Ann. Inst. Statist. Math.
30 (1978), Part A, 115-123

REDUCTION OF THE NUMBER OF ASSOCIATE CLASSES OF HYPERCUBIC ASSOCIATION SCHEMES

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(Received Dec. 16, 1977)

Abstract

The reduction of the number of associate classes of some hypercubic association schemes by clubbing certain associate classes has been studied in the paper. It has been found that the reduction of an m-class hypercubic association scheme for $v=2^m$ treatments into a 2-class association scheme is always possible. Further it is proved herein that the m-class hypercubic association scheme for $v=s^m$ treatments is reducible (i) to a 3-class association scheme, when $s=3$ and (ii) to a 2-class association scheme, when $s=4$, which really has $p_{11}^1=p_{11}^2$ and hence leads to a series of balanced incomplete block designs.

1. Introduction

If from an m-class association scheme A, another $m'\,(<m)$-class association scheme is constructible by clubbing one or more associate classes of A, we shall say A is a reducible association scheme and A' is the corresponding reduced association scheme. Partially balanced incomplete block (PBIB) designs based on a reducible association scheme A can, then, be better considered as based on A' with $m'\,(<m)$ associate classes, when the coincidence numbers (λ_i's) of the corresponding clubbed associate classes of A are equal. Reduction of associate classes of various association schemes is considered by Kageyama [2], [3], [4] in a series of papers, a survey of which is available in Kageyama [5]. Some necessary and sufficient conditions of PBIB designs of m-associate classes to be reducible to $m'<m$ associate classes are obtained by him. Kageyama [3] conjectures that a necessary and sufficient condition for the m-class hypercubic association scheme of $v=s^m$ treatments of Shah [9] and Kusumoto [6] to be reducible is that $s=2$, 3 or 4 and he finds that his conjecture holds true for $m=3$, 4 and 5. In this paper we have

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established that by clubbing certain associate classes, the m-class hypercubic association scheme for $v=2^m$ treatments, is always reducible to a 2-class association scheme. Further, it is proved that the m-class hypercubic association scheme of $v=s^m$ is reducible (i) to a 3-class association scheme, when $s=3$ and (ii) to a 2-class association scheme, when $s=4$ which really has $p_{11}^1=p_{11}^2$ and hence leads to a series of balanced incomplete block designs. Throughout this paper "treatment" has been written for "treatment combination".

2. Hypercubic association scheme

For the $v=s^m$ treatments $(x_1, x_2, \cdots, x_m)$, $(x_i=0, 1, 2, \cdots, s-1)$, $(i=1, 2, \cdots, m)$, Shah [9] introduced an m-class association scheme which was later called an m-class hypercubic association scheme by Kusumoto [6], by treating two treatments as ith associates $(i=1, 2, \cdots, m)$, if the levels of $(m-i)$ factors in their representation are identical but the levels of the remaining i factors are different in their representation. The parameters of this association scheme are as follows:

$$v=s^m, \qquad n_i=\binom{m}{i}(s-1)^i, \quad i=1, 2, \cdots, m,$$

$$p_{jk}^i=\sum_u \binom{m-i}{m-u}\binom{i}{u-j}\binom{i+j-u}{u-k}(s-1)^{u-i}(s-2)^{i+j+k-2u}$$

where $\sum_u$ is the summation over all the values of u such that $\max\{i, j, k\} \leqq u \leqq (1/2)(i+j+k)$. Otherwise, $p_{jk}^i=0$.

An interesting feature of the m-class hypercubic association scheme of $v=s^m$ treatments is that if the association scheme of the treatment $(0, 0, \cdots, 0)$ is obtained, i.e., if the arrangement of all the other treatments into different associate classes against $(0, 0, \cdots, 0)$ is obtained, the association scheme of any other treatment, say, α, is directly obtained by adding $\alpha \pmod s$ to the different associates of treatment $(0, 0, \cdots, 0)$.

3. Reduction of associate classes

In this section, we shall consider the reduction of associate classes of the m-class hypercubic association scheme for $v=s^m$ treatments, where $s=2$, 3, and 4. The three different cases are dealt with separately in the following. The reduced schemes may be called folded hypercubic association schemes.

3.1. *Case* I. $v=2^m$ *treatments*

THEOREM 1. *An m-class hypercubic association scheme for $v=2^m$*

treatments can always be reduced to a 2-class association scheme, say A_2, where A_2 is defined as follows. Call a treatment with an even (odd) number of level 1 *an even (odd)-lettered treatment. The two associate classes of* $(0, 0, \cdots, 0)$, *under A_2, the consist of all the even- and odd-lettered treatments; and those of any treatment β are obtained by adding β* (mod 2) *to the associate classes of* $(0, 0, \cdots, 0)$. *The scheme A_2 has the following parameters*:

$$v=2^m, \qquad n_1=2^{m-1}, \qquad n_2=2^{m-1}-1;$$

$$P_1=\begin{bmatrix} 0 & 2^{m-1}-1 \\ 2^{m-1}-1 & 0 \end{bmatrix}, \qquad P_2=\begin{bmatrix} 2^{m-1} & 0 \\ 0 & 2^{m-1}-2 \end{bmatrix}.$$

PROOF. Let us consider all the odd-lettered treatments of 2^m factorials as constituting the first associate class and the even-lettered treatments as the second associate class of the treatment $(0, 0, \cdots, 0)$. Then evidently,

$$n_1=\binom{m}{1}+\binom{m}{3}+\binom{m}{5}+\cdots=2^{m-1}, \qquad \text{and}$$

$$n_2=\binom{m}{2}+\binom{m}{4}+\binom{m}{6}+\cdots=2^{m-1}-1.$$

As in the case of an m-class hypercubic association scheme, here also we notice that if the arrangement of all other treatments into two associate classes against the treatment $(0, 0, \cdots, 0)$ is obtained, such arrangement (associate classes) for any other treatment, say, β, is directly obtained by just adding β (mod 2) to the associate classes of the treatment $(0, 0, \cdots, 0)$. Let a treatment α be the first associate of $(0, 0, \cdots, 0)$, then it must be a treatment containing an odd number of non-zero elements. Therefore, by virtue of the above property, the first associates of α are all the treatments having an even number of non-zero elements. Hence, in this case p_{11}^1 (=the number of treatments common between the first associates of $(0, 0, \cdots, 0)$ and of α, when they are mutually first associates)$=0$. Similarly, $p_{12}^1=p_{21}^1=2^{m-1}-1$ and $p_{22}^1=0$.

Following a similar argument, by considering the treatment α to be a second associate of $(0, 0, \cdots, 0)$, it can be easily shown that $p_{11}^2=2^{m-1}$, $p_{12}^2=p_{21}^2=0$ and $p_{22}^2=2^{m-1}-2$. Hence the theorem.

In this connection we may draw the attention of the reader to the reduction results of the hypercubic association schemes for 2^m treatments, due to Saha and Das [8]. In essence they establish that an m-class hypercubic association scheme for 2^m treatments can always be reduced to $m/2$ and $(m+2)/2$, or $(m+1)/2$-class association schemes, accordingly as m is even or odd.

3.2. *Case* II. $v=3^m$ *treatments*

THEOREM 2. *An m-class hypercubic association scheme for* 3^m *treatments can always be reduced to a 3-class association scheme. The ith associate class of the treatment* $(0, 0, \cdots, 0)$ *of the reduced 3-class association scheme is obtained by clubbing the* $(j+1)$*th associate classes of the m-class hypercubic association scheme where the j's satisfy*: $j \equiv (i-1)$, $\bmod 3$, *for* $i=1, 2, 3$ *and* $j=0, 1, 2, \cdots, (m-1)$. *The parameters of the reduced association scheme are as follows*:

When m is odd,

$$n_1=3^{m-1}+(-1)^{(m+3)/2}3^{(m-1)/2}, \qquad n_2=3^{m-1}+(-1)^{(m+1)/2}3^{(m-1)/2},$$
$$n_3=3^{m-1}-1,$$

$$P_1=\begin{bmatrix} 3^{m-2}+(-1)^{(m-1)/2}\cdot 2\cdot 3^{(m-3)/2}, & 3^{m-2}+(-1)^{(m+1)/2}3^{(m-3)/2}, & 3^{m-2}-1+2(-1)^{(m+3)/2} \\ , & 3^{m-2}+(-1)^{(m+1)/2}3^{(m-3)/2}, & 3^{m-2}+(-1)^{(m+1)/2}3^{(m-3)/2} \\ \mathit{Sym.} & , & 3^{m-2}+(-1)^{(m+1)/2}3^{(m-3)/2} \end{bmatrix},$$

$$P_2=\begin{bmatrix} 3^{m-2}+(-1)^{(m-1)/2}3^{(m-3)/2}, & 3^{m-2}+(-1)^{(m-1)/2}3^{(m-3)/2}, & 3^{m-2}+(-1)^{(m-1)/2}3^{(m-3)/2} \\ , & 3^{m-2}+(-1)^{(m+1)/2}\cdot 2\cdot 3^{(m-3)/2}, & 3^{m-2}-1+(-1)^{(m+1)/2}\cdot 2\cdot 3^{(m-3)/2} \\ \mathit{Sym.} & , & 3^{m-2}+(-1)^{(m-1)/2}3^{(m-3)/2} \end{bmatrix},$$

$$P_3=\begin{bmatrix} 3^{m-2}+(-1)^{(m-1)/2}3^{(m-1)/2}, & 3^{m-2}, & 3^{m-2} \\ , & 3^{m-2}+(-1)^{(m-3)/2}3^{(m-1)/2}, & 3^{m-2} \\ \mathit{Sym.} & , & 3^{m-2}-2 \end{bmatrix}.$$

When m is even

$$n_1=3^{m-1}+(-1)^{(m+2)/2}3^{(m-2)/2}=n_2, \qquad n_3=3^{m-1}+(-1)^{m/2}\cdot 2\cdot 3^{(m-2)/2}-1,$$

$$P_1=\begin{bmatrix} 3^{m-2}, & 3^{m-2}+(-1)^{(m+2)/2}3^{(m-2)/2}, & 3^{m-2}-1 \\ , & 3^{m-2}+(-1)^{(m+2)/2}3^{(m-2)/2}, & 3^{m-2}+(-1)^{m/2}3^{(m-2)/2} \\ \mathit{Sym.} & , & 3^{m-2}+(-1)^{m/2}3^{(m-2)/2} \end{bmatrix},$$

$$P_2=\begin{bmatrix} 3^{m-2}+(-1)^{(m-2)/2}3^{(m-2)/2}, & 3^{m-2}+(-1)^{(m-2)/2}3^{(m-2)/2}, & 3^{m-2}+(-1)^{m/2}3^{(m-2)/2} \\ & , \ 3^{m-2}, & 3^{m-2}-1 \\ Sym. & , & 3^{m-2}+(-1)^{m/2}3^{(m-2)/2} \end{bmatrix},$$

$$P_3=\begin{bmatrix} 3^{m-2}+(-1)^{(m-2)/2}3^{(m-2)/2}, & 3^{m-2}, & 3^{m-2} \\ & , \ 3^{m-2}+(-1)^{(m-2)/2}3^{(m-2)/2}, & 3^{m-2} \\ Sym. & , & 3^{m-2}-2+(-1)^{m/2}\cdot 2\cdot 3^{(m-2)/2} \end{bmatrix}.$$

PROOF. Let $A_0\,(=I_v)$, $A_1, A_2, \cdots, A_m$ be association matrices of an m-class hypercubic association scheme of $v=3^m$ treatments, i.e., $A_i=\|a^{\beta}_{\alpha i}\|$ and $a^{\beta}_{\alpha i}=1$ or 0 accordingly as the αth treatment and the βth treatment are ith associates or not $(i=0, 1, \cdots, m)$. We now define

$$(3.2.1)\qquad B_1=\sum_{i=0}^{[(m-1)/3]} A_{3i+1}\,, \qquad B_2=\sum_{i=0}^{[(m-2)/3]} A_{3i+2}\,, \qquad B_3=\sum_{i=0}^{[(m-3)/3]} A_{3i+3}$$

which will be later considered as new association matrices of the reduced 3-class association scheme required, where $[x]$ stands for the greatest integer of x. Then each treatment has exactly n_i ith associates $(i=1, 2, 3)$, where

$$n_1=\sum_{i=0}^{[(m-1)/3]}\binom{m}{3i+1}2^{3i+1}\,, \qquad n_2=\sum_{i=0}^{[(m-2)/3]}\binom{m}{3i+2}2^{3i+2}\,,$$
$$n_3=\sum_{i=0}^{[(m-3)/3]}\binom{m}{3i+3}2^{3i+3}\,,$$

the values of which can be shown to be (see Riordan [7], pp. 135, 161):

$$n_1=\begin{cases} 3^{m-1}+(-1)^{(m+3)/2}3^{(m-1)/2}\,, & \text{if } m \text{ is odd}\,, \\ 3^{m-1}+(-1)^{(m+2)/2}3^{(m-2)/2}\,, & \text{if } m \text{ is even}\,. \end{cases}$$

$$(3.2.2)\qquad n_2=\begin{cases} 3^{m-1}+(-1)^{(m+1)/2}3^{(m-1)/2}\,, & \text{if } m \text{ is odd}\,, \\ 3^{m-1}+(-1)^{(m+2)/2}3^{(m-2)/2}\,, & \text{if } m \text{ is even}\,. \end{cases}$$

$$n_3=\begin{cases} 3^{m-1}-1\,, & \text{if } m \text{ is odd}\,, \\ 3^{m-1}+(-1)^{m/2}\cdot 2\cdot 3^{(m-2)/2}-1\,, & \text{if } m \text{ is even}\,. \end{cases}$$

For the parameters of the second kind of the reduced association scheme, from (3.2.1), (3.2.2) and argument similar to Bose and Clatworthy [1], it is sufficient to show that for any pair of the v treatments which are ith associates, the numbers, p^i_{11}, p^i_{12}, p^i_{21}, p^i_{22} for $i=1, 2, 3$, are independent of the pair of treatments with which we start, and $p^i_{12}=p^i_{21}$. From relation (3.2.2) and some calculation, the matrix forms of B_1B_1, B_2B_2, B_1B_2 and B_2B_1, and the linearly independence of the original association matrices $A_0, A_1, \cdots, A_m$, yield that

$$p^1_{11}=\sum_{u=0}^{[(m-1)/3]}\binom{m-1}{3u}2^{3u}=\begin{cases}[3^{m-1}+2(-3)^{(m-1)/2}]/3 & \text{if } m \text{ is odd},\\ 3^{m-2} & \text{if } m \text{ is even};\end{cases}$$

$$p^1_{12}=p^1_{22}=\begin{cases}[3^{m-1}-(-3)^{m/2}]/3 \\ [3^{m-1}-(-3)^{(m-1)/2}]/3\end{cases}\left(=\sum_{u=0}^{[(m-2)/3]}\binom{m-1}{3u+1}2^{3u+1}\right)\begin{matrix}\text{if } m \text{ is even},\\ \text{if } m \text{ is odd};\end{matrix}$$

$$\begin{aligned}p^2_{11}&=\sum_{j=0}^{[(m-4)/3]}\binom{m-2}{3j+2}2^{3j+2}+2\sum_{j=0}^{[(m-5)/3]}\binom{m-2}{3j+3}2^{3j+3}+2\\ &=\begin{cases}3^{m-2}+(-1)^{(m-1)/2}3^{(m-3)/2} & \text{if } m \text{ is odd},\\ 3^{m-2}+(-1)^{(m-2)/2}3^{(m-2)/2} & \text{if } m \text{ is even};\end{cases}\\ &=p^2_{12};\end{aligned}$$

$$\begin{aligned}p^2_{22}&=\sum_{j=0}^{[(m-2)/3]}\binom{m-2}{3j}2^{3j}+2\sum_{j=0}^{[(m-3)/3]}\binom{m-2}{3j+1}2^{3j+1}\\ &=\begin{cases}3^{m-2}+(-1)^{(m+1)/2}\cdot 2\cdot 3^{(m-3)/2} & \text{if } m \text{ is odd},\\ 3^{m-2} & \text{if } m \text{ is even};\end{cases}\end{aligned}$$

$$\begin{aligned}p^3_{11}&=2\sum_{j=0}^{[(m-4)/3]}\binom{m-3}{3j+1}2^{3j+1}+6\sum_{j=0}^{[(m-5)/3]}\binom{m-3}{3j+2}2^{3j+2}+\sum_{u=0}^{[(m-4)/3]}\binom{m-3}{3u+1}2^{3u+1}\\ &=\begin{cases}3^{m-2}+(-3)^{(m-1)/2} & \text{if } m \text{ is odd},\\ 3^{m-2}+(-3)^{(m-2)/2} & \text{if } m \text{ is even};\end{cases}\end{aligned}$$

$$\begin{aligned}p^3_{12}&=3\sum_{u=0}^{[(m-4)/3]}\binom{m-3}{3u+1}2^{3u+1}+3\sum_{u=0}^{[(m-5)/3]}\binom{m-3}{3u+2}2^{3u+2}+3\sum_{u=0}^{[(m-6)/3]}\binom{m-3}{3u+3}2^{3u+3}+3\\ &=\begin{cases}3^{m-2} & \text{if } m \text{ is odd},\\ 3^{m-2} & \text{if } m \text{ is even};\end{cases}\end{aligned}$$

$$\begin{aligned}p^3_{22}&=2\sum_{j=0}^{[(m-5)/3]}\binom{m-3}{3j+2}2^{3j+2}+6\sum_{u=0}^{[(m-3)/3]}\binom{m-3}{3u}2^{3u}+\sum_{u=0}^{[(m-5)/3]}\binom{m-3}{3u+2}2^{3u+2}\\ &=\begin{cases}3^{m-2}+3(-3)^{(m-3)/2} & \text{if } m \text{ is odd},\\ 3^{m-2}+(-3)^{(m-2)/2} & \text{if } m \text{ is even}.\end{cases}\end{aligned}$$

Therefore, we can obtain all the parameters of the second kind as described in the theorem. Hence the theorem.

Some of the reduced 3-class association scheme can be further reduced to 2-class association schemes. We have obtained the following.

Remark 1. When m is even, the reduced 3-class association scheme in Theorem 2 is further reducible to 2-class association schemes by clubbing (i) 1st and 2nd associate classes, (ii) 2nd and 3rd associate classes, or (iii) 1st and 3rd associate classes. Parameters of the reduced association schemes are given as follows:

(i) $n_1=2\{3^{m-1}+(-1)^{(m+2)/2}3^{(m-2)/2}\}$, $n_2=3^{m-1}+(-1)^{m/2}\cdot 2\cdot 3^{(m-2)/2}-1$,
$p_{11}^1=4\cdot 3^{m-2}+(-1)^{(m-2)/2}\cdot 3^{m/2}$, $p_{12}^1=p_{21}^1=2\cdot 3^{m-2}-1+(-1)^{m/2}\cdot 3^{(m-2)/2}$,
$p_{22}^1=3^{m-2}+(-1)^{m/2}3^{(m-2)/2}$, $p_{11}^2=4\cdot 3^{m-2}+(-1)^{(m-2)/2}\cdot 2\cdot 3^{(m-2)/2}$,
$p_{12}^2=p_{21}^2=2\cdot 3^{m-2}$, $p_{22}^2=3^{m-2}-2+(-1)^{m/2}\cdot 2\cdot 3^{(m-2)/2}$.

(ii) $n_1=3^{m-1}+(-1)^{(m+2)/2}3^{(m-2)/2}$, $n_2=2\cdot 3^{m-1}+(-1)^{m/2}3^{(m-2)/2}-1$,
$p_{11}^1=3^{m-2}$, $p_{12}^1=p_{21}^1=2\cdot 3^{m-2}-1+(-1)^{(m+2)/2}3^{(m-2)/2}$,
$p_{22}^1=4\cdot 3^{m-2}+(-1)^{m/2}3^{(m-2)/2}$, $p_{11}^2=3^{m-2}+(-1)^{(m-2)/2}3^{(m-2)/2}$,
$p_{12}^2=p_{21}^2=2\cdot 3^{m-2}$, $p_{22}^2=4\cdot 3^{m-2}+(-1)^{m/2}3^{(m-2)/2}-2$.

(iii) $n_1=2\cdot 3^{m-1}+(-1)^{m/2}3^{(m-2)/2}-1$, $n_2=3^{m-1}+(-1)^{(m+2)/2}3^{(m-2)/2}$,
$p_{11}^1=4\cdot 3^{m-2}-2+(-1)^{m/2}3^{(m-2)/2}$, $p_{12}^1=p_{21}^1=2\cdot 3^{m-2}$,
$p_{22}^1=3^{m-2}+(-1)^{(m+2)/2}3^{(m-2)/2}$, $p_{11}^2=4\cdot 3^{m-2}+(-1)^{m/2}\cdot 2\cdot 3^{(m-2)/2}$,
$p_{12}^2=p_{21}^2=2\cdot 3^{m-2}-1+(-1)^{(m-2)/2}3^{(m-2)/2}$, $p_{22}^2=3^{m-2}$.

Note that cases (ii) and (iii) are the same after interchanging 1st and 2nd associates.

Remark 2. When m is odd, the reduced 3-class association scheme in Theorem 2 is not further reducible. This fact can be shown by use of Theorem 2.1 due to Kageyama [3].

3.3. *Case* III. $v=4^m$ *treatments*

THEOREM 3. *An m-class hypercubic association scheme for 4^m treatments can always be reduced to a 2-class association scheme by putting the odd- and even-lettered treatments in two different associate classes against the treatment* $(0, 0, \cdots, 0)$. *The parameters of the reduced association scheme are given by*

$$v=4^m, \quad n_1=2^{m-1}[2^m-(-1)^m], \quad n_2=2^{m-1}[2^m+(-1)^m]-1,$$

$$P_1=\begin{bmatrix} 4^{m-1}+(-1)^{m-1}2^{m-1}, & 4^{m-1}-1 \\ 4^{m-1}-1, & 4^{m-1}+(-1)^m 2^{m-1} \end{bmatrix},$$

$$P_2=\begin{bmatrix} 4^{m-1}+(-1)^{m-1}2^{m-1}, & 4^{m-1} \\ 4^{m-1}, & 4^{m-1}+(-1)^m 2^{m-1}-2 \end{bmatrix}.$$

PROOF. Let $A_0\,(=I_v)$, $A_1, A_2, \cdots, A_m$ be association matrices of an m-class hypercubic association scheme of $v=4^m$ treatments We now define

$$B_1=\sum_{i=0}^{[(m-1)/2]} A_{2i+1}\,, \qquad B_2=\sum_{i=0}^{[(m-2)/2]} A_{2i+2} \tag{3.3.1}$$

which will be later considered as new association matrices of the reduced 2-class association scheme required. It is then obvious that

$$\begin{aligned} n_1&=\sum_{i=0}^{[(m-1)/2]}\binom{m}{2i+1}3^{2i+1}=2^{m-1}[2^m-(-1)^m]\,, \\ n_2&=\sum_{i=0}^{[(m-2)/2]}\binom{m}{2i+2}3^{2i+2}=2^{m-1}[2^m+(-1)^m]-1\,. \end{aligned} \tag{3.3.2}$$

For the parameters of the second kind of the reduced association scheme, it is sufficient to show from (3.3.1), (3.3.2), and Theorems 3.1 and 3.2 of Bose and Clatworthy [1] that for any pair of the v treatments which are ith associates, the numbers, p_{11}^i for $i=1, 2$, of treatments common to the first associates of the first and the first associates of the second is independent of the pair of treatments with which we start, i.e., it is to determine p_{11}^1 and p_{11}^2 such that $B_1B_1=n_1I_v+p_{11}^1B_1+p_{11}^2B_2$. From (3.3.1) and (3.3.2), the matrix forms of B_1B_1, B_1 and B_2, and the linearly independence of matrices $A_0, A_1, \cdots, A_m$, yield that

$$p_{11}^1=p_{11}^2=2\sum_{u=0}^{[(m-1)/2]}\binom{m-1}{2u}3^{2u}=4^{m-1}+(-1)^{m-1}2^{m-1}\,.$$

Therefore, we can get all the remaining parameters of the second kind as mentioned in the theorem. Hence the theorem.

Remark 3. Since the reduced 2-class association scheme for $v=4^m$ treatments in Theorem 3 has $p_{11}^1=p_{11}^2$ (see Shrikhande and Singh [10]), it immediately leads us to an interesting class of symmetrical balanced incomplete block designs with the parameters:

$$v=b=4^m\,,\quad r=k=2^{m-1}[2^m-(-1)^m]\,,\quad \text{and}\quad \lambda=4^{m-1}+(-1)^{m-1}2^{m-1}\,.$$

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REFERENCES

[1] Bose, R. C. and Clatworthy, W. H. (1955). Some classes of partially balanced designs, *Ann. Math. Statist.*, **26**, 212-232.

[2] Kageyama, S. (1972). On the reduction of associate classes for certain PBIB designs, *Ann. Math. Statist.*, **43**, 1528-1540.

[3] Kageyama, S. (1974a). Note on the reduction of associate classes for PBIB designs, *Ann. Inst. Statist. Math.*, **26**, 163-170.

[4] Kageyama, S. (1974b). On the reduction of associate classes for the PBIB designs of a certain generalized type, *Ann. Statist.*, **2**, 1346-1350.

[5] Kageyama, S. (1974c). Reduction of associate classes for block designs and related combinatorial arrangements, *Hiroshima Math. J.*, **4**, 527-618.

[6] Kusumoto, K. (1965). Hypercubic designs, *Wakayama Medical Reports*, **9**, 123-132.

[7] Riordan, J. (1968). *Combinatorial Identities*, John Wiley and Sons, Inc., New York.

[8] Saha, G. M. and Das, M. N. (1971). Construction of partially balanced incomplete block designs through 2^m factorial and some new designs of two associate classes, *J. Comb. Theory*, **11**, 282-295.

[9] Shah, B. V. (1958). On balancing in factorial experiments, *Ann. Math. Statist.*, **29**, 766-779.

[10] Shrikhande, S. S. and Singh, N. K. (1962). On a method of constructing symmetrical balanced incomplete block designs, *Sankhyā*, A**24**, 25-32.