Ann. Inst. Statist. Math.
30 (1978), Part A, 15-26

DETECTING OPTIMUM TIME OF CONTROL ACTION FOR A MANUFACTURING SYSTEM

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(Received Aug. 5, 1977; revised Mar. 8, 1978)

Summary

A statistical method for detecting the optimum time of control for a linear system with quadratic loss, is discussed. When the initial state is unknown and only one control action is available, both the optimum time of control and the amount of control are determined from the current observations at that time. Some simulated results are presented and our heuristic method is shown to be fairly satisfactory for practical use. An idea of reducing the bias of our estimate for the optimum time, is also proposed.

1. Statement of the problem

The optimal solutions of linear (discrete or continuous) control systems, subject to quadratic loss in both state and control, have been investigated by many authors, see for example, Adorno [1], Kreindler and Jameson [2] and Kushner [4]. But all articles treated the class of successive control based on all observations at that time. There are also the cases where the observations have a considerable cost and it is desired to observe as less as possible. Kushner [3] considered the optimum timing problem of driving a linear system to some terminal position using a selected number of noise perturbed observations. We shall consider another type of the optimum timing problem. The results are the new look from the point that the estimation of optimum control times is adopted.

Suppose we are confronted with the problem of manufacturing N articles such that the quality characteristic X of each article is normally distributed with known variance σ^2. Initially, the expected value of X may be different from the desirable quality level μ_0, e.g.

$$\mathrm{E}\{X\}=\mu_0+\mu\ , \qquad \text{say}\ .$$

As the risk incurred in production under the above level, we consider the mean quadratic loss $\mathrm{E}(X-\mu_0)^2$. Then our task is to revise the level $\mathrm{E}(X)$ by the amount μ, which is unknown to us but is estimable from the current information of successively produced articles. Now suppose we are allowed only one control action. Hasty control makes use of rough estimation of initial level but hesitation causes the production of many article of poor quality.

In this note, we shall put forward a statistical method for obtaining the optimum time at which the control action should be taken, as well as the estimated value of μ adopted. Without loss of generality we can assume $\mu_0=0$. Then our problem is the following.

Find n and $\hat{\mu}=\hat{\mu}(X_1, X_2, \cdots, X_n)$ for which

$$R\equiv R(n, \hat{\mu}; \mu, \sigma^2, N, c)=\mathrm{E}\left\{\sum_{i=1}^{n} X_i^2+\sum_{i=n+1}^{N}(X_i-\hat{\mu})^2+c\hat{\mu}^2\right\}$$

is minimized. Here $c\,(\geqq 0)$ is the cost which is assumed to be proportional to the square of amount of the control.

In the sequel one heuristic method is presented and it will be shown with computer simulation that our appproach is fairly satisfactory for practical use. Furthermore when two control actions are available the similar problem is discussed.

2. Some analytical results

First we consider the problem for fixed n. With a slight loss of efficiency we restrict the following type estimator of μ

$$\hat{\mu}_n=\alpha_n\bar{X}(n)=\frac{\alpha_n}{n}\sum_{i=1}^{n} X_i\ .$$

To find the best α_n we write

$$\begin{aligned}R&=\mathrm{E}\left\{\sum_{i=1}^{N} X_i^2-2\alpha_n\bar{X}(n)\sum_{i=n+1}^{N} X_i+(N-n+c)\alpha_n^2\bar{X}(n)^2\right\}\\&=N(\mu^2+\sigma^2)-2(N-n)\mu^2\alpha_n+(N-n+c)[\mu^2+\sigma^2/n]\alpha_n^2\\&=(N-n+c)\left(\mu^2+\frac{\sigma^2}{n}\right)\left[\alpha_n-\frac{N-n}{(N-n+c)(1+\sigma^2/n\mu^2)}\right]^2\\&\quad+N(\mu^2+\sigma^2)-\frac{(N-n)^2\mu^2}{(N-n+c)(1+\sigma^2/n\mu^2)}\ .\end{aligned}$$

Then we may put

$$\alpha_n^*=\frac{N-n}{N-n+c}\,\frac{1}{1+d^2/n}$$

where $d=\sigma/|\mu|$. In this case we incur the risk

$$R=N(\mu^2+\sigma^2)-\psi(n, \mu)$$

where

$$(1) \qquad \psi(n, \mu)=\frac{(N-n)^2\mu^2}{(N-n+c)(1+d^2/n)} .$$

Next we shall find n for which $\psi(n, \mu)$ is maximized. Put

$$\phi(t)=\frac{(N-t)^2 t}{(N+c-t)(t+d^2)} .$$

Then

$$(N+c-t)^2(t+d^2)^2\phi'(t)=(N-t)h(t)$$

where

$$h(t)=(N-3t)(N+c-t)(t+d^2)+(N-t)t[(2t+d^2)-(N+c)]$$
$$=\tilde{h}(t)+2c^2d^2$$

$$(2) \qquad \begin{aligned} &\tilde{h}(t)=(t-\alpha_1)(t-\alpha_2)(t-\alpha_3) \\ &\alpha_1=N+2c \\ &\alpha_2=-d^2+d\sqrt{d^2+N-c} \\ &\alpha_3=-d^2-d\sqrt{d^2+N-c} . \end{aligned}$$

For $0<t<N$, $\phi'(t)$ and $h(t)$ have the same sign and $h(t)$ changes sign from + to − at $t=\alpha_2'$ which is slightly larger than zero point α_2 of $\tilde{h}(t)$ (see Fig. 1). Thus $\phi(t)$ has the maximum value at $t=\alpha_2'$. Then we shall define n^* is the smallest integer exceeding α_2. Thus we can

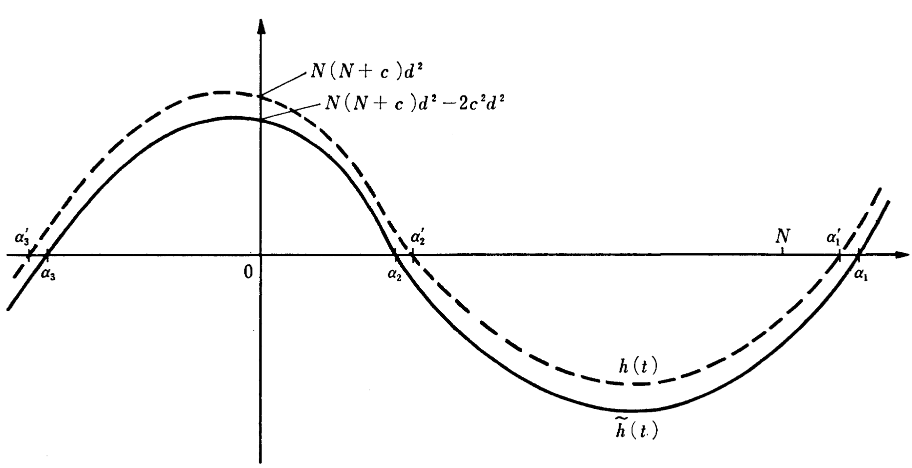

Fig. 1

consider that $\phi(n)$ has the nearly maximum $\phi(n^*, \mu^*)$ at $n=n^*$, where ϕ is defined by (1) and

$$\mu^*=\frac{N-n^*}{N-n^*+c}\frac{\mu}{1+d^2/n^*} .$$

Next we consider the estimation problem of n^*. Rewriting the relation (2) we have

$$(\alpha_2/d)^2+2\alpha_2=N-c$$

i.e.

$$(\alpha_2^2\mu^2+\alpha_2\sigma^2)+\alpha_2\sigma^2=\sigma^2(N-c) .$$

Since $X_1, X_2, \cdots$ are independently distributed $N(\mu, \sigma^2)$, as an estimator of n^* we can take $\hat{n}$ which is defined by the first exceeding time over the line $\sigma^2(N-c)$ of the random sequence

$$\left\{\left(\sum_{i=1}^{n} X_i\right)^2+\sigma^2 n:\ n=1, 2, \cdots\right\} .$$

Consequently, we first define $\hat{n}$ using the sequentially observed random variables $X_1, X_2, \cdots$, and immediately control

$$\hat{\mu}=u(\hat{n}, \bar{X}(\hat{n}), c) ,$$

where

$$u(n, \mu, c)=\frac{N-n}{N-n+c}\frac{\mu}{1+d^2/n} . \tag{3}$$

Secondary we consider the case where two control actions are allowed. For simplicity we put $c=0$ (c is the cost for the amount of control). Let m, n, be the first, the second control time and $u=\alpha\bar{X}_m$, $v=\beta\bar{X}_n$ be the amounts of control respectively. In such case our global risk becomes

$$\begin{aligned}R&=\mathrm{E}\left\{\sum_{i=1}^{N} X_i^2+(n-m)u^2+(N-n)v^2-2u\sum_{i=m+1}^{N} X_i-2v\sum_{i=n+1}^{N} X_i\right\}\\
&=N(\mu^2+\sigma^2)+(n-m)\left\{\left[\mu^2+\frac{\sigma^2}{m}\right]\alpha^2-2\mu^2\alpha\right\}\\
&\quad+(N-n)\left\{\left[\mu^2+\frac{\sigma^2}{n}\right]\beta^2-2\mu^2\beta\right\}\\
&=N(\mu^2+\sigma^2)+(n-m)\left(\mu^2+\frac{\sigma^2}{m}\right)\left[\alpha-\frac{1}{1+d^2/m}\right]^2\\
&\quad+(N-n)\left(\mu^2+\frac{\sigma^2}{n}\right)\left[\beta-\frac{1}{1+d^2/n}\right]^2-\frac{(n-m)\mu^2}{1+d^2/m}-\frac{(N-n)\mu^2}{1+d^2/n} .\end{aligned}$$

Then we may take

$$\alpha=\frac{1}{1+d^2/m}\,, \qquad \beta=\frac{1}{1+d^2/n}\,.$$

Next we shall find m, n for which

$$\frac{(n-m)\mu^2}{1+d^2/m}+\frac{(N-n)\mu^2}{1+d^2/n}$$

is maximized. Putting

$$\phi(s,\,t)=\frac{(t-s)s}{s+d^2}+\frac{(N-t)t}{t+d^2}\,,$$

then

$$\frac{\partial}{\partial s}\phi(s,\,t)=\frac{d^2t-s[s+2d^2]}{(s+d^2)^2}=0$$

$$\frac{\partial}{\partial t}\phi(s,\,t)=\frac{s}{s+d^2}+\frac{d^2N-t[t+2d^2]}{(t+d^2)^2}=0\,.$$

Equating these we have

$$\left(\frac{s+d^2}{d^2}\right)^3=\frac{N+d^2}{d^2}=\nu^3 \quad \text{(say)}\,, \qquad \frac{t+d^2}{d^2}=\left(\frac{s+d^2}{d^2}\right)^2=\nu^2\,.$$

Namely $\phi(s,t)$ is maximized at

(4) $$s=d^2(\nu-1)\,,$$

(5) $$t=d^2(\nu^2-1)\,,$$

where

$$\nu=\sqrt[3]{\frac{N+d^2}{d^2}}\,.$$

To estimate such s, t we rewrite the relations (4) and (5) as

$$s^3\mu^4+3\sigma^2s^2\mu^2+3\sigma^4s=\sigma^4N\,, \qquad t^3\mu^4+\sigma^2(3t^2-N^2)\mu^2+3\sigma^4t=2\sigma^4N\,.$$

For any constant c we define

$$T_n(c)=\frac{n^2}{n-1}\left[\left(\sum_{i=1}^{n}X_i^2\right)^2-\sum_{i=1}^{n}X_i^4\right]-(2n^2-c)\sigma^2\sum_{i=1}^{n}X_i^2+(n^2+3-c)n\sigma^4\,.$$

Since

$$\mathrm{E}\,\{\textstyle\sum X_i^2\}=n(\mu^2+\sigma^2)\,,$$

$$\mathrm{E}\,\{(\textstyle\sum X_i^2)^2-\sum X_i^4\}=\mathrm{E}\,\{\sum_{i\neq j}X_i^2X_j^2\}=n(n-1)(\mu^2+\sigma^2)^2\,,$$

then it is easily seen that

$$\mathrm{E}\{T_n(c)\} = n^3\mu^4 + cn\mu^2\sigma^2 + 3n\sigma^4 .$$

Define the two sequences of random variables by

$$Y_m = \frac{m^2}{m-1}\left[\left(\sum_{i=1}^{m} X_i^2\right)^2 - \sum_{i=1}^{m} X_i^4\right] - (2m-3)m\sigma^2 \sum_{i=1}^{m} X_i^2 + (m^2-3m+3)m\sigma^4$$

$$Z_n = \frac{n^2}{2(n-1)}\left[\left(\sum_{i=1}^{n} X_i^2\right)^2 - \sum_{i=1}^{n} X_i^4\right] - \frac{1}{2}\left(2n^2-3n+\frac{N^2}{n}\right)\sigma^2 \sum_{i=1}^{n} X_i^2 + \frac{1}{2}(n^3-3n^2+3n+N^2)\sigma^4 .$$

Then let $\hat{m}$ and $\hat{n}$ be the first exceeding times over the line $\sigma^4 N$ of the sequences $\{Y_m\}$ and $\{Z_n\}$ respectively. Furthermore the amounts of control are given by

$$\hat{u} = u(\hat{m}, \bar{X}(\hat{m}), 0) , \qquad \hat{v} = u(\hat{n}, \bar{X}(\hat{n}), 0) - \hat{u} ,$$

where the function u is defined by (3).

3. Some simulated results

We now show some empirical results with computer simulation. We first treat the case $c=0$ and $\sigma^2=1$. For given N and μ, we first determine the optimum time n^* of control in the case of only one control action (case I) and the couple (n_1^*, n_2^*) of times of the first and the second control in the case where two control actions are available (case II). Minimum attainable risk is given by

$$R_{\mathrm{I}} = N(\mu^2+1) - \frac{n^*(N-n^*)\mu^4}{n^*\mu^2+1} \qquad \text{(Case I)}$$

$$R_{\mathrm{II}} = N(\mu^2+1) - \frac{n_1^*(n_2^*-n_1^*)}{n_1^*\mu^2+1} - \frac{n_2^*(N-n_2^*)}{n_2^*\mu^2+1} \qquad \text{(Case II)}$$

Table 1

N	μ	n^*	n_1^*	n_2^*	R_{I}	R_{II}
100	0.5	17	8	32	108.2	105.9
100	1.0	10	4	21	118.2	111.0
100	2.0	5	2	14	138.1	119.4
200	1.0	14	5	34	226.4	214.6
500	1.0	22	7	63	542.8	520.8
1000	1.0	31	10	100	1061.3	1027.1

For $N=100$ and $\mu=0.5$, $\mu=1$, $\mu=2$ and for $(N, \mu)=(200, 1)$ and $(N, \mu)=(1000, 1)$, the values of n^*, n_1^*, n_2^*, R_{I} and R_{II} are given in Table 1. Next, for any sequence $\{X_1, X_2, \cdots, X_N\}$, define

$$L_{\text{I}}(n)=\sum_{i=1}^{N} X_i^2-2\hat{\mu}(n)\sum_{i=n+1}^{N} X_i+(N-n)\hat{\mu}(n)^2$$

$$L_{\text{II}}(n_1, n_2)=\sum_{i=1}^{N} X_i^2-2\hat{\mu}(n_1)\sum_{i=n_1+1}^{n_2} X_i-2\hat{\mu}(n_2)\sum_{i=n_2+1}^{N} X_i + (n_2-n_1)\hat{\mu}(n_1)^2+(N-n_2)\hat{\mu}(n_2)^2 ,$$

where

$$\hat{\mu}(n)=\frac{n\bar{X}(n)^3}{n\bar{X}(n)^2+1} .$$

Then, the optimum loss with optimum control action(s) can be expressed by

$$L_{\text{I}}(n^*) \qquad \text{(Case I)}$$

$$L_{\text{II}}(n_1^*, n_2^*) \qquad \text{(Case II)}$$

and the loss with estimated control action(s) by

$$L_{\text{I}}(\hat{n}) \qquad \text{(Case I)}$$

$$L_{\text{II}}(\hat{n}_1, \hat{n}_2) \qquad \text{(Case II)}$$

For each case, the results of averages of 1000 times of experiments are presented in Table 2. Note that for $(N, \mu)=(100, 0.5)$ $L_{\text{I}}(\hat{n})$ is smaller than $L_{\text{II}}(\hat{n}_1, \hat{n}_2)$, on an average. This seems to be cursed by instability of estimated times.

We also simulated the case of one control action with non-zero constant c for $N=100$, $\mu=1$ and $\sigma^2=1$. The results of averages of 1000 times of experiments are shown in Table 3.

Table 2

N	μ	$L_{\text{I}}(n^*)$	$L_{\text{II}}(n_1^*, n_2^*)$	$\hat{n}$	$\hat{n}_1$	$\hat{n}_2$	$L_{\text{I}}(\hat{n})$	$L_{\text{II}}(\hat{n}_1, \hat{n}_2)$
100	0.5	105.91	103.82	19.48	20.45	41.14	111.38	114.18
100	1.0	111.55	105.59	10.40	7.83	24.23	121.20	116.20
100	2.0	121.46	109.48	5.54	2.93	14.43	140.20	121.74
200	1.0	216.01	206.96	14.50	9.15	36.93	230.19	220.36
1000	1.0	1033.88	1012.68	31.92	13.25	102.60	1058.70	1034.95

Table 3

c	$n^*(c)$	$\hat{n}(c)$	$L(n^*)$	$L(\hat{n})$
0.2	10.0	10.46	119.18	117.71
0.4	9.0	10.45	119.35	117.91
0.6	9.0	10.44	119.53	118.09
0.8	9.0	10.44	119.70	118.25
1.0	9.0	10.43	119.88	118.43
2.0	8.0	10.37	120.73	119.28
3.0	8.0	10.32	121.56	120.12
4.0	8.0	10.27	122.37	120.92
5.0	8.0	10.23	123.17	121.68

4. Further consideration

In the later section, we have seen that our heuristic method is fairly satisfactory for practical use. But we note that there is further problem of estimating the time of control action. For, we found from the results of computer simulation that our estimate is slightly later, in the sense of average, than the optimum time of control action. Especially, this type of bias is more serious in the case where two control actions are available.

We shall state the problem of this type, in the general form:

Given the sequence

$$\{Z_i \mid i=1, 2, \cdots\}$$

of random variables whose means and variances, say $\mu_i = \mathrm{E}\{Z_i\}$, $\sigma_i^2 = \mathrm{V}\{Z_i\}$ are monotone increasing with respect to i. Of course, they are not independent. Define

$$n^* = n^*(L, \{\mu_i\}) = \min\{i:\ \mu_i \geqq L\} . \tag{6}$$

Find some unbiased estimator of n^*.

Since

$$\mathrm{E}\{Z_i\} = \mu_i , \qquad i=1, 2, \cdots$$

the estimator defined by

$$\hat{n} = \min\{i:\ Z_i \geqq L\}$$

may be reasonable. But this will be positively biased. For, put

$$t^* = \min\{t:\ \mu(t) \geqq L\} ,$$

$$t_1 = \min\{t:\ \mu(t) + \sigma(t) \geqq L\} ,$$

$$t_2 = \min\{t:\ \mu(t) - \sigma(t) \geqq L\} .$$

Then

$$\Pr\{\hat{n}\leqq t_1\}=\Pr\{X_{[t_1]}\geqq L\}=\Pr\{X_{[t_1]}\geqq\mu(t_1)+\sigma(t_1)\}\ ,$$
$$\Pr\{\hat{n}\geqq t_2\}=\Pr\{X_{-[-t_2]}\leqq L\}=\Pr\{X_{-[-t_2]}\leqq\mu(t_2)-\sigma(t_2)\}$$

and these two expressions would be nearly equal. On the other hand

$$t^*-t_1<t_2-t^*$$

because the functional defined by (6) is convex from the following Lemma.

LEMMA. *For any non-decreasing function μ, define*

$$L(\mu)=\inf\{t\mid \mu(t)\geqq L\}\ .$$

Then the functional L is convex in the following sense: For every pair μ, ν such that

$$\frac{\nu(t_2)-\nu(t_1)}{t_2-t_1}\geqq\frac{\mu(t_3)-\mu(t_2)}{t_3-t_2}\qquad(t_1<t_2<t_3)\ ,$$

put

$$\eta(t)=\alpha\mu(t)+(1-\alpha)\nu(t)\qquad(0<\alpha<1)\ .$$

Then

$$L(\eta)\geqq\alpha L(\mu)+(1-\alpha)L(\nu)\ .$$

PROOF. We write

$$t_\mu=L(\mu)\ ,\qquad t_\nu=L(\nu)\ ,\qquad t^*=\alpha t_\mu+(1-\alpha)t_\nu\ .$$

Then

$$\begin{aligned}\eta(t^*)&=\alpha\mu(t^*)+(1-\alpha)\nu(t^*)\\&=\alpha L-\alpha[\mu(t_\mu)-\mu(t^*)]+(1-\alpha)L+(1-\alpha)[\nu(t^*)-\nu(t_\nu)]\\&=L+\alpha(1-\alpha)(t_\mu+t_\nu)\left\{\frac{\nu(t^*)-\nu(t_\nu)}{t^*-t_\nu}-\frac{\mu(t_\mu)-\mu(t^*)}{t_\mu-t^*}\right\}\ .\end{aligned}$$

Since

$$t_\mu>t^*>t_\nu\ ,$$

then we have

$$\eta(t^*)\leqq L\ .$$

This means that

$$L(\eta)\geqq t^*=\alpha L(\mu)+(1-\alpha)L(\nu) .$$

We now propose some modified estimator. Given two increasing functions $\mu(t)$, $\sigma(t)$ for which

$$(7)\qquad \frac{\mu(t_3)-\mu(t_2)}{t_3-t_2}-\frac{\mu(t_2)-\mu(t_1)}{t_2-t_1}\leqq\frac{\sigma(t_3)-\sigma(t_2)}{t_3-t_2}+\frac{\sigma(t_2)-\sigma(t_1)}{t_2-t_1}\qquad (0<t_1<t_2<t_3)$$

and

$$\mathrm{E}\{Z_i\}=\mu(i) , \qquad \mathrm{V}\{Z_i\}=\sigma(i)^2 .$$

Define

$$\tilde{R}(h)=\frac{\mu(t^*+h)-\mu(t^*-h)}{\sigma(t^*+h)+\sigma(t^*-h)} ,$$

where t^* is the solution of

$$\mu(t)=L .$$

Noting that $\tilde{R}(0)=0$ and $\tilde{R}(h)$ is the increasing function of h, we can find h^* for which

$$(8)\qquad \tilde{R}(h^*)=1 .$$

Then we have

$$\mu(t^*-h^*)+\sigma(t^*-h^*)=\mu(t^*+h^*)-\sigma(t^*+h^*) .$$

Putting

$$\tilde{\gamma}=\frac{L}{\mu(t^*-h^*)+\sigma(t^*-h^*)}$$

then

$$\tilde{\gamma}[\mu(t)+\sigma(t)]=L \qquad \text{for } t=t^*-h^*$$

$$\tilde{\gamma}[\mu(t)-\sigma(t)]=L \qquad \text{for } t=t^*+h^* .$$

Consequently, we can expect that the random sequence

$$\{\tilde{\gamma}X_i \mid i=1, 2, \cdots\}$$

crosses the line L at near point of t^*. Then we define

$$\tilde{n}=\min\{i:\ \tilde{\gamma}X_i\geqq L\} .$$

This modified estimator would be almost unbiased for n^*.

We further note that the procedure of finding h^* by (8) is more complicated. Therefore replacing $\tilde{R}$ by

$$R(h)=\frac{2h\mu'(t^*)}{2\sigma(t^*)} ,$$

we equate

$$R(h^{**})=1$$

i.e.

$$h^{**}=\frac{\sigma(t^*)}{\mu'(t^*)} .$$

Then using

$$\gamma=\frac{L}{\mu(t^*-h^{**})+\sigma(t^*-h^{**})}$$

we can construct the following estimator

$$\hat{\hat{n}}=\min_i \{i:\ \gamma X_i \geqq L\} .$$

Now we are the case

$$\mu(t)=\mu^2t^2+2\sigma^2t , \qquad \sigma(t)=2\sigma t\sqrt{2\mu^2t+\sigma^2} ,$$

which satisfy the condition (7). Then

$$t^*=d^2(\tilde{t}-1)$$

$$h^{**}=d^2\sqrt{2\tilde{t}-1}\,(\tilde{t}-1)/\tilde{t}$$

$$\gamma=\frac{N-c}{(\delta+2+2\sqrt{2\delta+1})\delta d^2}$$

where

$$\tilde{t}=\frac{t^*}{d^2}+1=\sqrt{\frac{N-c}{d^2}+1}$$

$$\delta=\frac{t^*-h^{**}}{d^2}=(\tilde{t}-\sqrt{2\tilde{t}-1}\,)(\tilde{t}-1)/\tilde{t} .$$

For $N=100$ and $\mu=0.5$, $\mu=1$, $\mu=2$ and for $(N, \mu)=(200, 1)$ and $(N, \mu)=(1000, 1)$, the values of t^*, δ and γ are given in Table 4. For each case, the results of averages of 1000 times of experiments for our modified estimator $\hat{\hat{n}}$, are presented in Table 5. We can recognize that the bias of the estimator $\hat{n}$ is fairly removed by our modified estimator $\hat{\hat{n}}$.

Table 4

N	μ	t^*	δ	γ
100	0.5	16.396	1.661	1.925
100	1.0	9.050	5.114	1.415
100	2.0	4.756	13.088	1.198
200	1.0	13.177	8.316	1.285
1000	1.0	30.639	22.996	1.123

Table 5

N	μ	$\hat{n}$
100	0.5	13.67
100	1.0	8.69
100	2.0	5.06
200	1.0	12.83
1000	1.0	30.16

Remark. It is easily seen that the modified estimator $\hat{n}$ can not be constructed without the information of unknown parameter d. Then there are further problem for actual adoption of the control time $\hat{n}$.

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