Ann. Inst. Statist. Math.
30 (1978), Part A, 51-55

THE ASYMPTOTIC EXPANSION OF THE DISTRIBUTION OF ANDERSON'S STATISTIC FOR TESTING A LATENT VECTOR OF A COVARIANCE MATRIX

TAKESI HAYAKAWA

(Received Aug. 26, 1977; revised Dec. 21, 1977)

1. Introduction

For testing a hypothesis that a specified vector is a corresponding latent vector of the maximum latent root of a population covariance matrix, Mallows [4] gave the likelihood ratio criterion and showed that the limiting distribution of it is a central chi-square one with appropriate degrees of freedom. Kshirsagar [2] has studied another type of statistic for testing this hypothesis. Anderson [1] has proposed a test statistic for testing a hypothesis that a specified vector is a corresponding latent vector of the ith latent root of a population covariance matrix and showed that the limiting distribution is also central chi-square one.

In this paper we consider the asymptotic expansion of the distribution of Anderson's statistic for the case of a vector of the maximum latent root up to the terms of order $1/n$ and give some comments for the use of this statistic.

2. Asymptotic expansion of a distribution of Anderson's statistic

Consider testing the null hypothesis that γ_1, the vector corresponding to the maximum latent root λ_1 of a population covariance matrix Σ, i.e., $\Sigma\gamma_1=\lambda_1\gamma_1$, is a specified vector $\boldsymbol{a}$ ($\boldsymbol{a}'\boldsymbol{a}=1$) under the assumption that Σ has latent roots with multiplicity 1, where $\boldsymbol{a}'$ is a transpose of a p-dimensional vector $\boldsymbol{a}$.

Anderson [1] proposed a test statistic T for testing this hypothesis in the form of

$$T=n\{f_1\boldsymbol{a}'S^{-1}\boldsymbol{a}+\boldsymbol{a}'S\boldsymbol{a}/f_1-2\}\ , \tag{1}$$

where S has a Wishart distribution $W_p(\Sigma, n)$ and f_1 is the maximum latent root of a determinantal equation $\det(S-fI_p)=0$.

Let $\Gamma=[\boldsymbol{a}, \Gamma_2]$ be an orthogonal matrix of order p such that $\Gamma'\Sigma\Gamma$

$=\Lambda=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$, $\lambda_1>\lambda_2>\cdots>\lambda_p>0$. Since the statistic f_1 is invariant under a transformation $S=\Gamma G\Gamma'$, T is expressed as

$$T=n\{f_1\boldsymbol{a}'\Gamma G^{-1}\Gamma'\boldsymbol{a}+\boldsymbol{a}'\Gamma G\Gamma'\boldsymbol{a}/f_1-2\} \tag{2}$$
$$=n\{f_1\boldsymbol{e}_1'G^{-1}\boldsymbol{e}_1+\boldsymbol{e}_1'G\boldsymbol{e}_1/f_1-2\} ,$$

where $\boldsymbol{e}'=(1, 0, \cdots, 0)$ and G is a Wishart matrix with distribution $W_p(\Lambda, n)$.

Transforming a positive definite symmetric matrix G to a real symmetric matrix Z by

$$\frac{G}{n}=\Lambda^{1/2}\exp\left(\sqrt{\frac{2}{n}}Z\right)\Lambda^{1/2} , \tag{3}$$

which has been used in Nagao [5], we obtain the asymptotic expansion of the latent root f_1/n in terms of the elements of Z by the use of the formula (3) in Lawley [3] and (3.9) in Sugiura [6] as a following form:

$$\begin{aligned}\frac{f_1}{n}=\lambda_1\Bigg[1&+\sqrt{\frac{2}{n}}z_{11}+\frac{1}{n}\left\{z_{11}^{(2)}+2\sum_{j=2}^{p}a_{1j}z_{1j}^2\right\}\\&+\frac{1}{n}\sqrt{\frac{2}{n}}\left\{\frac{1}{3}z_{11}^{(3)}+2\sum_{j=2}^{p}a_{1j}z_{1j}z_{1j}^{(2)}+2\sum_{j=2}^{p}a_{1j}a_{j1}z_{11}z_{1j}^2\right.\\&\left.+2\sum_{j=2}^{p}a_{1j}^2z_{1j}^2z_{jj}+4\sum_{\substack{j<k\\ j\neq 1\\ k\neq 1}}a_{1j}a_{1k}z_{1j}z_{1k}z_{jk}\right\}+o(1/n^{3/2})\Bigg]\end{aligned} \tag{4}$$

where

$$a_{1j}=\lambda_j/(\lambda_1-\lambda_j) , \qquad j=2, \cdots, p$$
$$z_{\alpha\beta}^{(l)}=(\alpha, \beta)\text{th element of matrix } Z^l , \qquad l=2, 3, 4 .$$

By this transformation, Z has the following asymptotic expansion of a probability density function $f(Z)$:

$$f(Z)=\frac{1}{2^{p/2}\pi^{p(p+1)/4}}\,\mathrm{etr}\left(-\frac{1}{2}Z^2\right)\left\{1+\sqrt{\frac{2}{n}}A+\frac{B}{n}\right\}+O(1/n^{3/2}) , \tag{5}$$

where

$$A=-\frac{1}{6}\,\mathrm{tr}\,Z^3$$

$$B=\frac{p}{12}\,\mathrm{tr}\,Z^2-\frac{1}{12}(\mathrm{tr}\,Z)^2-\frac{1}{12}\,\mathrm{tr}\,Z^4+\frac{1}{36}(\mathrm{tr}\,Z^3)^2$$
$$-\frac{p}{24}(2p^2+3p-1) .$$

Using (3) and (4) we have the asymptotic expansion of T up to the terms of order $1/n$ as follows:

$$T=2\sum_{j=2}^{p} z_{1j}^2+\frac{1}{n}\left\{4\left(\sum_{j=2}^{p} a_{1j}z_{1j}^2\right)^2+4z_{11}^{(2)}\sum_{j=2}^{p} a_{1j}z_{1j}^2-4z_{11}^2\sum_{j=2}^{p} a_{1j}z_{1j}^2\right.$$
$$\left.+(z_{11}^{(2)})^2-\frac{4}{3}z_{11}z_{11}^{(3)}+\frac{1}{3}z_{11}^{(4)}\right\}+o(1/n)\ . \tag{6}$$

The characteristic function of T is defined as

$$M(t)=\mathrm{E}\,[e^{itT}]=\int \exp{(itT)}f(Z)dZ\ . \tag{7}$$

The term of exponential in the integrand of (7) is written as

$$\begin{aligned}
&-\frac{1}{2}\operatorname{tr} Z^2+2it\sum_{j=2}^{p} z_{1j}^2\\
&=-\frac{1}{2}\left\{\sum_{k=1}^{p} z_{kk}^2+2(1-2it)\sum_{k=2}^{p} z_{1k}^2+2\sum_{2\leqq j<k\leqq p} z_{jk}^2\right\}\\
&=-\frac{1}{2}Q(t,Z)\ ,
\end{aligned} \tag{8}$$

which implies

$$\begin{aligned}
\mathrm{E}_t\,(z_{kk}^{2l})&=\frac{(2l)!}{2^l l!}\\
\mathrm{E}_t\,(z_{1k}^{2l})&=\frac{(2l)!}{2^l l!}\left(\frac{1}{2(1-2it)}\right)^l\\
\mathrm{E}_t\,(z_{jk}^{2l})&=\frac{(2l)!}{2^l l!}\left(\frac{1}{2}\right)^l
\end{aligned} \tag{9}$$

and

$$\mathrm{E}_t\,[g(Z)]=\frac{(1-2it)^{(p-1)/2}}{2^{p/2}\pi^{p(p+1)/4}}\int g(Z)\exp\left(-\frac{1}{2}Q(t,Z)\right)dZ\ . \tag{10}$$

By the use of (9) and the laborious calculation, the characteristic function $M(t)$ is expressed as

$$M(t)=\frac{1}{(1-2it)^{f/2}}\left[1+\frac{1}{n}\left\{\left(\frac{1}{4}(p^2-1)+D\right)\frac{1}{(1-2it)^2}\right.\right.$$
$$\left.\left.-\frac{D}{1-2it}-\frac{1}{4}(p^2-1)\right\}+o(1/n)\right] \tag{11}$$

where

$$D=\sum_{j=2}^{p} a_{1j}^2+\frac{1}{2}\left(\sum_{j=2}^{p} a_{1j}\right)^2+\frac{1}{2}(p+1)\sum_{j=2}^{p} a_{1j}\ ,\qquad f=p-1\ .$$

Inverting this $M(t)$, we have the asymptotic expansion of the distribution of T under the null hypothesis H: $\boldsymbol{\gamma}_1=\boldsymbol{a}$ (specified vector):

$$\text{(12)} \qquad \mathrm{P}\{T\leqq x\}=P_f+\frac{1}{n}\left\{\left(\frac{1}{4}(p^2-1)+D\right)P_{f+4}-DP_{f+2} -\frac{1}{4}(p^2-1)P_f\right\}+o(1/n)$$

where $P_f=\mathrm{P}\{\chi_f^2\leqq x\}$ and χ_f^2 is a central chi-square random variable with $f=p-1$ degrees of freedom. The representation (12) shows that we are not able to find a correction factor ρ which makes the terms of order $1/n$ in asymptotic expansion of the distribution of T vanish. (12) is also expressed as follows:

$$\text{(13)} \qquad \mathrm{P}\{T\leqq x\}=P_f+\frac{1}{n}\left[\frac{1}{4}(p^2-1)\{P_{f+4}-P_f\} +D\{P_{f+4}-P_{f+2}\}\right]+o(1/n)\ .$$

Since all terms in D are positive, D is estimated by

$$\text{(14)} \qquad \frac{1}{2}(p^2-1)\frac{\xi_p}{(\xi_p-1)^2}\leqq D\leqq\frac{1}{2}(p^2-1)\frac{\xi_2}{(\xi_2-1)^2}$$

where

$$\xi_p=\lambda_1/\lambda_p>\lambda_1/\lambda_2=\xi_2>1\ .$$

This implies that the contribution of the term of order $1/n$ to the first approximation by P_f becomes critical if the two population latent roots λ_1 and λ_p are close, that is, all latent roots are same magnitude. This fact suggests that we have to check the equality of the latent roots before using this statistic.

Example 1. For $p=3$, $\lambda_1=3$, $\lambda_2=2$, $\lambda_3=1$, D in (11) becomes 99/8. Since the 5% point of a chi-square random variable with $p-1=2$ degrees of freedom is 5.99, we have

$$D\{\mathrm{P}(\chi_6^2\geqq 5.99)-\mathrm{P}(\chi_4^2\geqq 5.99)\}=2.776$$

$$\frac{1}{4}(p^2-1)\{\mathrm{P}(\chi_6^2\geqq 5.99)-\mathrm{P}(\chi_2^2\geqq 5.99)\}=0.748$$

and

$$\mathrm{P}\{T\geqq 5.99\}=0.05+3.524/n+o(1/n)\ .$$

Example 2. For $p=3$, $\lambda_1=10$, $\lambda_2=2$, $\lambda_3=1$, D becomes 0.862. We have

$$P\{T \geqq 5.99\} = 0.05 + 0.941/n + o(1/n) .$$

These examples show that the contribution of the term of order $1/n$ becomes small when the maximum latent root λ_1 is extremely large compared with the remaining latent roots.

Acknowledgement

The author expresses his gratitude to Professor T. W. Anderson for his advices and comments to work this study and thanks are due to a referee for revising an original version.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] Anderson, T. W. (1963). Asymptotic theory for principal component analysis, *Ann. Math. Statist.*, **34**, 122–148.
[2] Kshirsagar, A. M. (1961). The goodness-of-fit of a single (non-isotropic) hypothetical principal component, *Biometrika*, **48**, 397-407.
[3] Lawley, D. N. (1956). Tests of significance for the latent roots of covariance and correlation matrices, *Biometrika*, **43**, 128–136.
[4] Mallows, C. L. (1961). Latent vectors of random symmetric matrices, *Biometrika*, **48**, 133–149.
[5] Nagao, H. (1973). On some test criteria for covariance matrix, *Ann. Statist.*, **1**, 700–709.
[6] Sugiura, N. (1973). Derivatives of the characteristic root of a symmetric or a Hermitian matrix with two applications in multivariate analysis, *Commun. Statist.*, **1**, 393–417.