination of x and y;*

(2) *if $\dim\mathcal{K}\geqq 3$, then $z\in\mathcal{O}_{\mathcal{K}}$ is ひ_ν^v-between x and y if and only if*

$$\left(z, \frac{\nu}{1+\text{と}}(x+y)+w\right)\geqq\nu \tag{6.15}$$

for every $w\in\mathcal{K}$ satisfying the conditions

$$w\perp x, y\ , \qquad \|w\|^2=1-\frac{2\nu^2}{1+\text{と}}\ . \tag{6.16}$$

(The character と that has been introduced here is the Japanese hiragana *to*.) When the condition (6.13) holds, the two sets

$$\{\omega\in\mathcal{O}_{\mathcal{K}}\,|\,(x, \omega)\geqq\nu\}\ , \qquad \{\omega\in\mathcal{O}_{\mathcal{K}}\,|\,(y, \omega)\geqq\nu\}\ , \tag{6.17}$$

are non-overlapping, and the assertion of the theorem is thus a logical triviality. When (6.14) holds, the sets (6.17) do overlap, and in particular the assertion (1) of the theorem is then quite obvious geometrically and we can consider it proved. Assertion (2), on the other hand, is non-trivial and requires a detailed proof. Because of its length, we shall put that proof over until the second article in this series.

The present theorem does not yet give us, completely, the characterization of ひ_ν^v-betweenness that we desire. For this we must transform the assertion (2). Toward this objective, let us express an element $z\in\mathcal{O}_{\mathcal{K}}$ in the form

$$z=ax+by+u\,, \qquad u\perp x, y\,. \tag{6.18}$$

Of course, we have

$$a^2+b^2+2\xi ab+\|u\|^2=1\,. \tag{6.19}$$

In terms of this component form for z, (6.15) becomes

$$\nu(a+b)+(u, w)\geqq\nu\,. \tag{6.20}$$

Finally, for brevity let us set

$$\rho\overset{\text{def.}}{=}\sqrt{1-\frac{2\nu^2}{1+\xi}}\,. \tag{6.21}$$

Now, if (6.15) holds for our present z, and if $u\neq\theta$, then (6.20) will hold with the vector

$$-\frac{\rho}{\|u\|}u \tag{6.22}$$

substituted for w. This gives us the inequality

$$\rho\|u\|\leqq\nu[(a+b)-1]\,. \tag{6.23}$$

If $u=\theta$ then (6.20) tells us immediately that

$$a+b\geqq1\,, \tag{6.24}$$

and therefore (6.23) is verified also in this case. Hence, we see that (6.23) is a necessary condition for (6.15). But it is also sufficient. To see this, suppose w is any vector orthogonal to both x and y, and of norm ρ. Then

$$|(u, w)|\leqq\rho\|u\|\leqq\nu[(a+b)-1] \qquad \text{by (6.23)}\,. \tag{6.25}$$

Therefore

$$\nu(a+b)+(u, w)\geqq\nu(a+b)-\nu[(a+b)-1]=\nu\,; \tag{6.26}$$

that is, (6.20) holds. Since (6.20) is equivalent to (6.15), the sufficiency is hereby proved.

Let us observe that with (6.15) holding for our z, the coefficients a and b, in the case of $y\neq x$, are necessarily non-negative. We see this by noting first that (6.23) implies (6.24), and therefore we have

$$a^2+b^2+2ab\geqq1\,. \tag{6.27}$$

On the other hand, from (6.19) we have

$$a^2+b^2+2\xi ab\leqq1\,. \tag{6.28}$$

For the case $y \neq x$ we have $と < 1$, and so these last two inequalities imply that $ab \geqq 0$. This result taken together with (6.24) implies, as asserted, that a and b are both non-negative.

In the case of $y = x$ we have $と = 1$, so that (6.19) implies $|a+b| \leqq 1$, while (6.23) still implies $a+b \geqq 1$. It follows that $a+b=1$, and consequently also, by (6.19), $u=\theta$. Thus, in the case of $y=x$, the only $z \in \mathcal{O}_{\mathcal{K}}$ which is ひ$^{\nu}_{\nu}$-between x and x is x. This result is no surprise. It can be seen directly from the defining relation (6.5) through precisely the same detailed argument that was used to establish the third assertion in (6.8).

The necessary and sufficient condition (6.23) has been derived here for the case $\dim \mathcal{K} \geqq 3$. We shall now show that it is valid just as well in the cases $\dim \mathcal{K} \leqq 2$. Statement (1) in the above theorem describes explicitly ひ$^{\nu}_{\nu}$-betweenness in these two cases. Now, if z is a non-negative linear combination of x and y, then $u=\theta$. Therefore, by (6.19),

$$a+b \geqq \sqrt{a^2+b^2+2\,と\,ab} = 1 \,, \tag{6.29}$$

and so we see that (6.23) is verified. Thus, (6.23) is a necessary condition for ひ$^{\nu}_{\nu}$-betweenness in the cases $\dim \mathcal{K} \leqq 2$. To prove the sufficiency, consider first the case of $\dim \mathcal{K} = 2$. In this case, if $y \neq x$ then $u=\theta$. (6.23) gives (6.24), and the argument employed above applies to show that a and b are non-negative. If $y=x$ then again the reasoning exhibited above (in the preceding paragraph) applies to show that the only possible z is x itself, that is, a non-negative linear combination of x and x. In the case of $\dim \mathcal{K} = 1$, the only possible situation, by virtue of (6.14), is $y=x$, and once again the argument of the preceding paragraph applies to give the result. This completes the sufficiency proof.

We are now able, with these results, to state the characterizing theorem that we were after:

THEOREM 6.2. *Let x and y be elements of $\mathcal{O}_{\mathcal{K}}$, and let $と$ denote the inner product (x, y). Let $\nu > 0$ and ρ be defined by* (6.21).

If the condition

$$\nu > \sqrt{\frac{1+と}{2}} \tag{6.30}$$

holds, then every $z \in \mathcal{O}_{\mathcal{K}}$ is ひ$^{\nu}_{\nu}$-between x and y.

If, on the other hand, the inequality

$$\nu \leqq \sqrt{\frac{1+と}{2}} \tag{6.31}$$

holds, then a necessary and sufficient condition that $z \in \mathcal{O}_{\mathcal{K}}$ *be* $ひ^{v}_{\nu}$*-between* x *and* y *is that* z *be of the form*

$$z = ax + by + u\,, \qquad u \perp x, y \tag{6.32}$$

with the following condition fulfilled:

$$\rho\|u\| \leqq \nu[(a+b)-1]\,. \tag{6.33}$$

When, under (6.31), $z \in \mathcal{O}_{\mathcal{K}}$ *is* $ひ^{v}_{\nu}$*-between* x *and* y, *with* $y \neq x$, *the coefficients* a *and* b *are necessarily non-negative. If* $y = x$ *then* $a+b=1$ *and* $u=\theta$; *that is,* x *is the only element of* $\mathcal{O}_{\mathcal{K}}$ *that is* $ひ^{v}_{\nu}$*-between* x *and* x.

We now go on to carry out the program we outlined earlier for obtaining the characterization of $ひ^{v}_{\nu}$-betweenness in all of $\mathcal{K}$ from its characterization—in this last theorem—in $\mathcal{O}_{\mathcal{K}}$. For this purpose, if x and y are any elements of $\mathcal{K}$ we define

$$と(x, y) \overset{\text{def.}}{=} \begin{cases} \dfrac{(x, y)}{\|x\| \cdot \|y\|} & \text{if } x \neq \theta \text{ and } y \neq \theta\,, \\ 1 & \text{if either } x = \theta \text{ or } y = \theta\,, \end{cases} \tag{6.34}$$

and

$$\rho(x, y) \overset{\text{def.}}{=} \sqrt{1 - \frac{2\nu^2}{1 + と(x, y)}} \qquad \text{in the case } \frac{1 + と(x, y)}{2} \geqq \nu^2\,. \tag{6.35}$$

We consider first the case of all three elements, x, y and z, of $\mathcal{K}$ non-vanishing. According to our earlier discussion, z is $ひ^{v}_{\nu}$-between x and y if and only if $z/\|z\|$ is $ひ^{v}_{\nu}$-between $x/\|x\|$ and $y/\|y\|$ in $\mathcal{O}_{\mathcal{K}}$. Thus, if

$$\nu > \sqrt{\frac{1 + と(x, y)}{2}} \tag{6.36}$$

then z is $ひ^{v}_{\nu}$-between x and y for every z in $\mathcal{K} - \{\theta\}$. If, on the other hand,

$$\nu \leqq \sqrt{\frac{1 + と(x, y)}{2}} \tag{6.37}$$

then z is $ひ^{v}_{\nu}$-between x and y if and only if

$$\frac{z}{\|z\|} = a\frac{x}{\|x\|} + b\frac{y}{\|y\|} + u\,, \qquad u \perp x, y\,, \tag{6.38}$$

with a, b and u satisfying (6.33). On making the substitutions

$$a = \frac{\|x\|}{\|z\|}\alpha\,, \qquad b = \frac{\|y\|}{\|z\|}\beta\,, \qquad u = \frac{1}{\|z\|}v\,, \tag{6.39}$$

we see that our statement becomes the following: z is $\mho^{v}_{\nu}$-between x and y if and only if

$$z=\alpha x+\beta y+v\ , \qquad v\perp x, y\ , \tag{6.40}$$

with α, β and v satisfying the condition

$$\rho(x, y)\cdot\|v\|\leqq\nu[\alpha\|x\|+\beta\|y\|-\sqrt{\|\alpha x+\beta y\|^2+\|v\|^2}]\ . \tag{6.41}$$

By (6.39) we see also—according to the preceding theorem—that if $x/\|x\|$ and $y/\|y\|$ are unequal, that is, if y is not a positive multiple of x, then α and β are necessarily non-negative. And if y is a positive multiple, γ, of x, then $\alpha+\gamma\beta>0$ and $v=\theta$; that is, only the positive multiples of x in $\mathcal{K}-\{\theta\}$ are $\mho^{v}_{\nu}$-between x and x.

What we have now obtained, up to this point, is a characterization of $\mho^{v}_{\nu}$-betweenness in $\mathcal{K}-\{\theta\}$. But the fact is that the form of the conditions just obtained provides a characterization for all of $\mathcal{K}$. In other words, there is actually no need to adjoin explicitly to these conditions the statement (6.8): that statement is already inherent in these conditions. We shall prove this after first stating the full characterization theorem.

THEOREM 6.3. *Let x and y be elements of $\mathcal{K}$. Let $\nu>0$, and let $\natural(x, y)$ and $\rho(x, y)$ be defined by* (6.34) *and* (6.35), *respectively.*

If the condition

$$\nu>\sqrt{\frac{1+\natural(x, y)}{2}} \tag{6.42}$$

holds, then every $z\in\mathcal{K}$ is $\mho^{v}_{\nu}$-between x and y.

If, on the other hand, the inequality

$$\nu\leqq\sqrt{\frac{1+\natural(x, y)}{2}} \tag{6.43}$$

holds, then a necessary and sufficient condition that $z\in\mathcal{K}$ be $\mho^{v}_{\nu}$-between x and y is that z be of the form

$$z=\alpha x+\beta y+v\ , \qquad v\perp x, y\ , \tag{6.44}$$

with α, β and v satisfying the condition

$$\rho(x, y)\cdot\|v\|\leqq\nu[\alpha\|x\|+\beta\|y\|-\sqrt{\|\alpha x+\beta y\|^2+\|v\|^2}]\ . \tag{6.45}$$

When, under (6.43), *z is $\mho^{v}_{\nu}$-between x and y, with x and y both $\neq\theta$ and one not a positive multiple of the other, then the coefficients α and β are necessarily non-negative. If x is $\neq\theta$ and y is a positive multiple, γ, of x, then $\alpha+\gamma\beta\geqq0$ and $v=\theta$; that is, only the non-negative multiples* *of x are ひ_ν^v-between x and x.*

To see that this theorem covers the case of one or more of x, y and z vanishing—as well as the case of all three of these vectors in $\mathcal{K}-\{\theta\}$—let us first consider the situation of $z=\theta$ while x and y are non-θ. If (6.42) holds, then the theorem asserts that θ is between x and y. According to the first assertion in (6.8) this is correct. If (6.43) holds, we see that $\alpha x+\beta y$ and v in (6.44) are vanishing and that, on that account, (6.45) is satisfied. Therefore, the theorem asserts again in this case that θ is between x and y. And again this statement is correct according to (6.8). Notice that when (6.43) holds then necessarily $\text{と}(x, y)>-1$. Therefore if x and y are not in the relation of one being a positive multiple of the other, then they are linearly independent. Hence, for $z=\theta$ both α and β vanish. Thus, the first of the last two statements of the theorem is seen to be valid in the present case. If y is a positive multiple, γ, of x, then for $z=\theta$ we have $\alpha+\gamma\beta=0$ and $v=\theta$; and so also the last statement of the theorem is valid.

We consider next the case of $x=y=\theta$. We begin by noticing that when either x or y is θ then $\text{と}(x, y)=1$ and therefore we have (6.43) holding. Thus, in the present case we put $x=y=\theta$ in (6.45), and we find the implication that $v=\theta$. That is, according to the theorem, θ is the only element that is ひ_ν^v-between θ and θ. By (6.8) we see that this is the correct statement.

Finally, we look at the case of $x\neq\theta$ and $y=\theta$. Setting $y=\theta$ in (6.45) we see that the right-hand side is negative unless $v=\theta$ and $\alpha\geqq 0$. Thus, the theorem asserts that in the present case only the nonnegative multiples of x are ひ_ν^v-between x and y. Again, (6.8) shows this to be the correct statement.

This completes the proof of Theorem 6.3.

7. Concerning the past literature on "betweenness"

The classical literature on "betweenness" uses this term more generally than we are using it here. This came about in the following way. Huntington and Kline [9] presented a very thorough postulational analysis of the elementary case of betweenness for points on the real line. This simple case has many formally distinct properties of the following general type: a collection of specific instances of betweenness among triplets of the points of a given set of k points implies another specific instance of betweenness among some triplet of these k points. Among the twelve postulates that Huntington and Kline study there are eight which are precisely such properties for a 4-point set.

Such a property came to be called a *transitivity*. When subsequent authors began to give attention to triplet relations in other spaces than the real line—triplet relations that conform to the general intent (—if not to the letter—) of the other four postulates of Huntington and Kline—they employed the term "betweenness" quite freely: the prevailing spirit was not "What transitivity shall be singled out as the basis for a general definition of betweenness?" but rather "What transitivities does this or this 'betweenness' have?"

Pitcher and Smiley [12] continued the Huntington and Kline study on the real line by examining also transitivites on a 5-point set, of which there are thirty-eight. Our *hereditary property*, condition (iii) of Definition 2.1 above, is the transitivity T_5 in the Pitcher and Smiley list of thirty-eight. This property seems to us to lay reasonably exclusive claim to the privileged position as the basis of a firm general definition of *betweenness*.

It is instructive to examine some of the classical examples of "betweenness" from the point of view of betweenness; that is, to see if or when these general triplet relations are betweenness relations according to our Definition 2.1. For this discussion we shall introduce a terminological device to assure greater ease and clarity of expression: the rather liberal classical idea of "betweenness" we shall call *betwixtness*. Thus, we are going to scrutinize various cases of betwixtness to see which of them are cases of betweenness.

Smiley [14] pursues his betwixtness studies by inquiring into the relationships among three particular kinds of betwixtness. These are the notions that we shall look at here. They are defined as follows. If $\mathscr{X}$ is a real vector space, the point z is said to be *algebraically betwixt* the points x and y if

$$z=\alpha x+(1-\alpha)y\ , \qquad \alpha\in[0,1]\,. \tag{7.1}$$

If $\mathscr{X}$ is a semimetric space, with semimetric δ, the point z is said to be *metrically betwixt* the points x and y if

$$\delta(x,y)=\delta(x,z)+\delta(z,y)\ . \tag{7.2}$$

Finally, if $\mathscr{X}$ is a lattice, the point z is said to be *lattice betwixt* the points x and y if

$$(x\wedge z)\vee(z\wedge y)=z=(x\vee z)\wedge(z\vee y)\ . \tag{7.3}$$

It is immediately clear that algebraic betwixtness is a betweenness relation. Theorem 1 in Smiley [14] asserts that if $\mathscr{X}$ is a seminormed real vector space then it is strictly convex if and only if algebraic and metric betwixtness coincide in $\mathscr{X}$. (The semimetric is understood to be the seminorm difference.) We can therefore state that if $\mathcal{X}$ is a strictly convex, seminormed real vector space then metric betwixtness is a betweenness relation. Another case in which this is so is that of $\mathcal{X}$ a metric ptolemaic space; this fact is proved in Pitcher and Smiley [12]. (A metric space is said to be *ptolemaic* if, for every tetrahedron, the three products of the lengths of opposite edges are the sides of some euclidean triangle; that is, they satisfy the triangle inequalities.) There are still other, distinct cases in which metric betwixtness is a betweenness. We shall present two such examples.

Consider $\mathcal{X}$ to be the normed real vector space of ordered pairs of real numbers, $\langle a, b\rangle$, with norm defined by $\|\langle a, b\rangle\|=\max[|a|, |b|]$. We may immediately check that this space is neither strictly convex nor ptolemaic. The two particular points $x=\langle 2, 1\rangle$ and $y=\langle 1, 1\rangle$ satisfy $\|x+y\|=\|x\|+\|y\|$, but they are not linearly dependent; thus, $\mathcal{X}$ is not strictly convex. If we take the tetrahedral vertices $x=\langle 0, 0\rangle$, $y=\langle 2, -4\rangle$, $z=\langle 0, 4\rangle$ and $w=\langle 9, -1\rangle$, then, with δ denoting the norm difference metric (that is, $\delta(x, y)=\|\langle x_1, x_2\rangle-\langle y_1, y_2\rangle\|=\max[|x_1-y_1|, |x_2-y_2|]$, where $x=\langle x_1, x_2\rangle$ and $y=\langle y_1, y_2\rangle$), we have

$$(7.4)\quad \delta(x, y)\cdot\delta(z, w)=36\ , \quad \delta(x, z)\cdot\delta(y, w)=28\ , \quad \delta(x, w)\cdot\delta(y, z)=72\ ,$$

so that

$$(7.5)\qquad \delta(x, w)\cdot\delta(y, z)>\delta(x, y)\cdot\delta(z, w)+\delta(x, z)\cdot\delta(y, w)\ ;$$

and this shows that $\mathcal{X}$ is not ptolemaic. Now, to show that metric betwixtness in this present $\mathcal{X}$ is a betweenness relation, we may proceed most effectively by giving a characterization of the set of points z satisfying (7.2) for given x and y. To this end we shall employ a rotated coordinate system in $\mathcal{X}$; that is, a point of $\mathcal{X}$ will be denoted by $\langle p_1+p_2, p_1-p_2\rangle$. Notice that

$$(7.6)\qquad \|\langle p_1+p_2, p_1-p_2\rangle\|=\max[|p_1+p_2|, |p_1-p_2|]=|p_1|+|p_2|\ .$$

If we set $x=\langle p_1+p_2, p_1-p_2\rangle$, $y=\langle q_1+q_2, q_1-q_2\rangle$ and $z=\langle r_1+r_2, r_1-r_2\rangle$, then the equation (7.2) becomes

$$(7.7)\qquad |p_1-q_1|+|p_2-q_2|=|p_1-r_1|+|p_2-r_2|+|r_1-q_1|+|r_2-q_2|\ .$$

By virtue of the obvious inequalities that hold among the terms in this equation, the fact of the equality gives us two equations:

$$(7.8)\qquad |p_1-q_1|=|p_1-r_1|+|r_1-q_1|\ , \qquad |p_2-q_2|=|p_2-r_2|+|r_2-q_2|\ .$$

And these equations state the following: the number r_1 is between the numbers p_1 and q_1, and the number r_2 is between the numbers p_2 and q_2. This, then, is the characterization of the z's that satisfy (7.2). Geometrically it is seen that this set of z's is just the 45°-inclined rectangle that has the line segment from x to y as one of its diagonals. From either this geometric statement or the analytical characterization out of (7.8) it is now quite clear that this metric betwixtness fulfills our Definition 2.1 and so is a betweenness relation.

Our second example comes from Schoenberg [13]. Let $\mathcal{X}$ be the metric space of the numbers in the closed interval $[-1, 1]$ with metric

$$\delta(x, y)=\begin{cases} |x-y| & \text{if } xy\geqq 0\,, \\ |x|+|y|-x^2y^2 & \text{if } xy<0\,. \end{cases} \tag{7.9}$$

That this space is not ptolemaic is seen by

$$\delta\left(-1, \frac{1}{2}\right)\delta\left(-\frac{1}{2}, 1\right)=\frac{25}{16}>\frac{19}{16} \tag{7.10}$$
$$=\delta\left(-1, -\frac{1}{2}\right)\delta\left(\frac{1}{2}, 1\right)+\delta(-1, 1)\delta\left(-\frac{1}{2}, \frac{1}{2}\right)\,.$$

It is not difficult to show that (7.2) is satisfied if and only if either (i) $xy\geqq 0$ and the number z is between the numbers x and y, or (ii) $xy<0$ and z is either x or y. This result makes it evident that metric betwixtness in this space is a betweenness relation.

It is of interest to see how this betweenness may be represented according to Theorem 5.1 with a relation R of membership form (see (5.8)). If Ω of (5.8) is taken to be the class of all closed sub-intervals of $\mathcal{X}=[-1, 1]$, then the resulting betweenness relation is ordinary linear betweenness in this interval. But if Ω is enlarged to include also all two-point subsets of $\mathcal{X}$ in which the two points are numbers of opposite sign, then the betweenness generated is precisely the present metric betwixtness.

Metric betwixtness is not always a betweenness relation. Menger [10] presented his example of a "railroad" space to demonstrate this fact. This space consists of five points, u, v, x, y, z, and the metrization is as follows:

$$\begin{cases} \delta(u, x)=\delta(v, y)=2\,, \\ \delta(x, z)=\delta(y, z)=3\,, \\ \delta(u, z)=\delta(v, z)=5\,, \\ \delta(u, y)=\delta(v, x)=\delta(x, y)=6\,, \\ \delta(u, v)=8\,. \end{cases} \tag{7.11}$$

It can be verified that z is not metrically betwixt u and v, while x and y are both betwixt u and v, and z is betwixt x and y. Thus, the hereditary property does not hold. We may go on to observe that this 5-point space can be isometrically imbedded in the 3-dimensional real vector-space with norm $\|\langle a, b, c\rangle\| = \max[|a|, |b|, |c|]$. Specifically, the following identification can be made:

$$
(7.12) \qquad \begin{cases} u=\langle 0,0,0\rangle\,, & v=\langle 8,6,5\rangle\,, \quad x=\langle 2,0,2\rangle\,, \\ y=\langle 6,6,5\rangle\,, & z=\langle 3,3,5\rangle\,. \end{cases}
$$

Obviously an isometric imbedding can be made as well in the max-norm vector-space of any dimension greater than 3. Thus, we can state this: in the max-norm real vector spaces $\mathcal{X}$, metric betwixtness is a betweenness relation for $\dim \mathcal{X}=1$ or 2, and is not a betweenness relation for $\dim \mathcal{X} \geqq 3$.

Finally, with respect to lattice betwixtness, Lemma 8.2 and Theorem 9.3 of Pitcher and Smiley [12] state the whole story, namely: lattice betwixtness is a betweenness relation if and only if the lattice is distributive. (Their Lemma 8.2 establishes their postulates α and β, which imply that our (2.1)-(i) and (ii) are fulfilled; and their Theorem 9.3 proves the hereditary property.)

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