Ann. Inst. Statist. Math.
29 (1978), Part A, 57-62

ASYMPTOTIC DISTRIBUTIONS OF THE LATENT ROOTS WITH MULTIPLE POPULATION ROOTS IN MULTIPLE DISCRIMINANT ANALYSIS

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(Received Dec. 14, 1977)

Abstract

Asymptotic expansions are derived for the confluent hypergeometric function ${}_1F_1(a; c; R, S)$ with two argument matrices, which appears in the joint density function of the latent roots in multiple discriminant analysis, when R is "large" and each of the latent roots of R assumes the general multiplicity. Laplace's method and a partial differential equation method are utilized in the derivation.

1. Introduction

Let S_1 and S_2 be distributed independently as noncentral Wishart $W_m(n_1, \Sigma; \Omega)$ and Wishart $W_m(n_2, \Sigma)$ respectively. Tne joint density function of the latent roots, $1>b_1>b_2>\cdots>b_m>0$ of the matrix $B=S_1(S_1+S_2)^{-1}$ is expressed by James [10] as

$$(1.1)\quad \frac{\pi^{m^2/2}\Gamma_m((n_1+n_2)/2)}{\Gamma_m(n_1/2)\Gamma_m(n_2/2)\Gamma_m(m/2)} \operatorname{etr}(-\Omega/2) \prod_{i=1}^m b_i^{(n_1-m-1)/2} \prod_{i=1}^m (1-b_i)^{(n_2-m-1)/2} \cdot \prod_{\substack{i=1\\i<j}}^m \prod_{j=1}^m (b_i-b_j)\, {}_1F_1((n_1+n_2)/2; n_1/2; \Omega/2, B)\,,$$

where $\operatorname{etr} A \underset{\text{def}}{\equiv} \exp(\operatorname{tr} A)$ and ${}_1F_1$ is a confluent hypergeometric function with two argument matrices. The roots $b_1, b_2, \cdots, b_m$ play an important role in multiple discriminant analysis used for discriminating among several groups (see e.g. Kshirsagar [11]). The latent roots of Ω can be considered as "distances" along m different directions among the groups, and various functions of $b_1, b_2, \cdots, b_m$ have been proposed as measures for the "distance". When the first k b_i's are significant, the corresponding latent vectors give the sample discriminant functions.

The power series expansion in terms of zonal polynomials for the ${}_1F_1$ function, due to Herz [7], Constantine [4] and James [10], converges very slowly for "large" Ω. In this connection, it is useful to derive asymptotic expansions for the function ${}_1F_1(a; c; R, S)$ for "large" R. Throughout this paper, R and S are $m\times m$ diagonal matrices such that

$$(1.2)\qquad R=\begin{pmatrix} NR_1 & 0 \\ 0 & R_2 \end{pmatrix}, \qquad S=\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}=\mathrm{diag}\,(s_1, s_2, \cdots, s_m)\,, \quad s_1>s_2>\cdots>s_m>0\,,$$

where R_1, S_1, and R_2, S_2 are $m_1\times m_1$ and $m_2\times m_2$ diagonal matrices respectively $(m_1+m_2=m)$.

Limiting terms and further terms of asymptotic expansions for the ${}_1F_1$ function, for large N, have been derived by Constantine and Muirhead [6] up to and including the terms of order N^{-2} for the cases (i) when R_1 has distinct roots $(m_1\leqq m)$ and (ii) when $m_1=m$ and the smallest root of R_1 is multiple. A multivariate extension of Laplace's method for integrals and partial differential equations (pde's) satisfied by the ${}_1F_1$ function were utilized for the derivation.

In this paper, we derive asymptotic expansions for the ${}_1F_1$ function, up to and including the terms of order N^{-2}, extending Constantine and Muirhead's result [6] to the case when each of the latent roots of R_1 and R_2 assumes the general multiplicity.

2. The limiting term for ${}_1F_1(a; c; R, S)$

Throughout this paper, it is assumed that the m_1 roots of R_1 satisfy

$$(2.1)\qquad \underbrace{r_1=\cdots=r_1}_{q_1}>\underbrace{r_2=\cdots=r_2}_{q_2}>\cdots>\underbrace{r_t=\cdots=r_t}_{q_t}>0\,,$$

where $\sum_{u=1}^{t} q_u=m_1$, and let $k_1=1$, $k_i=\sum_{u=1}^{i-1} q_u+1$ $(i=2,\cdots,t+1)$. The situation (2.1) gives the general multiple root assumption in multiple discriminant analysis; r_i is regarded as a simple root when $q_i=1$.

In this section, we give the limiting term for the function ${}_1F_1(a; c; R, S)$ under the general multiple root assumption (2.1), which is an extension of Theorem 3.3 of Constantine and Muirhead [6].

THEOREM 2.1. *Let R and S be $m\times m$ matrices defined in* (1.2) *and* (2.1). *Then as $N\to\infty$ we have*

$$(2.2)\qquad {}_1F_1(a; c; R, S)\sim\pi^{-m_1(m+m_2)/2}\Gamma_m(m/2)\Gamma_{m_1}(c)\,[\Gamma_{m_2}(m_2/2)\Gamma_{m_1}(a)]^{-1}$$
$$\cdot\prod_{u=1}^{t}\left[\frac{\pi^{q_u^2/2}}{\Gamma_{q_u}(q_u/2)}\right]\det\,(NR_1S_1)^{a-c}\,\mathrm{etr}\,(NR_1S_1)$$
$$\cdot\,{}_1F_1(a-m_1/2; c-m_1/2; R_2, S_2)$$
$$\cdot\prod_{u=1}^{t}\prod_{i=k_u}^{k_{u+1}-1}\prod_{j=k_{u+1}}^{m}\left(\frac{\pi}{NC_{ij}}\right)^{1/2},$$

where the C_{ij} are given by

$$(2.3)\qquad C_{ij}=\begin{cases}(r_u-r_v)(s_i-s_j) & i=k_u,\cdots,k_{u+1}-1,\ u=1,\cdots,t \\ & j=k_v,\cdots,k_{v+1}-1,\ v=1,\cdots,t\,, \\ r_u(s_i-s_j) & i=k_u,\cdots,k_{u+1}-1,\ u=1,\cdots,t \\ & j=m_1+1,\cdots,m\,.\end{cases}$$

PROOF. We can prove the theorem similarly as for Theorem 3.3 of Constantine and Muirhead [6] with some modifications. Here we apply the maximization procedures due to Chattopadhyay and Pillai [1] and Chattopadhyay, Pillai and Li [2] for the case of general multiple roots. It is proved that an extension of Corollary 2.1 of [6], required for the proof, is given by

$$(2.4)\quad \int_{H_1\in V(m_1,m)}\operatorname{etr}(NR_1H_1'SH_1)(dH_1)\sim 2^{m_1}\prod_{u=1}^{t}\left[\frac{\pi^{q_u^2/2}}{\Gamma_{q_u}(q_u/2)}\right]\operatorname{etr}(NR_1S_1)\cdot\prod_{u=1}^{t}\prod_{i=k_u}^{k_{u+1}-1}\prod_{j=k_{u+1}}^{m}\left(\frac{\pi}{NC_{ij}}\right)^{1/2}.$$

3. The asymptotic expansions for the ${}_1F_1$ function

In this section, a pde method is utilized to derive an asymptotic expansion, for large N, for the function ${}_1F_1(a;c;NR,S)$, i.e. for the case, $m_2=0$, under the general multiple root assumption (2.1). This then yields an asymptotic expansion for the ${}_1F_1$ function for a general case when $m_2>0$ and the roots of R_2 satisfy

$$(3.1)\qquad \underbrace{r_{t+1}=\cdots=r_{t+1}}_{q_{t+1}}>\underbrace{r_{t+2}=\cdots=r_{t+2}}_{q_{t+2}}>\cdots>\underbrace{r_{t+p}=\cdots=r_{t+p}}_{q_{t+p}}\geqq 0\,,$$

where $\sum_{j=1}^{p}q_{t+j}=m_2$, and let $k_{t+1}=m_1+1$, $k_{t+i}=m_1+\sum_{j=1}^{i-1}q_{t+j}+1$ $(i=2,\cdots,p+1)$.

A pde satisfied by ${}_1F_1(a;c;NR,S)$ under the assumption (2.1) can be obtained, similarly for the ${}_0F_0$ function in Chikuse [3], based on the pde due to Constantine and Muirhead [5], valid for the distinct roots of R. It is summarized in

LEMMA 3.1. *The function ${}_1F_1(a;c;NR,S)$ under the assumption (2.1) satisfies the pde*

$$(3.2)\quad \sum_{i=1}^{m}s_i\frac{\partial^2F}{\partial s_i^2}+\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{s_i}{s_i-s_j}\frac{\partial F}{\partial s_i}+\left(c-\frac{m-1}{2}\right)\sum_{i=1}^{m}\frac{\partial F}{\partial s_i}-N\sum_{u=1}^{t}r_u^2\frac{\partial F}{\partial r_u}=Na\sum_{u=1}^{t}q_ur_uF\,.$$

We can put, from (2.2),

$$ {}_1F_1(a;c;NR,S)=f(N;R,S;a,c)G(N;R,S;a,c)\,, \tag{3.3} $$

where the function f is given by the right-hand side of (2.2), with $m_2=0$, R and S replacing R_1 and S_1 respectively and the terms involving m_2, R_2 and S_2 ignored. We know that $\lim_{N\to\infty} G=1$ and G satisfies the "boundary" conditions

$$ \text{(i)}\quad G(N;R,S;a,c)=G(N;S,R;a,c) \quad\text{and}\quad \text{(ii)}\quad G(N;R,S;a,c)=H(NR,S;a,c)\,. \tag{3.4} $$

Substituting (3.3) in (3.2) gives the pde satisfied by G

$$ \begin{aligned} &\sum_{i=1}^{m} s_i\frac{\partial^2 G}{\partial s_i^2}+\left(2a-c-\frac{m-1}{2}\right)\sum_{i=1}^{m}\frac{\partial G}{\partial s_i}+2N\sum_{u=1}^{t} r_u\sum_{i=k_u}^{k_{u+1}-1} s_i\frac{\partial G}{\partial s_i}\\ &\quad+\sum_{u=1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\sum_{\substack{j=k_u\\ j\neq i}}^{k_{u+1}-1}\frac{s_i}{s_i-s_j}\frac{\partial G}{\partial s_i}-N\sum_{u=1}^{t} r_u^2\frac{\partial G}{\partial r_u}\\ &\quad+\left[\frac{1}{4}\sum_{u=1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\sum_{j=k_{u+1}}^{m}\frac{s_i+s_j}{(s_i-s_j)^2}+(a-c)\left(a-\frac{m+1}{2}\right)\sum_{i=1}^{m}\frac{1}{s_i}\right]G\\ &\quad=0\,. \end{aligned} \tag{3.5} $$

We look for a solution of (3.5) of the form

$$ G=1+Q_1N^{-1}+Q_2N^{-2}+\cdots\,, \tag{3.6} $$

where the Q_i are independent of N. Substituting (3.6) in (3.5) and solving the equations derived by equating coefficients of like powers of N^{-1} on both sides, with the boundary condition (3.4), gives, after an enormous amount of calculation,

$$ Q_1=\frac{1}{4}\sum_{u=1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\sum_{j=k_{u+1}}^{m}\frac{1}{C_{ij}}+(a-c)\left(a-\frac{m+1}{2}\right)\sum_{u=1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\frac{1}{r_us_i}\,, \tag{3.7} $$

$$ \begin{aligned} Q_2=&\frac{1}{32}\left(\sum_{u=1}^{t-1}\sum_{v=u+1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\sum_{j=k_v}^{k_{v+1}-1}\frac{1}{C_{ij}}\right)^2+\frac{1}{4}\sum_{u=1}^{t-1}\sum_{v=u+1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\sum_{j=k_v}^{k_{v+1}-1}\frac{1}{C_{ij}^2}\\ &+\frac{1}{8}\sum_{u=1}^{t}\sum_{\substack{i=k_u\\ i<j}}^{k_{u+1}-1}\sum_{j=k_u}^{k_{u+1}-1}\sum_{\substack{v=1\\ v\neq u}}^{t}\sum_{l=k_v}^{k_{v+1}-1}\frac{1}{(r_u-r_v)^2(s_i-s_l)(s_j-s_l)}\\ &+\frac{1}{2}(a-c)\left(a-\frac{m+1}{2}\right)\Bigg[(a-c)\left(a-\frac{m+1}{2}\right)\left(\sum_{u=1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\frac{1}{r_us_i}\right)^2\\ &-\left(2a-c-\frac{m+3}{2}\right)\sum_{u=1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\frac{1}{r_u^2s_i^2}+\sum_{u=1}^{t}\sum_{\substack{i=k_u\\ i<j}}^{k_{u+1}-1}\sum_{j=k_u}^{k_{u+1}-1}\frac{1}{r_u^2s_is_j}\\ &+\frac{1}{2}\left(\sum_{u=1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\frac{1}{r_us_i}\right)\left(\sum_{u=1}^{t}\sum_{i=k_u}^{k_{u+1}-1}\sum_{j=k_{u+1}}^{m}\frac{1}{C_{ij}}\right)\Bigg]\,, \end{aligned} \tag{3.8} $$

where the C_{ij} are given by (2.3). The asymptotic expansion is summarized in

THEOREM 3.1. *The function* ${}_1F_1(a; c; NR, S)$, *under the assumption* (2.1), *with* $m_1=m$, *can be expanded for large* N *as*

$$(3.9)\quad {}_1F_1(a; c; NR, S)=f(N; R, S; a, c)[1+Q_1N^{-1}+Q_2N^{-2}+O(N^{-3})]\ ,$$

where $f(N; R, S; a, c)$ *is given by* (2.2), *with* $m_2=0$, R *and* S *replacing* R_1 *and* S_1 *respectively and the terms involving* m_2, R_2 *and* S_2 *ignored, and* Q_1 *and* Q_2 *are given by* (3.7) *and* (3.8), *respectively.*

(3.9) yields an asymptotic expansion for the general case when $m_2>0$ and the roots of R_2 satisfy (3.1). A technique, similar to that used by Constantine and Muirhead ([6], p. 383) yields the following

COROLLARY 3.1. *The function* ${}_1F_1(a; c; R, S)$, *where* R *and* S *are* $m\times m$ *matrices defined by* (1.2), (2.1) *and* (3.1), *may be expanded for large* N *as*

$$(3.10)\qquad {}_1F_1(a; c; R, S)=f^*(N; R, S; a, c)[1+Q_1^*N^{-1}+O(N^{-2})]\ ,$$

where f^* *and* Q_1^* *are given by* (2.2) *and* (3.7) *respectively with the* D_{ij} *replacing the* C_{ij} *with*

$$(3.11)\quad D_{ij}=\begin{cases} C_{ij} & i, j=1,\cdots, m_1,\ \text{given by (2.3)}\ , \\ \left(r_u-\dfrac{r_v}{N}\right)(s_i-s_j) & \begin{array}{l} i=k_u,\cdots, k_{u+1}-1,\ u=1,\cdots, t \\ j=k_v,\cdots, k_{v+1}-1,\ v=t+1,\cdots, t+p\ . \end{array} \end{cases}$$

The term of order N^{-2} *is lengthy and hence omitted here.*

The results given in this paper agree with those derived for special cases by Constantine and Muirhead [6].

The expansions derived in this and previous sections may be substituted in (1.1) to give asymptotic expansions for the joint distribution of the latent roots $b_1, b_2,\cdots, b_m$ of the matrix $B=S_1(S_1+S_2)^{-1}$.

It is easily shown that the m latent roots of the matrix $S_1S_2^{-1}/N$ for $\Omega/2=NR$ (i.e. $m_2=0$) have, asymptotically for large N, the joint density function of the latent roots of the matrix of the form $2RT$, where T^{-1} is distributed as Wishart $W_m(n_2, I)$. Hence the m latent roots of the matrix $B=S_1(S_1+S_2)^{-1}$ are asymptotically distributed like the latent roots of the matrix $(I+\Omega^{-1}U)^{-1}$, where U is distributed as Wishart $W_m(n_2, I)$ and $\Omega=2NR$.

Acknowledgment

The author would like to express her thanks to Dr. A. G. Constantine at C.S.I.R.O. for his suggestion which lead to greater simplicity of the formulae.

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