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ON THE REDUCTION TO A COMPLETE CLASS IN MULTIPLE DECISION PROBLEMS (2)

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Summary

As a criterion for the reduction to a complete class of decision rule in case where actions, samples and states are finite in number, "regret-relief ratio" criterion and "incremental loss-gain ratio" criterion were introduced in 2-state of nature case [2]. In this paper, "generalized regret-relief ratio" criterion in k-state of nature case is introduced as an extension of "regret-relief ratio" criterion and its usefulness is shown with an example.

1. Introduction

Consider a following decision problem. Let $L(\theta, a)$ be a loss function incurred if an action a is taken when the state of nature is θ and let x be an information about to guess the true state of nature. In this paper, we have assumed only the case the state space Θ contains k points, the action space A contains n points and the sample space contains m points. Hence, $\Theta=\{\theta_1,\cdots,\theta_k\}$, $A=\{a_1,\cdots,a_n\}$ and $X=\{x_1,\cdots,x_m\}$. For each θ, there is a corresponding cumulative distribution function $F(x|\theta)$, which represents the distribution of X when the true state is θ.

A non-randomized decision rule $d\in D$ and a randomized decision rule $\delta\in D^*$ are defined respectively as follows,

$$d(x)=a \qquad \text{and} \qquad \delta=\sum_{i=1}^{t}\pi_i d_i$$

where $\pi_i\geqq 0$ for all i and $\sum_{i=1}^{t}\pi_i=1$. For any d and δ, the risk can be defined by

$$R(\theta, d)=\mathrm{E}_\theta L(\theta, d(x))=\int L(\theta, d(x))dF(x|\theta)=\sum_{i=1}^{m}L(\theta, d(x_i))f(x_i|\theta)$$

and

$$R(\theta, \delta)=\sum_{i=1}^{t} \pi_i R(\theta, d_i) .$$

Let $w_i \in W$ be a prior probability of θ_i. If the prior distribution is known, one may use the minimal risk criterion for selecting the best decision rule requires to minimize the expected risk defined by

$$r(\delta)=\mathrm{E}_w R(\theta, \delta)=\sum_{s=1}^{k} w_s R(\theta_s, \delta) .$$

However, if W is not known, a reasonable way in choosing a decision rule is to use the relation of natural ordering. A decision rule δ is said to be better than a decision rule δ^* if $R(\theta, \delta)\leqq R(\theta, \delta^*)$ for all $\theta \in \Theta$ and $R(\theta, \delta)<R(\theta, \delta^*)$ for at least one $\theta \in \Theta$. (Natural ordering)

In such a case we say that δ dominates δ^*. A class C of decision rules, $C\subset D^*$, is said to be complete, if given any rule $\delta \in D^*$ not in C, there exists a rule $\delta_0 \in C$ that is better than δ.

For the reduction to the complete class of decision rule, we have introduced a new criterion, that is, "regret-relief ratio criterion," in 2-state of nature case, [2]. The basic idea of this criterion is following; Let $\underline{l}(\theta, a_k)$ be regret of action a_k in θ which is caused by failing to take the best action in θ and let $\bar{l}(\theta, a_k)$ be relief of action a_k in θ which is given by avoiding the worst action in θ. Hence,

$$\underline{l}(\theta, a_k)=L(\theta, a_k)-\min_{a\in A} L(\theta, a) , \qquad \text{and}$$

$$\bar{l}(\theta, a_k)=\max_{a\in A} L(\theta, a)-L(\theta, a_k) .$$

A reasonable decision maker may not take action a_k if $\bar{l}(\theta, a_k)<l(\theta, a_k)$. In this case, regret-relief ratio $r(\theta, a_k)$ is greater than 1,

$$r(\theta, a_k)=\frac{l(\theta, a_k)}{\bar{l}(\theta, a_k)}>1 .$$

In 2-state of nature case, this idea is summarized as "regret-relief ratio criterion" (Theorem 2 in [2]).

The purpose of the paper is to extend this idea from 2-state of nature case to k-state of nature case.

2. Generalized regret-relief ratio criterion

A new notation is introduced to simplify our discussion. In case that $\Theta=\{\theta_1,\cdots, \theta_k\}$ and $A=\{a_1,\cdots, a_n\}$, let $a_{j(s)}$ be any action with $\min_{a\in A} L(\theta_s, a)$. For instance, in 3-state and 5-action case like

$$L(\theta_1, a_1)<L(\theta_1, a_2)<L(\theta_1, a_3)<L(\theta_1, a_4)<L(\theta_1, a_5)$$

$$L(\theta_2, a_1) > L(\theta_2, a_2) > L(\theta_2, a_3) < L(\theta_2, a_4) < L(\theta_2, a_5)$$

$$L(\theta_3, a_1) > L(\theta_3, a_2) > L(\theta_3, a_3) > L(\theta_3, a_4) > L(\theta_3, a_5)$$

one would know

$$a_{j(1)} = a_1 , \quad a_{j(2)} = a_3 \text{ and } a_{j(3)} = a_5 .$$

Let us define generalized regret $\underline{l}^*(\theta, a)$, generalized relief $\bar{l}^*(\theta, a)$ and generalized regret-relief ratio $\gamma^*(\theta, a)$ of action a as follows.

DEFINITION 1 (Generalized regret). The magnitude of regret caused by failing to take an action $a_{j(s)}$ which satisfies

$$L(\theta, a_j) > L(\theta, a_{j(s)})$$

is called generalized regret $\underline{l}_s^*(\theta, a_j)$ of action a_j and defined by

$$\underline{l}_s^*(\theta, a_j) = L(\theta, a_j) - L(\theta, a_{j(s)}) . \tag{2.1}$$

DEFINITION 2 (Generalized relief). The magnitude of relief given by avoiding an action $a_{j(s)}$ which satisfies

$$L(\theta, a_j) < L(\theta, a_{j(s)})$$

is called generalized relief $\bar{l}_s^*(\theta, a_j)$ of action a_j and defined by

$$\bar{l}_s^*(\theta, a_j) = L(\theta, a_{j(s)}) - L(\theta, a_j) . \tag{2.2}$$

DEFINITION 3 (Generalized regret-relief ratio). The generalized regret-relief ratio $\gamma_{s,t}^*(\theta, a_j)$ of action a_j is defined by

$$\gamma_{s,t}^*(\theta, a_j) = \frac{\underline{l}_s^*(\theta, a_j)}{\bar{l}_t^*(\theta, a_j)} . \tag{2.3}$$

Using generalized regret-relief ratio, we have obtained the following theorem.

THEOREM. *In the case that* $\Theta = \{\theta_1, \cdots, \theta_k\}$, $X = \{x_1, \cdots, x_m\}$ *and* $A = \{a_1, \cdots, a_n\}$, *assume that* $f(x_i | \theta) > 0$ $(i = 1, \cdots, m;\ \theta \in \Theta)$ *and that there exist integers* $1 = n(1) < \cdots < n(k) = n$ *such that*

$$\begin{aligned} &L(\theta_s, a_h) > L(\theta_s, a_{h'}) \qquad h < h' < n(s) \quad \textit{and} \\ &L(\theta_s, a_h) < L(\theta_s, a_{h'}) \qquad n(s) < h < h' . \end{aligned} \tag{2.4}$$

Then, for each a_j $(\neq a_{j(s)},\ s = 1, \cdots, k)$, *the following holds. If the condition*

$$\gamma_{t,t+1}^*(\theta_\alpha, a_j) \cdot \gamma_{t,t+1}^*(\theta_\beta, a_j) > 1 \tag{2.5}$$

is satisfied, for integers t, α and β such that

$$j(t)<j<j(t+1)\ ,$$

$$\gamma^*_{t,t+1}(\theta_\alpha, a_j)=\min_{s\leq t}\{\gamma^*_{t,t+1}(\theta_s, a_j)\}\ , \tag{2.6}$$

and

$$\gamma^*_{t,t+1}(\theta_\beta, a_j)=\min_{s>t}\{\gamma^*_{t,t+1}(\theta_s, a_j)\}\ . \tag{2.7}$$

then any non-randomized decision rule which takes action a_j is dominated by some randomized decision rule.

PROOF. Since any non-randomized decision rule $d \in D$ is defined by assigning $d(x_i)=a^i$ $(\in A)$ $(i=1, 2, \cdots, m)$, we write a non-randomized decision rule d in the form of an ordered m-tuple

$$d\equiv(a^1, \cdots, a^m)\ .$$

For $d=(a^1, \cdots, a^m)$, i $(1\leqq i\leqq m)$ and $a_j \in A$, we define a new non-randomized decision rule

$$d^{(i)} * a_j=(a^1, \cdots, a^{i-1}, \underset{\substack{\uparrow \\ i\text{th place}}}{a_j}, a^{i+1}, \cdots, a^m)\ .$$

That is, a non-randomized decision rule with a_j in ith place and remaining $a^1, \cdots, a^{i-1}, a^{i+1}, \cdots, a^m$ unchanged from d. We have to show, for each i $(i=1, \cdots, m)$, that some randomized decision rule $\delta^{(i)}$ which is some mixture of $d^{(i)} * a_{j(t)}$ and $d^{(i)} * a_{j(t+1)}$ dominates $d^{(i)} * a_j$. Note that $a_{j(t)}$ and $a_{j(t+1)}$ are the best actions when θ_t and θ_{t+1} are true respectively. First, there exists a number q $(0<q<1)$ which satisfies the following equation;

$$R(\theta_\alpha, d^{(i)} * a_j)=(1-q)\cdot R(\theta_\alpha, d^{(i)} * a_{j(t)})+q\cdot R(\theta_\alpha, d^{(i)} * a_{j(t+1)})\ . \tag{2.8}$$

For let q be

$$\begin{aligned} q&=\frac{R(\theta_\alpha, d^{(i)} * a_j)-R(\theta_\alpha, d^{(i)} * a_{j(t)})}{R(\theta_\alpha, d^{(i)} * a_{j(t+1)})-R(\theta_\alpha, d^{(i)} * a_{j(t)})} \\ &=\frac{f(x_i|\theta_\alpha)\{L(\theta_\alpha, a_j)-L(\theta_\alpha, a_{j(t)})\}}{f(x_i|\theta_\alpha)\{L(\theta_\alpha, a_{j(t+1)})-L(\theta_\alpha, a_{j(t)})\}} \\ &=\frac{L(\theta_\alpha, a_j)-L(\theta_\alpha, a_{j(t)})}{L(\theta_\alpha, a_{j(t+1)})-L(\theta_\alpha, a_{j(t)})}\ . \end{aligned}$$

Since $L(\theta_\alpha, a_{j(t)})<L(\theta_\alpha, a_j)<L(\theta_\alpha, a_{j(t+1)})$, q satisfies (2.8) and $0<q<1$.

Suppose $\delta^{(i)}$ is the randomized decision rule of $d^{(i)} * a_{j(t)}$ and $d^{(i)} * a_{j(t+1)}$ in the ratio of $(1-q):q$. Then

$$R(\theta_\alpha, \delta^{(i)}) = (1-q)\cdot R(\theta_\alpha, d^{(i)} * a_{j(t)}) + q\cdot R(\theta_\alpha, d^{(i)} * a_{j(t+1)})$$
$$= R(\theta_\alpha, d^{(i)} * a_j)$$

and, for $s \neq \alpha$

$$\begin{aligned} R(\theta_s, \delta^{(i)}) &= (1-q)\cdot R(\theta_s, d^{(i)} * a_{j(t)}) + q\cdot R(\theta_s, d^{(i)} * a_{j(t+1)}) \\ &= \frac{1}{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_{j(t)})} \\ &\quad \cdot [\{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_j)\} R(\theta_s, d^{(i)} * a_{j(t)}) \\ &\quad + \{L(\theta_\alpha, a_j) - L(\theta_\alpha, a_{j(t)})\} R(\theta_s, d^{(i)} * a_{j(t+1)})] \,. \end{aligned}$$

Hence

$$\begin{aligned} (2.9)\quad & R(\theta_s, d^{(i)} * a_j) - R(\theta_s, \delta^{(i)}) \\ &= \frac{1}{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_{j(t)})} [\{L(\theta_\alpha, a_j) - L(\theta_\alpha, a_{j(t)})\} \\ &\quad \cdot \{R(\theta_s, d^{(i)} * a_j) - R(\theta_s, d^{(i)} * a_{j(t+1)})\} - \{L(\theta_\alpha, a_{j(t+1)}) \\ &\quad - L(\theta_\alpha, a_j)\} \{R(\theta_s, d^{(i)} * a_{j(t)}) - R(\theta_s, d^{(i)} * a_j)\}] \\ &= \frac{f(x_i|\theta_s)}{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_{j(t)})} [\{L(\theta_\alpha, a_j) - L(\theta_\alpha, a_{j(t)})\} \\ &\quad \cdot \{L(\theta_s, a_j) - L(\theta_s, a_{j(t+1)})\} - \{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_j)\} \\ &\quad \cdot \{L(\theta_s, a_{j(t)}) - L(\theta_s, a_j)\}] \\ &= \frac{f(x_i|\theta_s)}{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_{j(t)})} \{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_j)\} \\ &\quad \cdot \{L(\theta_s, a_{j(t)}) - L(\theta_s, a_j)\} \\ &\quad \cdot \left\{ \frac{L(\theta_\alpha, a_j) - L(\theta_\alpha, a_{j(t)})}{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_j)} \cdot \frac{L(\theta_s, a_j) - L(\theta_s, a_{j(t+1)})}{L(\theta_s, a_{j(t)}) - L(\theta_s, a_j)} - 1 \right\} . \end{aligned}$$

i) *In case* $s \leqq t$.

Since

$$\frac{L(\theta_\alpha, a_j) - L(\theta_\alpha, a_{j(t)})}{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_j)} = \min_{s \leqq t} \left\{ \frac{L(\theta_s, a_j) - L(\theta_s, a_{j(t)})}{L(\theta_s, a_{j(t+1)}) - L(\theta_s, a_j)} \right\}$$

then for $s \leqq t$ $(s \neq \alpha)$, we have

$$\frac{L(\theta_\alpha, a_j) - L(\theta_\alpha, a_{j(t)})}{L(\theta_\alpha, a_{j(t+1)}) - L(\theta_\alpha, a_j)} < \frac{L(\theta_s, a_j) - L(\theta_s, a_{j(t)})}{L(\theta_s, a_{j(t+1)}) - L(\theta_s, a_j)} \,.$$

Both sides are positive. So

$$\frac{L(\theta_\alpha, a_j)-L(\theta_\alpha, a_{j(t)})}{L(\theta_\alpha, a_{j(t+1)})-L(\theta_\alpha, a_j)}\cdot\frac{L(\theta_s, a_{j(t+1)})-L(\theta_s, a_j)}{L(\theta_s, a_j)-L(\theta_s, a_{j(t)})}<1 ,$$

and hence

$$\frac{L(\theta_\alpha, a_j)-L(\theta_\alpha, a_{j(t)})}{L(\theta_\alpha, a_{j(t+1)})-L(\theta_\alpha, a_j)}\cdot\frac{L(\theta_s, a_j)-L(\theta_s, a_{j(t+1)})}{L(\theta_s, a_{j(t)})-L(\theta_s, a_j)}-1<0 .$$

Since $f(x_i|\theta_s)>0$, $L(\theta_\alpha, a_{j(t+1)})-L(\theta_\alpha, a_{j(t)})>0$, $L(\theta_\alpha, a_{j(t+1)})-L(\theta_\alpha, a_j)>0$ and $L(\theta_s, a_{j(t)})-L(\theta_s, a_j)<0$. In (2.9), we get for $s\leqq t$

$$R(\theta_s, d^{(i)} * a_j)-R(\theta_s, \delta^{(i)})>0 .$$

ii) *In case* $s>t$.

From the condition

$$\gamma^*_{t,t+1}(\theta_\alpha, a_j)\cdot\gamma^*_{t,t+1}(\theta_\beta, a_j)>1$$

we get

$$\frac{L(\theta_\alpha, a_j)-L(\theta_\alpha, a_{j(t)})}{L(\theta_\alpha, a_{j(t+1)})-L(\theta_\alpha, a_j)}\cdot\frac{L(\theta_\beta, a_j)-L(\theta_\beta, a_{j(t+1)})}{L(\theta_\beta, a_{j(t)})-L(\theta_\beta, a_j)}-1>0 .$$

Since

$$\frac{L(\theta_\beta, a_j)-L(\theta_\beta, a_{j(t+1)})}{L(\theta_\beta, a_{j(t)})-L(\theta_\beta, a_j)}=\min_{s>t}\left\{\frac{L(\theta_s, a_j)-L(\theta_s, a_{j(t+1)})}{L(\theta_s, a_{j(t)})-L(\theta_s, a_j)}\right\} ,$$

for $s>t$, we have

$$\frac{L(\theta_\alpha, a_j)-L(\theta_\alpha, a_{j(t)})}{L(\theta_\alpha, a_{j(t+1)})-L(\theta_\alpha, a_j)}\cdot\frac{L(\theta_s, a_j)-L(\theta_s, a_{j(t+1)})}{L(\theta_s, a_{j(t)})-L(\theta_s, a_j)}-1>0 .$$

Because $L(\theta_s, a_{j(t)})-L(\theta_s, a_j)>0$, we get for $s>t$

$$R(\theta_s, d^{(i)} * a_j)-R(\theta_s, \delta^{(i)})>0 .$$

Hence

$$R(\theta_\alpha, d^{(i)} * a_j)=R(\theta_\alpha, \delta^{(i)}) \qquad \text{for some } \alpha$$

and

$$R(\theta_s, d^{(i)} * a_j)>R(\theta_s, \delta^{(i)}) \qquad \text{for } s\neq\alpha ,$$

which completes the proof.

3. Some example

Example. Consider a decision problem of 4-state, m-sample and 8-action. A loss function is given by Table 1 and $f(x_i|\theta_s)>0$ for $i=$

$1, \cdots, m,\ s=1, \cdots, 4.$

Table 1. $L(\theta, a)$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
θ_1	0	2	3	5	9	10	13	14
θ_2	4	3	2	0	3	4	7	10
θ_3	8	7	6	3	1	0	3	6
θ_4	9	8	7	5	4	3	2	0

In this problem, the number of all possible non-randomized decision rules is 8^m. Since $a_{j(1)}=a_1$, $a_{j(2)}=a_4$, $a_{j(3)}=a_6$ and $a_{j(4)}=a_8$, the assumption of Theorem is satisfied. For action a_2, generalized regret-relief ratios are

$$\gamma^*_{1,2}(\theta_\alpha, a_2)=\gamma^*_{1,2}(\theta_1, a_2)=\frac{2}{3}$$

$$\begin{aligned}\gamma^*_{1,2}(\theta_\beta, a_2)&=\min_{s>1}\ \{\gamma^*_{1,2}(\theta_s, a_2)\}\\&=\min\ \{\gamma^*_{1,2}(\theta_2, a_2), \gamma^*_{1,2}(\theta_3, a_2), \gamma^*_{1,2}(\theta_4, a_2)\}\\&=\min\left\{\frac{3}{1}, \frac{4}{1}, \frac{3}{1}\right\}=3\ .\end{aligned}$$

Hence,

$$\gamma^*_{1,2}(\theta_\alpha, a_2)\cdot\gamma^*_{1,2}(\theta_\beta, a_2)=\frac{2}{3}\cdot 3=2>1\ .$$

By Theorem, we know that the non-randomized decision rules which take action a_2 for any x_i $(i=1, \cdots, m)$ are dominated.

Similarly as a_2, for a_3, a_5, a_7 we get the following result. For a_3, we get

$$\gamma^*_{1,2}(\theta_\alpha, a_3)=\gamma^*_{1,2}(\theta_1, a_3)=\frac{3}{2}\ ,$$

$$\begin{aligned}\gamma^*_{1,2}(\theta_\beta, a_3)&=\min_{s>1}\ \{\gamma^*_{1,2}(\theta_s, a_3)\}\\&=\min\ \{\gamma^*_{1,2}(\theta_2, a_3), \gamma^*_{1,2}(\theta_3, a_3), \gamma^*_{1,2}(\theta_4, a_3)\}\\&=\min\left\{\frac{2}{2}, \frac{3}{2}, \frac{2}{2}\right\}=1\end{aligned}$$

and

$$\gamma^*_{1,2}(\theta_\alpha, a_3)\cdot\gamma^*_{1,2}(\theta_\beta, a_3)=\frac{3}{2}\cdot 1=\frac{3}{2}>1\ .$$

For a_5, we get

$$\gamma^*_{2,3}(\theta_\alpha, a_5) = \min\{\gamma^*_{2,3}(\theta_1, a_5), \gamma^*_{2,3}(\theta_2, a_5)\} = \min\left\{\frac{4}{1}, \frac{3}{1}\right\} = 3,$$

$$\gamma^*_{2,3}(\theta_\beta, a_5) = \min\{\gamma^*_{2,3}(\theta_3, a_5), \gamma^*_{2,3}(\theta_4, a_5)\} = \min\left\{\frac{1}{2}, \frac{1}{1}\right\} = \frac{1}{2},$$

and

$$\gamma^*_{2,3}(\theta_\alpha, a_5)\cdot\gamma^*_{2,3}(\theta_\beta, a_5) = 3\cdot\frac{1}{2} = \frac{3}{2} > 1 .$$

For a_7, we get

$$\gamma^*_{3,4}(\theta_\alpha, a_7) = \min\{\gamma^*_{3,4}(\theta_1, a_7), \gamma^*_{3,4}(\theta_2, a_7), \gamma^*_{3,4}(\theta_3, a_7)\} = \min\left\{\frac{3}{1}, \frac{3}{3}, \frac{3}{3}\right\} = 1,$$

$$\gamma^*_{3,4}(\theta_\beta, a_7) = \gamma^*_{3,4}(\theta_4, a_7) = 2,$$

and

$$\gamma^*_{3,4}(\theta_\alpha, a_7)\cdot\gamma^*_{3,4}(\theta_\beta, a_7) = 1\cdot 2 = 2 > 1 .$$

Therefore, by Theorem, all of the non-randomized decision rules which take action a_3, a_5 or a_7 for any x_i $(i=1,\cdots, m)$ are dominated. Hence the number of non-randomized decision rule in the complete class is reduced from 8^m to 4^m.

We conclude that for the problem of this example, generalized regret-relief ratio criterion is very effective in the reduction to a complete class.

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