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UNIQUE FACTORIZATION OF PRODUCTS OF BIVARIATE NORMAL CUMULATIVE DISTRIBUTION FUNCTIONS

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1. Introduction

In a previous paper ([1]), the authors have considered the following problem: If the random variables $X_1, \cdots, X_n$ are known to be independent and normally distributed with unknown means and variances, does the distribution of $\max\{X_1, \cdots, X_n\}$ uniquely determine the parameters? A partial extension to the bivariate normal case was also considered, with the restriction that all covariances be nonnegative. It was shown that there is essentially unique identification of parameters in all these cases.

In the present paper, we treat the bivariate normal case with $n=2$ but without any restriction on the covariance matrices. If $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ are independent, normally distributed 2-dimensional random vectors with cumulative distribution functions (cdf's) Φ_1 and Φ_2, respectively, the cdf of $\max\{\boldsymbol{X}_1, \boldsymbol{X}_2\}$ is $\Phi_1\Phi_2$; $\max\{\boldsymbol{X}_1, \boldsymbol{X}_2\}$ is defined as the vector whose first component is the maximum of the first components of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ and whose second component is the maximum of the second components of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. Hence, the question is:

If Φ_1, Φ_2, F_1, F_2 are bivariate normal cdf's, such that $\Phi_1\Phi_2=F_1F_2$, does it follow that either $(\Phi_1, \Phi_2)=(F_1, F_2)$ or $(\Phi_1, \Phi_2)=(F_2, F_1)$? We prove that this is so, except in the trivial case of zero covariances.

Our proof consists of detailed treatment of several different cases and, hence, is long and involved. We encourage the reader to look for a simpler unified proof.

2. Bivariate normal distributions with zero means

We treat here the case of normal distributions with zero means,

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because the univariate results of the previous paper mentioned above lead to the identification of means and variances; the intrinsically new aspect of the bivariate problem is that of the identification of the covariances and the correspondence of the covariances with the parameters of the marginal distributions. The correspondence can be done in terms of the variances.

THEOREM 2.1. *If Φ_1, Φ_2, F_1, F_2 are bivariate normal cdf's with zero means, such that*

$$\Phi_1(x, y)\Phi_2(x, y)=F_1(x, y)F_2(x, y)\ , \tag{2.1}$$

then one of the following relations holds:
(i) $(\Phi_1, \Phi_2)=(F_1, F_2)$, (ii) $(\Phi_1, \Phi_2)=(F_2, F_1)$, or
(iii) $\Phi_i(x, y)=A_i(x)B_i(y)$, $i=1, 2$ *and* $F_i(x, y)=A_i(x)B_j(y)$, $i, j=1, 2$, $i\neq j$,
(iv) $\Phi_i(x, y)=A_i(x)B_i(y)$, $i=1, 2$ *and* $F_i(x, y)=A_j(x)B_i(y)$, $i, j=1, 2$, $i\neq j$.

PROOF. Let the variances and correlation in Φ_i be $(\sigma_i^2, \rho_i, \tau_i^2)$, $i=1, 2$. The variances of the x-components on the r.h.s. of (2.1) are σ_1^2 and σ_2^2, and those of the y-components are τ_1^2 and τ_2^2. Therefore we may assume, without loss of generality, that the parameters of F_i are (σ_i^2, r_i, t_i^2), $i=1, 2$, where $(t_1, t_2)=(\tau_1, \tau_2)$ or (τ_2, τ_1). Let

$$\begin{aligned}\gamma(x, y)&=\frac{\partial^2}{\partial x\partial y}[\Phi_1(x, y)\Phi_2(x, y)]\\&=\phi_1\Phi_2+\frac{\partial}{\partial x}\Phi_1\frac{\partial}{\partial y}\Phi_2+\frac{\partial}{\partial y}\Phi_1\frac{\partial}{\partial x}\Phi_2+\Phi_1\phi_2\\&=\phi_1(x, y)\Phi_2(x, y)\\&\quad+\frac{1}{\tau_1\sigma_2}n\left(\frac{y}{\tau_1}\right)n\left(\frac{x}{\sigma_2}\right)N\left[\left(\frac{x}{\sigma_1}-\frac{\rho_1 y}{\tau_1}\right)\Big/\sqrt{1-\rho_1^2}\right]\\&\quad\cdot N\left[\left(\frac{y}{\tau_2}-\frac{\rho_2 x}{\sigma_2}\right)\Big/\sqrt{1-\rho_2^2}\right]+\frac{1}{\sigma_1\tau_2}n\left(\frac{x}{\sigma_1}\right)n\left(\frac{y}{\tau_2}\right)\\&\quad\cdot N\left[\left(\frac{y}{\tau_1}-\frac{\rho_1 x}{\sigma_1}\right)\Big/\sqrt{1-\rho_1^2}\right]N\left[\left(\frac{x}{\sigma_2}-\frac{\rho_2 y}{\tau_2}\right)\Big/\sqrt{1-\rho_2^2}\right]\\&\quad+\Phi_1(x, y)\phi_2(x, y)\ ,\end{aligned} \tag{2.2}$$

where $n(\cdot)$ and $N(\cdot)$ are univariate standard normal pdf and cdf, respectively. Also,

$$g(x, y)=\frac{\partial^2}{\partial x\partial y}[F_1(x, y)F_2(x, y)]\ . \tag{2.3}$$

I. First consider the simple case $\sigma_1=\sigma_2$, $\tau_1=\tau_2$. Then $\tau_1=\tau_2=t_1=t_2$, so that the problem can be reduced to standard form by scale transformations on x and y. In this case, (2.2) and (2.3) give us

$$\begin{aligned}(2.4)\qquad \gamma(x, y)=&\frac{1}{2\pi\sqrt{1-\rho_1^2}}\exp\left(-\frac{x^2-2\rho_1xy+y^2}{2(1-\rho_1^2)}\right)\Phi_2(x, y)\\ &+n(x)n(y)N\left[\frac{(x-\rho_1y)}{\sqrt{1-\rho_1^2}}\right]N\left[\frac{(y-\rho_2x)}{\sqrt{1-\rho_2^2}}\right]\\ &+n(x)n(y)N\left[\frac{(y-\rho_1x)}{\sqrt{1-\rho_1^2}}\right]N\left[\frac{(x-\rho_2y)}{\sqrt{1-\rho_2^2}}\right]\\ &+\Phi_1(x, y)\frac{1}{2\pi\sqrt{1-\rho_2^2}}\exp\left(-\frac{x^2-2\rho_2xy+y^2}{2(1-\rho_2^2)}\right)\\ =&g(x, y)\ .\end{aligned}$$

On putting $y=x$, we have

$$\begin{aligned}(2.5)\qquad \frac{\gamma(x, x)}{n^2(x)}=&\frac{1}{\sqrt{1-\rho_1^2}}\exp\left(\frac{\rho_1}{1+\rho_1}x^2\right)\Phi_2(x, x)+2N\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}}x\right)\\ &\cdot N\left(\sqrt{\frac{1-\rho_2}{1+\rho_2}}x\right)+\frac{1}{\sqrt{1-\rho_2^2}}\exp\left(\frac{\rho_2}{1+\rho_2}x^2\right)\Phi_1(x, x)\ .\end{aligned}$$

We shall consider separately the cases $\max(\rho_1, \rho_2, r_1, r_2)>, =$ and <0.

(i) Suppose $\max(\rho_1, \rho_2, r_1, r_2)=\rho_1>0$ and let $r_1\geqq r_2$. Then as $x\to\infty$

$$(2.6)\qquad \frac{\gamma(x, x)}{n^2(x)}\Big/\exp\left(\frac{\rho_1}{1+\rho_1}x^2\right)\to\begin{cases}\dfrac{1}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1>\rho_2\ ,\\ \dfrac{2}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1=\rho_2\ .\end{cases}$$

On the other hand

$$(2.7)\qquad \frac{g(x, x)}{n^2(x)}\Big/\exp\left(\frac{\rho_1}{1+\rho_1}x^2\right)\to\begin{cases}0 & \text{if } \rho_1>r_1\geqq r_2\ ,\\ \dfrac{1}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1=r_1>r_2\ ,\\ \dfrac{2}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1=r_1=r_2\ .\end{cases}$$

Hence, if $\rho_1=\rho_2$, then $r_1=r_2=\rho_1=\rho_2$ and $\Phi_1=\Phi_2=F_1=F_2$. If $\rho_1>\rho_2$, then $r_1=\rho_1$ and $r_2=\rho_2$; thus $\Phi_1=F_1$ and $\Phi_2=F_2$.

(ii) Now suppose $\max(\rho_1, \rho_2, r_1, r_2)=\rho_1=0$. If $\rho_2=0$, (2.5)$\to 4$; then $g(x, x)/n^2(x)\to 4$, which implies $r_1=r_2=0$. If $\rho_2<0$, (2.5)$\to 3$; then $g(x, x)/n^2(x)\to 3$, which implies $r_1=0>r_2=\rho_2$.

(iii) Finally, suppose $\max(\rho_1, \rho_2, r_1, r_2)=\rho_1<0$. Then (2.5)$\to 2$. Consider

$$(2.8)\qquad \frac{\gamma(x, x)}{n^2(x)}-2=\frac{1}{\sqrt{1-\rho_1^2}}\exp\left(\frac{\rho_1}{1+\rho_1}x^2\right)\Phi_2(x, x)$$

$$+\frac{1}{\sqrt{1-\rho_2^2}}\exp\left(\frac{\rho_2}{1+\rho_2}x^2\right)\Phi_1(x, x)$$
$$-2\bar{N}\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}}x\right)-2\bar{N}\left(\sqrt{\frac{1-\rho_2}{1+\rho_2}}x\right)$$
$$+2\bar{N}\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}}x\right)\bar{N}\left(\sqrt{\frac{1-\rho_2}{1+\rho_2}}x\right),$$

where $\bar{N}(x)=1-N(x)$. Using the one-dimensional Mills' ratio, we have

$$\bar{N}\left(\sqrt{\frac{1-\rho_i}{1+\rho_i}}x\right)\sim\sqrt{\frac{1+\rho_i}{1-\rho_i}}\,\frac{1}{x}\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1-\rho_i}{2(1+\rho_i)}x^2\right). \tag{2.9}$$

Then, as $x\to\infty$,

$$\left[\frac{\gamma(x, x)}{n^2(x)}-2\right]\sqrt{1-\rho_1^2}\exp\left(\frac{-\rho_1}{1+\rho_1}x^2\right)\to\begin{cases}1 & \text{if } \rho_1>\rho_2\,, \\ 2 & \text{if } \rho_1=\rho_2\,.\end{cases} \tag{2.10}$$

Similarly, as $x\to\infty$,

$$\left[\frac{g(x, x)}{n^2(x)}-2\right]\sqrt{1-\rho_1^2}\exp\left(\frac{-\rho_1}{1+\rho_1}x^2\right)\to\begin{cases}0 & \text{if } \rho_1>r_1\,, \\ 1 & \text{if } r_1=\rho_1>\rho_2\,, \\ 2 & \text{if } r_1=r_2=\rho_2\,.\end{cases} \tag{2.11}$$

We obtain identification.

II. Having disposed of the case $\sigma_1=\sigma_2$, $\tau_1=\tau_2$, we now consider the situation where there is at least one inequality. Without loss of generality, we may suppose $\sigma_1>\sigma_2$. Then in Equation (2.2), if we keep y fixed and let $x\to\infty$, we have

$$\frac{\gamma(x, y)\sigma_1}{n(x/\sigma_1)}\to\begin{cases}0 & \text{if } \rho_1>0\,, \\ \frac{1}{\tau_1}n\left(\frac{y}{\tau_1}\right)N\left(\frac{y}{\tau_2}\right)+\frac{1}{\tau_2}n\left(\frac{y}{\tau_2}\right)N\left(\frac{y}{\tau_1}\right) & \text{if } \rho_1=0\,, \\ \frac{1}{\tau_2}n\left(\frac{y}{\tau_2}\right) & \text{if } \rho_1<0\,.\end{cases} \tag{2.12}$$

In the same way, from (2.3) we get

$$\frac{g(x, y)\sigma_1}{n(x/\sigma_1)}\to\begin{cases}0 & \text{if } r_1>0\,, \\ \frac{1}{t_1}n\left(\frac{y}{t_1}\right)N\left(\frac{y}{t_2}\right)+\frac{1}{t_2}n\left(\frac{y}{t_2}\right)N\left(\frac{y}{t_1}\right) & \text{if } r_1=0\,, \\ \frac{1}{t_2}n\left(\frac{y}{t_2}\right) & \text{if } r_1<0\,.\end{cases} \tag{2.13}$$

Consequently, from (2.1) we see that

(A) $\rho_1>0 \Rightarrow r_1>0$,

(B) $\rho_1=0 \Rightarrow r_1=0$,

(C) $\rho_1<0 \Rightarrow r_1<0$ and $t_2=\tau_2$ (so that $t_1=\tau_1$).

In Case (A), let $y=ax$, where $a=\rho_1\tau_1/\sigma_1$, and $x\to\infty$; then, noting that $1/\sigma^2-2\rho a/(\sigma\tau)+a^2/\tau^2=(1-\rho^2)/\sigma^2+(\rho/\sigma-a/\tau)^2$, from (2.2) we obtain

$$\frac{\gamma(x, ax)}{n(x/\sigma_1)}\sigma_1 \to \frac{1}{\tau_1\sqrt{2\pi(1-\rho_1^2)}} . \tag{2.14}$$

On the other hand, looking at the expression for (2.3) similar to that for (2.2), we observe that

$$f_i(x, ax)=\frac{1}{\sigma_i t_i\sqrt{2\pi(1-r_i^2)}}n\left(\frac{x}{\sigma_i}\right)\exp\left[-\frac{(a/t_i-r_i/\sigma_i)^2x^2}{2(1-r_i^2)}\right] , \tag{2.15}$$

$$\frac{\gamma(x, ax)}{n(x/\sigma_1)}\sigma_1 \to \begin{cases} 0 & \text{if } a/t_1\neq r_1/\sigma_1 , \\ \dfrac{1}{t_1\sqrt{2\pi(1-r_1^2)}} & \text{if } a/t_1=r_1/\sigma_1 . \end{cases} \tag{2.16}$$

Hence, on account of (2.1), recalling that $a=\rho_1\tau_1/\sigma_1$, we conclude that

$$\frac{\rho_1\tau_1}{\sigma_1 t_1}=\frac{r_1}{\sigma_1} \quad \text{and} \quad t_1^2(1-r_1^2)=\tau_1^2(1-\rho_1^2) . \tag{2.17}$$

This implies $t_1=\tau_1$ and $r_1=\rho_1$. So, in Case (A), $\Phi_1=F_1$ and hence $\Phi_2=F_2$.

Next, in Case (B), the original equation (2.1) becomes

$$N\left(\frac{x}{\sigma_1}\right)N\left(\frac{y}{\tau_1}\right)\Phi_2(x, y)=N\left(\frac{x}{\sigma_1}\right)N\left(\frac{y}{t_1}\right)F_2(x, y) . \tag{2.18}$$

If we now remove the common factor $N(x/\sigma_1)$ from both sides, differentiate with respect to x and remove the common factor $(1/\sigma_2)n(x/\sigma_2)$ from both sides, we are left with

$$N\left(\frac{y}{\tau_1}\right)N\left[\frac{y/\tau_2-\rho_2(x/\sigma_2)}{\sqrt{1-\rho_2^2}}\right]=N\left(\frac{y}{t_1}\right)N\left[\frac{y/t_2-r_2(x/\sigma_2)}{\sqrt{1-r_2^2}}\right] . \tag{2.19}$$

If $\rho_2=0$, the lhs of (2.19) is independent of x, and hence $r_2=0$; in this case, both sides of (2.1) are products of univariate cdfs, and there is no unique matching of (x, y) pairs. On the other hand, if $\rho_2\neq 0$ then $r_2\neq 0$, and setting $y=0$ yields $\rho_2=r_2$. If we now let $x=\sigma_2 y/(\rho_2 t_2)$, we obtain $t_2=\tau_2$. Hence, $\Phi_1=F_1$ and $\Phi_2=F_2$.

Finally, in Case (C), $\rho_1<0$, $r_1<0$, $t_i=\tau_i$, $i=1, 2$. If we let

$$\bar{\Phi}(x, y)=\int_{\substack{u>x\\ v>y}}\phi(u, v)dudv , \tag{2.20}$$

and note that $\Phi(-x, -y)=\bar{\Phi}(x, y)$, (2.1) becomes

$$\prod_{i=1}^{2} \bar{\Phi}_i(x, y)=\prod_{i=1}^{2} \bar{F}_i(x, y)\ , \qquad x, y>0\ . \tag{2.21}$$

We can now use the bivariate Mills' ratio (Savage [3], Ruben [2]) for asymptotic expressions for the quantities in (2.21) as x and $y\to\infty$: If $x, y\to\infty$ in such a manner that $(x/\sigma_i-\rho_i(y/\tau_i))$ and $(y/\tau_i-\rho_i(x/\sigma_i))$ are both positive, then

$$\bar{\Phi}_i(x, y)\left(\frac{x}{\sigma_i}-\rho_i\frac{y}{\tau_i}\right)\left(\frac{y}{\tau_i}-\rho_i\frac{x}{\sigma_i}\right)\Big/[\phi(x, y)\sigma_i\tau_i(1-\rho_i^2)^2]\to 1\ . \tag{2.22}$$

If we now set $y=cx$ and let $x\to\infty$, then (2.22) holds for $\bar{\Phi}_1$ and $\bar{F}_1$ for all $c>0$, and also holds for $\bar{\Phi}_2$ and $\bar{F}_2$ at least for all c in an interval of positive length containing the point τ_2/σ_2. Hence, we have

$$\prod_{i=1}^{2}\frac{\phi_i(x, cx)(1-\rho_i^2)^2}{x^2(1/\sigma_i-c\rho_i/\tau_i)(c/\tau_i-\rho_i/\sigma_i)}\Big/\prod_{i=1}^{2}\frac{f_i(x, cx)(1-r_i^2)^2}{x^2(1/\sigma_i-cr_i/\tau_i)(c/\tau_i-r_i/\sigma_i)}\to 1 \tag{2.23}$$

as $x\to\infty$, for all c in an interval of positive length containing the point τ_2/σ_2. But

$$\prod_{i=1}^{2}\left[\frac{\phi_i(x, cx)}{f_i(x, cx)}\right]=\prod_{i=1}^{2}\frac{(1-r_i^2)^{1/2}}{(1-\rho_i^2)^{1/2}}\exp\left[-\frac{1}{2}Q(c)x^2\right]\ , \tag{2.24}$$

where $Q(c)$ is a quadratic polynomial, and (2.23) implies that the rhs of (2.24) has a finite positive limit for all c in an interval of positive length. This can happen only if $Q(c)\equiv 0$; the limit is then $\prod_{i=1}^{2}\sqrt{(1-r_i^2)}/\sqrt{(1-\rho_i^2)}$. Thus we have

$$\sum_{i=1}^{2}\left[\frac{1}{\sigma_i^2}-\frac{2\rho_i c}{\sigma_i\tau_i}+\frac{c^2}{\tau_i^2}\right](1-\rho_i^2)^{-1}=\sum_{i=1}^{2}\left[\frac{1}{\sigma_i^2}-\frac{2r_i c}{\sigma_i\tau_i}+\frac{c^2}{\tau_i^2}\right](1-r_i^2)^{-1}\ , \tag{2.25}$$

and

$$\prod_{i=1}^{2}(1-\rho_i^2)^{-3/2}\left(\frac{1}{\sigma_i}-\frac{c\rho_i}{\tau_i}\right)\left(\frac{c}{\tau_i}-\frac{\rho_i}{\sigma_i}\right)=\prod_{i=1}^{2}(1-r_i^2)^{-3/2}\left(\frac{1}{\sigma_i}-\frac{cr_i}{\tau_i}\right)\left(\frac{c}{\tau_i}-\frac{r_i}{\sigma_i}\right)\ , \tag{2.26}$$

both relations holding for all c in an interval of positive length. Consequently, from (2.25) we obtain

$$\frac{1}{\sigma_1^2}\left(\frac{1}{1-\rho_1^2}-\frac{1}{1-r_1^2}\right)+\frac{1}{\sigma_2^2}\left(\frac{1}{1-\rho_2^2}-\frac{1}{1-r_2^2}\right)=0\ , \tag{2.27}$$

$$\frac{1}{\tau_1^2}\left(\frac{1}{1-\rho_1^2}-\frac{1}{1-r_1^2}\right)+\frac{1}{\tau_2^2}\left(\frac{1}{1-\rho_2^2}-\frac{1}{1-r_2^2}\right)=0\ , \tag{2.28}$$

$$(2.29)\qquad \frac{1}{\sigma_1\tau_1}\left(\frac{\rho_1}{1-\rho_1^2}-\frac{r_1}{1-r_1^2}\right)+\frac{1}{\sigma_2\tau_2}\left(\frac{\rho_2}{1-\rho_2^2}-\frac{r_2}{1-r_2^2}\right)=0 .$$

If $r_1=\rho_1$, then $\Phi_1=F_1$, and hence $\Phi_2=F_2$. So, it remains only to investigate the possibility $r_1\neq\rho_1$; in this case, (2.27)$\Rightarrow r_2\neq\rho_2$, and from (2.27) and (2.28) we have

$$(2.30)\qquad \frac{\tau_1}{\sigma_1}=\frac{\tau_2}{\sigma_2}=\tau , \qquad \text{say} .$$

But from (2.26) we know that the polynomials in c on the two sides of the equation have the same zeros. The zeros of the lhs are $\{\tau/\rho_1, \tau\rho_1, 0\}$ if $\rho_2=0$, and $\{\tau\rho_1, \tau/\rho_1, \tau\rho_2, \tau/\rho_2\}$ if $\rho_2\neq 0$; and those of the rhs are $\{\tau/r_1, \tau r_1, 0\}$ if $r_2=0$, and $\{\tau r_1, \tau/r_1, \tau r_2, \tau/r_2\}$ if $r_2\neq 0$. Hence, the assumption that $r_1\neq\rho_1<0$ leads to the conclusion $r_1=\rho_2$ and $r_2=\rho_1$. This, together with (2.27), contradicts the assumption that $\sigma_1>\sigma_2$. Thus in Case (C) also, we must have $r_1=\rho_1$, $r_2=\rho_2$, so that $\Phi_1=F_1$, $\Phi_2=F_2$.

Q.E.D.

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