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THE SET-COMPOUND ONE-STAGE ESTIMATION IN THE NONREGULAR FAMILY OF DISTRIBUTIONS OVER THE INTERVAL $(0, \theta)$

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1. Introduction

Let f be a Lebesgue measurable function with $0<f(\cdot)\leqq 1$. With ξ, Lebesgue measure we define

$$(1.1)\qquad q(\theta)=\left(\int_0^\theta f d\xi\right)^{-1}.$$

Letting $p_\theta=dP_\theta/d\xi$ we denote by $\mathcal{P}(f)$ the family of probability measures given by

$$(1.2)\qquad \mathcal{P}(f)=\{P_\theta \text{ with } p_\theta=q(\theta)(0, \theta)f,\ \forall \theta \in \Omega=(0, \infty)\}$$

where we denote the indicator function of a set A by $[A]$ or simply A itself. The word "nonregular" in title was quoted from Ferguson ([1], p. 130) in which he refers the exponential families of distributions to regular families.

The component problem is an estimation of θ based on X distributed according to P_θ, with squared-error loss. For a prior distribution G on the parameter space Ω, let $R(G)$ denote the Bayes risk vs G in the component problem.

The set-compound decision problem consists of a set of n independent statistical decision problems each having the same structure of the component problem. The loss is taken to be the average of the component losses. Let $X_1,\cdots,X_n$ be n independent random variables with each X_j having the distribution $P_{\theta_j}\in\mathcal{P}(f)$. The jth component decision t_j for θ_j depends on all n observations $\boldsymbol{X}=(X_1,\cdots,X_n)$. Namely we estimate θ_j by $t_j(\boldsymbol{X})$ and thus $\boldsymbol{\theta}$ by $\boldsymbol{t}(\boldsymbol{X})=(t_1(\boldsymbol{X}),\cdots,t_n(\boldsymbol{X}))$. With G_n denoting the empiric distribution of $\theta_1,\cdots,\theta_n$, the modified regret of the decision procedure $\boldsymbol{t}$ is of form

$$(1.3)\qquad D(\boldsymbol{\theta}, \boldsymbol{t})=E\left\{n^{-1}\sum_{j=1}^n (\theta_j-t_j(\boldsymbol{X}))^2\right\}-R(G_n)$$

where E is the expectation with respect to $\boldsymbol{X}$.

If $\sup\{|D(\boldsymbol{\theta}, \boldsymbol{t})|: \boldsymbol{\theta} \in \Omega^n\} = O(n^{-\delta})$ for $\delta > 0$, then we will say $\boldsymbol{t}$ has a rate δ.

Robbins [9] gives an original and general formulation of the compound decision problem. His formulation is the set version of the compound problem rather than the sequence case (cf. Hannan [4], Gilliland [3], Singh [10] etc.). In Nogami [6] the author remarked that a bootstrap procedure based on a direct estimate of the component Bayes procedure vs G_n (or G) is called a one-stage procedure, while a procedure based on component procedures Bayes vs an estimate of G_n (or G) is called a two-stage procedure. Oaten [8] (cf. also [7]) showed in his part II the existence of set-compound two-stage procedures based on a partition of the sample space under some assumptions (among others) on P_θ, the loss function and infinite state space Ω. By now there has been done a few works in the set-compound problem when Ω is infinite. Fox [2] exhibited empirical Bayes risk convergenc $o(1)$ under the uniform distributions over the interval $(0, \theta)$ and $[\theta, \theta+1)$ for $\theta \in (0, \infty)$ and $\theta \in (-\infty, \infty)$, respectively, in the squared-error loss estimation (SELE) problem. This paper is a continuation of the author's Ph. D. thesis [6] and means a generalization and an extension of Fox's work [2], respective to a family $\mathcal{P}(f)$ of distributions over the interval $(0, \theta)$ and to the set-compound SELE problem with rates. In this paper we propose two one-stage procedures.

Section 2 gives an alternative form of a Bayes estimate in the component problem which leads us to two one-stage estimates. In Section 3 we exhibit the one-stage estimate $\hat{\boldsymbol{\phi}}$ with the best rate 1/3. In Section 4 we propose another one-stage estimate $\hat{\hat{\boldsymbol{\phi}}}$ and obtain convergence rates under the additional condition that $1/f$ satisfies Lipshitz condition. Under this condition for f the author developed a one-stage procedure in Chapter III of Nogami [6] in the k-extended problem for the family of distributions over the interval $[\theta, \theta+1)$ and has obtained a rate 1/4 by usage of Theorem 2 of Hoeffding [5].

We might observe that the method appeared in this paper can improve the convergence rate up to 1/3. (This is done already.)

The different device from the author's previous work [6] is taken for the parameter space Ω; instead of assuming the boundedness of Ω we assume that for each n, all $\theta_1, \cdots, \theta_n$ lie in the bounded interval $(0, \beta_n]$ where $\beta_n \to \infty$ as $n \to \infty$, and assume that $q(\theta) \leqq m_n$ for all $\theta \in (0, \beta_n]$ where $m_n \to \infty$ as $n \to \infty$. This idea originally comes from Singh [10].

Notational conventions. P_j, p_j and $\boldsymbol{P}$ abbreviate P_{θ_j}, p_{θ_j} and $\times_{j=1}^{n} P_j$, respectively. A distribution function also represents the corresponding measure. We often let $P(h)$ or $P(h(z))$ denote $\int h(z)dP(z)$. G abbreviates the empiric distribution G_n of $\theta_1,\cdots,\theta_n$. For any function h, $h]_a^b$ means $h(b)-h(a)$. $\vee$ and $\wedge$ denote the supremum and the infimum, respectively. $\doteq$ denote the defining property. When we refer to (a.b) in Section a, we simply write (b). $\boldsymbol{P}_x$ means the conditional expectation of $X_1,\cdots,X_{j-1},X_{j+1},\cdots,X_n$ given $X_j \doteq x$. As usual, let $\bar{z}=n^{-1}\sum_{i=1}^{n} z_i$.

2. The Bayes estimate in the component problem

We observe a sample of size n, $X_1,\cdots,X_n$ with each X_j taken from $P_j \in \mathscr{P}(f)$. Let $0<\bigvee_{i=1}^{n}\theta_i \leqq \beta < +\infty$ where $\beta \doteq \beta_n \to +\infty$ as $n\to\infty$. Assume

$$q(\theta)\leqq m \qquad \text{for all } \theta\in(0,\beta] \tag{2.1}$$

where $(0<)$ $m\doteq m_n \to +\infty$ as $n\to\infty$ and $1\leqq m\beta$. Then, by this and the definition (1.1) of q,

$$f(y)\geqq 1/(\beta m) \qquad \text{for all } y\in(0,\beta]\,. \tag{2.2}$$

We also note that (2.1) implies

$$(0<)m^{-1}\leqq \bigwedge_{i=1}^{n}\theta_i\,.$$

As the Bayes response vs G in the component problem we have the version of the conditional expectation:

$$\phi(y)=G(\theta p_\theta(y))/G(p_\theta(y))=\int_{y+}^{\infty}\theta q(\theta)dG(\theta)\Big/\int_{y+}^{\infty}q(\theta)dG(\theta) \tag{2.3}$$

where the affix + is taken to represent the right limit of the integration. The following lemma is the analogue of Lemma 7 in Section 1.1 of Nogami [6].

LEMMA 2.1. *Let τ be a signed measure and g a measurable function. Let $I=(y,\infty)$ be an interval such that $\int Ig\,d\tau\neq 0$. Define by τ_y the signed measure with density $Ig\Big/\int Ig\,d\tau$ wrt τ. Then,*

$$\int s\,d\tau_y(s)=y+\int_0^\infty \tau_y(t+y,\infty)dt\,.$$

PROOF. By Fubini's theorem applied to the lhs of the second equality below,

$$\int (s-y)d\tau_y(s)=\int\int_0^{s-y}dt\,d\tau_y(s)=\int_0^\infty \tau_y(t+y,\infty)dt\,.$$

Let Q be the measure with the density q wrt G, then by above Lemma 2.1 applied to rhs (3),

$$\phi(y)=y+\psi(y) \tag{2.4}$$

where

$$\psi(y)=\int_0^{\infty} Q(t+y,\infty)dt/Q(y,\infty)\ .$$

In Sections 3 and 4 we shall obtain two estimates of $\boldsymbol{\theta}$ through estimating $\boldsymbol{\phi}$.

3. The estimate $\hat{\boldsymbol{\phi}}$

In this section we shall propose the estimate $\hat{\boldsymbol{\phi}}$ with a rate 1/3 whose component at y is of form (2.4) with Q there replaced by its estimates.

For simplicity we let $\bar{u}(y)=Q(y,\infty)$. We first estimate $\bar{u}(y)$ by

$$\hat{\bar{u}}(y)=(nh)^{-1}\sum_{j=1}^{n}[y-h<X_j\leq y]/f(X_j) \tag{3.1}$$

where $h\doteq h_n\to 0$ as $n\to\infty$. Then, in view of (2.4), estimate $\phi(y)$ by

$$\hat{\phi}(y)=(y+\hat{\psi}(y))\wedge\beta \tag{3.2}$$

where

$$\hat{\psi}(y)=\int_0^{\infty}\hat{\bar{u}}(y+t)dt/\hat{\bar{u}}(y)\ . \tag{3.3}$$

Note that the jth component $\hat{\phi}_j(\boldsymbol{X})$ of $\hat{\boldsymbol{\phi}}$ at $\boldsymbol{X}$ equals $\hat{\phi}(X_j)$.

From (1.3)

$$nD(\boldsymbol{\theta},\hat{\boldsymbol{\phi}})=\sum_{j=1}^{n}\boldsymbol{P}\{(\theta_j-\hat{\phi}(X_j))^2-(\theta_j-\phi(X_j))^2\}\ . \tag{3.4}$$

Since $a^2-b^2=(a-b)(a+b)$ and all of the θ_j, $\phi(X_j)$ and $\hat{\phi}(X_j)$ are in the interval $(0,\beta]$,

$$(2\beta)^{-1}n|D(\boldsymbol{\theta},\hat{\boldsymbol{\phi}})|\leq\sum_{j=1}^{n}\boldsymbol{P}|\hat{\phi}(X_j)-\phi(X_j)|\ . \tag{3.5}$$

To get a bound of n^{-1}(4) we shall obtain a bound of rhs (5).

Fix j and let $x=X_j$. Applying Lemma A.2 of Singh [10] and weakening the resulted bound leads to

$$\boldsymbol{P}_x|\hat{\phi}(x)-\phi(x)|\leq 2(\bar{u}(x))^{-1}\left\{\int_0^{\beta+1}\boldsymbol{P}_x|\bar{u}(x+t)-\hat{\bar{u}}(x+t)|dt\right. \tag{3.6}$$

$$+2\beta \boldsymbol{P}_x |\bar{u}(x)-\bar{\hat{u}}(x)|\Big\} .$$

But, with $\bar{u}_j \doteq (n-1)^{-1} \sum_{j\neq i=1}^{n} u_i$,

$$\begin{aligned}(3.7)\qquad & \boldsymbol{P}_x|\bar{u}(x+t)-\bar{\hat{u}}(x+t)|-\boldsymbol{P}_x|\bar{u}_j(x+t)-\bar{\hat{u}}_j(x+t)| \\ & \leqq n^{-1}\{u_j(x+t)+(hf(x+t))^{-1}\}\leqq m(1+\beta h^{-1})n^{-1}\end{aligned}$$

where the last inequality follows by (1.1) and $(m\beta)^{-1}\leqq f(x)$.

Before obtaining the bound for rhs (5) we introduce the following lemma:

LEMMA 3.1.

$$(3.8)\qquad \sum_{j=1}^{n} P_j(\bar{u}(X_j))^{-1}\leqq n\beta .$$

PROOF. Since $f\leqq 1$ applied to the lhs of the inequality,

$$\text{lhs (8)}=\int_{\{\bar{u}(y)>0\}} nf(y)dy\leqq n\beta .$$

By two applications of (7) and an application of Lemma 3.1 and weakening the resulted bound we obtain

$$\begin{aligned}(3.9)\quad & \{2(3\beta+1)\}^{-1}(\text{rhs (5)}) \\ & \leqq \bigvee_{t\geqq 0} \sum_{j=1}^{n} P_j\{(\bar{u}(X_j))^{-1}\boldsymbol{P}_x|\bar{u}_j(X_j+t)-\bar{\hat{u}}_j(X_j+t)|\}+m(1+\beta h^{-1})\beta .\end{aligned}$$

Applying a triangular inequality and then Hölder's inequality results in the inequality

$$(3.10)\qquad \boldsymbol{P}_x|\bar{u}_j(y)-\bar{\hat{u}}_j(y)|\leqq|\bar{u}_j(y)-\boldsymbol{P}_x\bar{\hat{u}}_j(y)|+\sigma_n(y)$$

where $\sigma_n^2(y)$ = variance of $\bar{\hat{u}}_j(y)$. To get an upper bound of the first term of rhs (9) we shall obtain bounds for $\bigvee_y \sigma_n(y)$ and $\bigvee_{t\geqq 0} \sum_{j=1}^{n}$ {(first term of rhs (10) at $X_j+t)/\bar{u}(X_j)$} and utilize Lemma 3.1.

LEMMA 3.2. *For every y,*

$$\sigma_n(y)\leqq m\sqrt{\beta}\,((n-1)h)^{-1/2} .$$

PROOF. By the definition of $\sigma_n^2(y)$,

$$\begin{aligned}(3.11)\qquad & ((n-1)h)^2\sigma_n^2(y)\leqq \sum_{j\neq i=1}^{n} \boldsymbol{P}_x(\hat{u}_i(y))^2\leqq(n-1)\int_{y-h}^{y} \bar{u}(z)/f(z)dz \\ & \leqq m^2\beta(n-1)h\end{aligned}$$

where the last inequality follows by $\bar{u}_j(z)\leqq m$ and $1/f(z)\leqq\beta m$.

LEMMA 3.3. *For all* $t \geqq 0$,

$$\sum_{j=1}^{n} P_j\{|\bar{u}_j(X_j+t)-P_x\hat{\bar{u}}_j(X_j+t)|/\bar{u}(X_j)\} \leqq (1/2)mhn . \tag{3.12}$$

PROOF. Since $P_x\hat{\bar{u}}_j(y)=h^{-1}\int_{y-h}^{y}\bar{u}_j(z)dz=\int_0^1\bar{u}_j(y-hs)ds$, for every $t \geqq 0$ lhs (12) equals to

$$\sum_{j=1}^{n}\int_0^\infty\left|\int_0^1 \bar{u}_j]_{x+t-hs}^{x+t}ds\right|p_j(x)/\bar{u}(x)dx \tag{3.13}$$

or equivalently,

$$\sum_{j=1}^{n}\int_0^\infty\int_0^1(n-1)^{-1}\sum_{j\neq i=1}^{n}q(\theta_i)[x+t-hs<\theta_i\leqq x+t]ds\,p_j(x)/\bar{u}(x)dx .$$

Thus, interchanging integrations and also average operations over respective i and j gives

$$\text{lhs}\,(12)=\sum_{i=1}^{n}q(\theta_i)\int_0^1\int_0^\infty[\theta_i-t\leqq x<\theta_i-t+hs](n-1)^{-1} \cdot\sum_{i\neq j=1}^{n}p_j(x)/\bar{u}(x)\,dxds . \tag{3.14}$$

Since $(n-1)^{-1}\sum_{i\neq j=1}^{n}p_j(x)/\bar{u}(x)\leqq f(x)\leqq 1$, by as imple computation and (1.1), $(14)\leqq(1/2)mhn$.

We go back to the inequality (9). In view of (10) and from Lemmas 3.2 and 3.3 together with an application of Lemma 3.1 we get

$$(\text{first term of rhs}\,(9))\leqq m\beta^{3/2}((n-1)h)^{-1/2}n+(1/2)mhn . \tag{3.15}$$

Therefore, in view of (5), we finally obtain

THEOREM 3.1. *If* $0<\bigvee_{i=1}^{n}\theta_i\leqq\beta$ *and* q *satisfies* (2.1), *then for all* $\boldsymbol{\theta}\in[m^{-1},\beta]^n$,

$$|D(\boldsymbol{\theta},\hat{\boldsymbol{\phi}})|\leqq 4\beta(3\beta+1)\{2m\beta^{3/2}(nh)^{-1/2}+(1/2)mh+m(1+\beta h^{-1})\beta n^{-1}\} .$$

Two corollaries below are presented to show that by certain choices of m and β $\hat{\boldsymbol{\phi}}$ in Theorem 3.1 can have a rate 1/3− (in Corollary 3.1) and a rate 1/3 (in Corollary 3.2) which is the best rate obtained from the bound in Theorem 3.1.

COROLLARY 3.1. *When* $m=\beta=(\log n)^{2/27}$ *under the same assumption of Theorem* 3.1, *it follows that with a choice of* $h=n^{-1/3}$

$$|D(\boldsymbol{\theta},\hat{\boldsymbol{\phi}})|=O((n^{-1}\log n)^{1/3}) , \qquad \text{uniformly in } \boldsymbol{\theta}\in[m^{-1},\beta]^n .$$

COROLLARY 3.2. *Let m, β and q be as in Theorem* 3.1. *When m and β are positive constants such that $1 \leqq m\beta$, by a choice of $h = n^{-1/3}$*

$$|D(\boldsymbol{\theta}, \hat{\boldsymbol{\phi}})| = O(n^{-1/3}), \qquad \textit{uniformly in } \boldsymbol{\theta} \in [m^{-1}, \beta]^n .$$

In the next section we shall exhibit another one-stage procedure with a rate 1/3.

4. The estimate $\hat{\hat{\boldsymbol{\phi}}}$

Let us set the additional condition on f such that for a given finite positive constant M,

$$\vee \{(y-z)^{-1}|(f(y))^{-1} - (f(z))^{-1}| : \; y < z\} \leqq M . \tag{4.1}$$

This condition will be used to attain a convergence rate for the modified regret $D(\boldsymbol{\theta}, \hat{\hat{\boldsymbol{\phi}}})$.

The structure of $\hat{\hat{\boldsymbol{\phi}}}$ is similar to $\hat{\boldsymbol{\phi}}$ in Section 3. We first estimate $\bar{u}(y)$ by

$$\hat{\bar{v}}(y) = (nh)^{-1} \sum_{j=1}^{n} [y-h < X_j \leqq y] / f(y) \tag{4.2}$$

where h is as in (3.1). In view of (2.4), estimate $\phi(y)$ by

$$\hat{\hat{\phi}}(y) = (y + \hat{\hat{\varphi}}(y)) \wedge \beta \tag{4.3}$$

where

$$\hat{\hat{\varphi}}(y) = \int_0^\infty \hat{\bar{v}}(y+t) dt / \hat{\bar{v}}(y) . \tag{4.4}$$

Note that the jth component $\hat{\hat{\phi}}_j(\boldsymbol{X})$ of $\hat{\hat{\boldsymbol{\phi}}}(\boldsymbol{X})$ is $\hat{\hat{\phi}}(X_j)$.

To get an upper bound of $|D(\boldsymbol{\theta}, \hat{\hat{\boldsymbol{\phi}}})|$ we proceed in the same way as in Section 3. We use inequalities (3.5) through (3.10) by replacing $\hat{\varphi}$ and $\hat{u}$ by $\hat{\hat{\varphi}}$ and $\hat{v}$, respectively. Let $\sigma^2(y)$ be variance of $\hat{\bar{v}}_j(y)$. What we need is to obtain bounds for $\bigvee_y \sigma(y)$ and $\bigvee_{t \geqq 0} \sum_{j=1}^{n} P_j\{|\bar{u}_j(X_j+t) - \boldsymbol{P}_x \hat{\bar{v}}_j(X_j+t)| / \bar{u}(X_j)\}$.

LEMMA 4.1. *For every y,*

$$\sigma(y) \leqq m^{3/2} \beta ((n-1)h)^{-1/2} .$$

PROOF. As in the proof of Lemma 3.2,

$$((n-1)h)^2 \sigma^2(y) \leqq ((n-1)/f^2(y)) \int_{y-h}^{y} \bar{u}_j(z) f(z) dz \leqq (n-1) m^3 \beta^2 h$$

where the last inequality follows by $f \leqq 1$, (2.1) and two usages of (2.2).

LEMMA 4.2. *For all* $t \geqq 0$,

$$(4.5)\quad \sum_{j=1}^{n} P_j\{|\bar{u}_j(X_j+t)-\boldsymbol{P}_x\bar{v}_j(X_j+t)|/\bar{u}(X_j)\} \leqq (1/2)mhn\{1+M(\beta+1)\} .$$

PROOF. Before proving the lemma, note that Lipshitz condition (1) for $1/f$ implies that

$$(4.6)\qquad |1-(f(y)/f(z))| \leqq M|y-z| \qquad \text{for real } y \text{ and } z .$$

Since $\boldsymbol{P}_x\bar{v}_j(y) = h^{-1}\int_{y-h}^{y} \bar{u}_j(z)(f(z)/f(y))\,dz = \int_0^1 \bar{u}_j(y-sh)(f(y-sh)/f(y))\,ds$, it follows that for every $t \geqq 0$,

$$(4.7)\qquad \begin{aligned} \text{lhs}\,(5) &\leqq \sum_{j=1}^{n}\int_0^\infty\int_0^1 \{|[\bar{u}_j]_{x+t-sh}^{x+t}|+\bar{u}_j(x+t-sh) \\ &\qquad \cdot |1-(f(x+t-sh)/f(x+t))|\}\,dsdx \\ &\leqq \sum_{j=1}^{n}\int_0^\infty\int_0^1 |[\bar{u}_j]_{x+t-sh}^{x+t}|\,ds\; p_j(x)/\bar{u}(x)\,dx \\ &\qquad +(1/2)Mmhn\int_{\{\bar{u}(x)>0\}} f(x)dx \end{aligned}$$

where the last inequality follows from (6), $\bar{u}_j(\cdot) \leqq m$ and $\sum_{j=1}^{n} p_j(x)/\bar{u}(x) = nf(x)$ applied to double integrations of the second term in the curly bracket. Therefore, the bound in the lemma follows from the bound in Lemma 3.3 (notice lhs (3.12)=(3.13)) and $f \leqq 1$.

As in (3.15),

$$\begin{aligned} \bigvee_{t\geqq 0}\sum_{j=1}^{n} P_j\{(\bar{u}(X_j))^{-1}\boldsymbol{P}_x|\bar{u}_j(X_j+t)-\bar{v}(X_j+t)|\} \\ \leqq m^{3/2}\beta^2((n-1)h)^{-1/2}n+(1/2)mhn(1+M(\beta+1)) . \end{aligned}$$

Thus, we get the following theorem:

THEOREM 4.1. *Under the same assumption as Theorem* 3.1 *plus* (4.1),

$$\begin{aligned} |D(\boldsymbol{\theta}, \hat{\boldsymbol{\phi}})| \leqq 4\beta(3\beta+1)\{2m^{3/2}\beta^2(nh)^{-1/2}+(1/2)mh(1+M(\beta+1)) \\ +m(1+\beta h^{-1})\beta n^{-1}\} \end{aligned}$$

for all $\boldsymbol{\theta} \in [m^{-1}, \beta]^n$.

Remark. From the bound in Theorem 4.1 we can see that by a choice of $h=(m\beta^2n^{-1})^{1/3}$,

$$|D(\boldsymbol{\theta}, \hat{\boldsymbol{\phi}})| = O((m^4\beta^{11}n^{-1})^{1/3}) , \qquad \text{uniformly in } \boldsymbol{\theta} \in [m^{-1}, \beta]^n .$$

Thus, if m and β are constants such that $1 \leqq m\beta$, then we get $\hat{\hat{\phi}}$ with a rate 1/3 which is the best rate attained from the bound in Theorem 4.1.

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