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ON A SPECTRAL ESTIMATE OBTAINED BY AN AUTOREGRESSIVE MODEL FITTING

MITUAKI HUZII

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1. Introduction

Let $X(t)$ be a real-valued stationary normal process with a discrete time parameter t. For simplicity, we shall assume $\mathrm{E}\,X(t)=0$ and denote $R(h)=\mathrm{E}\,X(t+h)X(t)$ and $\rho_h=R(h)/R(0)$. Let us assume $X(t)$ is observed at $t=1, 2, \cdots, N$ and has a spectral density $f(\lambda)$, where λ is a frequency parameter and $-1/2\leqq\lambda\leqq 1/2$.

In this paper, we shall discuss efficient estimation of $f(\lambda)$. If we can assume $X(t)$ is an autoregressive process of order K, K being a known positive integer, we can obtain, easily, the maximum likelihood estimate of $f(\lambda)$ when N is sufficiently large. But for an actual process, we usually do not know the value of K. Recently, the estimates obtained by fitting an autoregressive model have been developed and discussed by many authors, i.e., Akaike [1], Parzen [8], Gersch and Sharpe [5] and Jones [7].

In this paper, we treat a process expressed as an autoregressive process of infinite order satisfying some conditions. We construct an estimate by fitting an autoregressive model of finite order K. In Section 2 we discuss the asymptotic bias of this estimate for a fixed K when N tends to infinity. In Section 3 we consider K as a function of N, tending to infinity as N tends to infinity. Berk [3] has discussed a similar situation. He has shown the consistency and the asymptotic normality of the estimate when N tends to infinity. Although we shall discuss the statistical properties of the same estimate, the process under consideration here satisfies stronger conditions than his, and under our conditions we show that this estimate has a property of efficiency as N tends to infinity.

2. An autoregressive approximation and bias

In the following we shall assume, furthermore, $X(t)$ satisfies the following assumption:

ASSUMPTION 1. $X(t)$ satisfies the relation

$$(1) \qquad \sum_{k=0}^{\infty} a_k X(t-k)=\xi(t) , \qquad a_0=1 ,$$

where $\{\xi(t)\}$ are mutually independent random variables each of which has the distribution $N(0, \sigma_\xi^2)$ and $\{a_k\}$ are constants such that

$$|a_k| \leqq \alpha^k , \qquad 0<\alpha<1/2 ,$$

for every $k \geqq 1$.

In this case, $X(t)$ has a backward moving average representation

$$X(t)=\sum_{k=0}^{\infty} G_k \xi(t-k) ,$$

where $\{G_k\}$ are constants.

As is shown in Huzii [6], we have the following result under the above assumptions.

LEMMA 1. *We have*

$$|\rho_h| \leqq C(2\alpha)^h \qquad \textit{and} \qquad |G_h| \leqq (2\alpha)^h/2$$

for any h, $h \geqq 1$, *where* $C=1/2^2(1-(2\alpha)^2)$.

Now we shall discuss the estimation of the spectral density $f(\lambda)$. We shall regard $X(t)$ as an autoregressive process of order K, K being a positive integer, and obtain $\{a_k^{(K)};\ 1 \leqq k \leqq K\}$ which minimize

$$Q=\sum_{t=K+1}^{N} \left(X(t)+\sum_{k=1}^{K} a_k^{(K)} X(t-k)\right)^2 .$$

Let us denote $a_k^{(K)}$,which minimizes Q, as $\hat{a}_k^{(K)}$ for $1 \leqq k \leqq K$. If $X(t)$ is an autoregressive process of order K, the $\hat{a}_k^{(K)}$'s are asymptotically maximum likelihood estimates of autoregressive coefficients. But here $X(t)$ is not an autoregressive process of finite order. So this argument does not hold. The $\hat{a}_k^{(K)}$'s are the solutions of the simultaneous equation

$$(2) \qquad \sum_{k=1}^{K} \hat{a}_k^{(K)} \hat{R}(k, l)=-\hat{R}(0, l) , \qquad 1 \leqq l \leqq K ,$$

where

$$\hat{R}(k, l)=\frac{1}{N-K} \sum_{t=K+1}^{N} X(t-k)X(t-l) .$$

Using these $\hat{a}_k^{(K)}$'s, we shall construct the estimate $\hat{f}(\lambda)$ of $f(\lambda)$ as follows

$$\hat{f}^{(K)}(\lambda)=\frac{\hat{\sigma}_\xi^2(K)}{\left(\sum_{k=0}^{K}\hat{a}_k^{(K)}\cos 2\pi k\lambda\right)^2+\left(\sum_{k=1}^{K}\hat{a}_k^{(K)}\sin 2\pi k\lambda\right)^2},$$

where $\hat{a}_0^{(K)}=1$ and

$$\hat{\sigma}_\xi^2(K)=\frac{1}{N-K}\sum_{t=K+1}^{N}\left(X(t)+\sum_{k=1}^{K}\hat{a}_k^{(K)}X(t-k)\right)^2.$$

Now let us evaluate the bias of the estimate $\hat{f}^{(K)}(\lambda)$ for a sufficiently large and fixed K. We shall denote $\hat{\boldsymbol{a}}(K)=(\hat{a}_1^{(K)}, \hat{a}_2^{(K)},\cdots,\hat{a}_K^{(K)})'$ and $\hat{\boldsymbol{R}}_K=(\hat{R}(0,1),\hat{R}(0,2),\cdots,\hat{R}(0,K))'$. And let $\hat{\boldsymbol{R}}$ be the $K\times K$ matrix whose (k,l) element is $\hat{R}(k,l)$. Then the simultaneous equation (2) can be written

$$\hat{\boldsymbol{R}}\hat{\boldsymbol{a}}(K)=-\hat{\boldsymbol{R}}_K\,. \tag{3}$$

Now let $\boldsymbol{Q}_J$ be the $J\times J$ matrix whose (k,l) element is ρ_{k-l}. Using the result of Lemma 1, we can show $\hat{R}(k,l)$ converges in probability to $R(k-l)$ as N tends to infinity. Let us denote the $K\times K$ matrix, whose (k,l) element is $R(k-l)$, as $\boldsymbol{R}$. Then for a sufficiently large and fixed K, every element of $\hat{\boldsymbol{R}}$ converges in probability to the corresponding element of $\boldsymbol{R}$. Let us put $\boldsymbol{R}_K=(R(1),R(2),\cdots,R(K))'$. Then every element of $\hat{\boldsymbol{R}}_K$ converges in probability to the corresponding element of $\boldsymbol{R}_K$. We can consider $|\boldsymbol{R}|\neq 0$ for any K, $K\geqq 1$. Using the relation (3), we can show the distribution of $\sqrt{N}(\hat{\boldsymbol{a}}(K)+\boldsymbol{R}^{-1}\boldsymbol{R}_K)$ converges to the normal distribution with mean vector $\mathbf{0}$ and a finite covariance matrix. This can be shown by the same method as in Anderson [2], Chap. 5. Using this result, we shall evaluate the asymptotic bias of the estimate $\hat{f}^{(K)}(\lambda)$ as $N\to\infty$. In the first place, we shall evaluate the value $\boldsymbol{a}(K)=-\boldsymbol{R}^{-1}\boldsymbol{R}_K$. Let us put $\boldsymbol{a}(K)=(a_1^{(K)},a_2^{(K)},\cdots,a_K^{(K)})'$. Then we have the following lemma.

LEMMA 2. *We have, for* $1\leqq k\leqq K$,

$$|a_k^{(K)}-a_k|\leqq C_1\alpha^K,$$

where C_1 *is a constant being independent of* K.

PROOF. Let us put $\boldsymbol{\rho}_K=(\rho_1,\rho_2,\cdots,\rho_K)'$ for simplicity. Then we have $\boldsymbol{a}(K)=-\boldsymbol{Q}_K^{-1}\boldsymbol{\rho}_K$. We can express, by using the result of Wise [9], the (k,l) element $q_{k,l}^{-1}$ of $\boldsymbol{Q}_K^{-1}$ as $\tilde{q}_{k,l}^{-1}+\varepsilon_{k,l}(K)$, where

$$\tilde{q}_{k,l}^{-1}=\begin{cases}\sum_{j=1}^{l} a_{k-j}a_{l-j}\ , & \text{if } k>l\\ 1+\sum_{j=1}^{k-1} a_j^2\ , & \text{if } k=l\\ \tilde{q}_{l,k}^{-1}\ , & \text{if } k<l\end{cases}$$

and $\varepsilon_{k,l}(K)$ is uniformly bounded for either $k<K-K_0$ or $l<K-K_0$, K_0 being a fixed positive integer, when K is sufficiently large. So taking into account of Assumption 1, we have

$$|q_{k,l}^{-1}|\leqq q\ ,$$

where q is a constant and independent of K (see Huzii [6]). Now we have the relation

$$\sum_{k=0}^{\infty} a_k\rho_{k-j}=0\ ,\qquad j=1,2,3,\cdots,$$

by (1). This can be written as

$$(4)\qquad \sum_{k=1}^{K} a_k\rho_{k-j}=-\rho_j-\sum_{k=K+1}^{\infty} a_k\rho_{k-j}\ ,\qquad j=1,2,\cdots.$$

Putting $\boldsymbol{a}=(a_1,a_2,\cdots,a_K)'$, $\delta_j=\sum_{k=K+1}^{\infty} a_k\rho_{k-j}$ and $\boldsymbol{\Delta}=(\delta_1,\delta_2,\cdots,\delta_K)'$, we have

$$(5)\qquad \boldsymbol{a}=-\boldsymbol{Q}_K^{-1}\boldsymbol{\rho}_K-\boldsymbol{Q}_K^{-1}\boldsymbol{\Delta}$$

by using (4). From this relation, we obtain

$$\boldsymbol{a}(K)-\boldsymbol{a}=\boldsymbol{Q}_K^{-1}\boldsymbol{\Delta}\ .$$

Let us evaluate δ_j for $1\leqq j\leqq K$.

$$|\delta_j|\leqq\sum_{k=K+1}^{\infty}|a_k||\rho_{k-j}|\leqq\sum_{k=K+1}^{\infty}\alpha^k C(2\alpha)^{k-j}=C\frac{\alpha^{K+1}(2\alpha)^{K+1-j}}{1-2\alpha^2}\ .$$

So we have

$$|a_k^{(K)}-a_k|\leqq Cq\sum_{j=1}^{K}\frac{\alpha^{K+1}(2\alpha)^{K+1-j}}{1-2\alpha^2}\leqq\frac{qC\alpha^{K+1}(2\alpha)}{(1-2\alpha^2)(1-2\alpha)}=C_1\alpha^K\ ,$$

for $1\leqq k\leqq K$, where $C_1=2qC\alpha^2/(1-2\alpha^2)(1-2\alpha)$. We can easily find that, when N tends to infinity, the mean value of the limiting distribution of $\sqrt{N}(\hat{f}^{(K)}(\lambda)-f^{(K)}(\lambda))$ is zero, where

$$f^{(K)}(\lambda)=\frac{\sigma_\xi^2(K)}{\left(\sum_{k=0}^{K} a_k^{(K)}\cos 2\pi k\lambda\right)^2+\left(\sum_{k=1}^{K} a_k^{(K)}\sin 2\pi k\lambda\right)^2}$$

and

$$\sigma_\xi^2(K)=R(0)+\sum_{k=1}^{K} a_k^{(K)}R(k) \ .$$

Now we have

$$\sigma_\xi^2-\sigma_\xi^2(K)=\sum_{k=K+1}^{\infty} a_kR(k)-\sum_{k=1}^{K}(a_k^{(K)}-a_k)R(k)$$

and

$$\begin{aligned}
&\Big(\sum_{k=0}^{\infty} a_k\cos 2\pi k\lambda\Big)^2+\Big(\sum_{k=1}^{\infty} a_k\sin 2\pi k\lambda\Big)^2\\
&\quad-\Big(\sum_{k=0}^{K} a_k^{(K)}\cos 2\pi k\lambda\Big)^2-\Big(\sum_{k=1}^{K} a_k^{(K)}\sin 2\pi k\lambda\Big)^2\\
&=\Big(\sum_{k=K+1}^{\infty} a_k\cos 2\pi k\lambda\Big)^2+\Big(\sum_{k=K+1}^{\infty} a_k\sin 2\pi k\lambda\Big)^2\\
&\quad+2\Big(\sum_{k=0}^{K} a_k\cos 2\pi k\lambda\Big)\Big(\sum_{k=K+1}^{\infty} a_k\cos 2\pi k\lambda\Big)\\
&\quad+2\Big(\sum_{k=1}^{K} a_k\sin 2\pi k\lambda\Big)\Big(\sum_{k=K+1}^{\infty} a_k\sin 2\pi k\lambda\Big)\\
&\quad-2\Big(\sum_{k=0}^{K} a_k\cos 2\pi k\lambda\Big)\Big(\sum_{k=0}^{K}(a_k^{(K)}-a_k)\cos 2\pi k\lambda\Big)\\
&\quad-2\Big(\sum_{k=1}^{K} a_k\sin 2\pi k\lambda\Big)\Big(\sum_{k=1}^{K}(a_k^{(K)}-a_k)\sin 2\pi k\lambda\Big)\\
&\quad-\Big(\sum_{k=0}^{K}(a_k^{(K)}-a_k)\cos 2\pi k\lambda\Big)^2\\
&\quad-\Big(\sum_{k=1}^{K}(a_k^{(K)}-a_k)\sin 2\pi k\lambda\Big)^2 .
\end{aligned}$$

And also we have, for example,

$$\Big|\sum_{k=K+1}^{\infty} a_k\cos 2\pi k\lambda\Big|\leqq \alpha^{K+1}/(1-\alpha)$$

and

$$\Big|\sum_{k=0}^{K}(a_k^{(K)}-a_k)\cos 2\pi k\lambda\Big|\leqq C_1K\alpha^K .$$

Therefore, we have

$$\begin{aligned}
&\bigg|\Big\{\Big(\sum_{k=0}^{K} a_k^{(K)}\cos 2\pi k\lambda\Big)^2+\Big(\sum_{k=0}^{K} a_k^{(K)}\sin 2\pi k\lambda\Big)^2\Big\}\\
&\quad-\Big\{\Big(\sum_{k=0}^{\infty} a_k\cos 2\pi k\lambda\Big)^2+\Big(\sum_{k=1}^{\infty} a_k\sin 2\pi k\lambda\Big)^2\Big\}\bigg|\leqq C_2K\alpha^K ,
\end{aligned}$$

where C_2 is a constant being independent of K. So we can show

$$|f^{(K)}(\lambda)-f(\lambda)|\leqq C_3K\alpha^K ,$$

where C_3 is a constent being independent of K. We obtain the following theorem.

THEOREM 1. *Let $X(t)$ be a stationary normal process satisfying the Assumption 1. Then*

$$|f^{(K)}(\lambda)-f(\lambda)|\leqq C_3K\alpha^K$$

for sufficiently large K, where C_3 is a constant, independent of K.

3. Asymptotic efficiency of $\hat{f}^{(K)}(\lambda)$ in a sense

In this section, we shall consider K to be a function of N. Recently, Berk [3] has shown that $\hat{f}^{(K)}(\lambda)$ is consistent and asymptotically normal when N tends to infinity under the condition that K tends to infinity and K^3/N tends to 0 and some other conditions. Here we make the following assumption.

ASSUMPTION 2. K is a function of N such that K tends to infinity and K^4/N and $N^9K^2\alpha^K$ tend to zero when N tends to infinity.

This assumption is stronger than Berk's condition. We shall define $f_K(\lambda)$ as

$$f_K(\lambda)=\frac{\sigma_\xi^2}{\left(\sum_{k=0}^{K} a_k\cos 2\pi k\lambda\right)^2+\left(\sum_{k=1}^{K} a_k\sin 2\pi k\lambda\right)^2} .$$

Now we have

$$\hat{f}^{(K)}(\lambda)-f_K(\lambda)=\frac{\partial f_K(\lambda)}{\partial\sigma_\xi^2}(\hat{\sigma}_\xi^2(K)-\sigma_\xi^2)+\sum_{k=1}^{K}\frac{\partial f_K(\lambda)}{\partial a_k}(\hat{a}_k^{(K)}-a_k)$$
$$+O\left\{\left[(\hat{\sigma}_\xi^2(K)-\sigma_\xi^2)^2+\sum_{k=1}^{K}(\hat{a}_k^{(K)}-a_k)^2\right]^{1/2}\right\} ,$$

where the last term converges in probability to zero more rapidly than the first two terms (see Berk [3]). So we can consider

$$\hat{f}^{(K)}(\lambda)-f_K(\lambda)\sim\frac{\partial f_K(\lambda)}{\partial\sigma_\xi^2}(\hat{\sigma}_\xi^2(K)-\sigma_\xi^2)+\sum_{k=1}^{K}\frac{\partial f_K(\lambda)}{\partial a_k}(\hat{a}_k^{(K)}-a_k) .$$

Let us put $\boldsymbol{f}^{(K)}=(\partial f_K(\lambda)/\partial\sigma_\xi^2, \partial f_K(\lambda)/\partial a_1,\cdots, \partial f_K(\lambda)/\partial a_K)'$.

For an autoregressive process of order K, an efficiency of the estimate $\hat{f}^{(K)}(\lambda)$ will be defined as

$$e_K=\frac{\mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f(\lambda))^2}{\boldsymbol{f}^{(K)\prime}\boldsymbol{W}^{(K)}(X)\boldsymbol{f}^{(K)}} ,$$

where $W^{(K)}(X)$ is the information matrix being defined later. In this section, we shall show

$$\lim_{N\to\infty} e_K = 1$$

under the Assumptions 1 and 2.

For this purpose, we shall consider an autoregressive process $X_K(t)$ of order K, which approximates $X(t)$. In the following, Lemmata 3 and 4 will be used to show $\lim_{N\to\infty} e_K=1$ for $X_K(t)$. Lemma 5 will be used to show the information matrix for $X_K(t)$ is asymptotically equal to that of $X(t)$. Combining these facts, we shall show the result.

At the beginning, let us evaluate asymptotic variances of the estimates. We have

$$|f_K(\lambda)-f(\lambda)| \leqq C_4\alpha^K ,$$

where C_4 is a constant being independent of K. In the following we shall consider the estimation of $f_K(\lambda)$ instead of $f(\lambda)$. We have

$$|f^{(K)}(\lambda)-f_K(\lambda)| \leqq C_3' K\alpha^K ,$$

where C_3' is a constant, independent of K. Now we have

$$\begin{aligned}
\mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f(\lambda))^2 = {} & \mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f^{(K)}(\lambda))^2 \\
& +(f^{(K)}(\lambda)-f_K(\lambda))^2+(f_K(\lambda)-f(\lambda))^2 \\
& +2(f^{(K)}(\lambda)-f_K(\lambda))\,\mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f^{(K)}(\lambda)) \\
& +2(f^{(K)}(\lambda)-f_K(\lambda))(f_K(\lambda)-f(\lambda)) \\
& +2(f_K(\lambda)-f(\lambda))\,\mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f^{(K)}(\lambda)) .
\end{aligned}$$

But when we consider the case in which N is sufficiently large, we can ignore the terms

$$2(f^{(K)}(\lambda)-f_K(\lambda))\,\mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f^{(K)}(\lambda))$$

and

$$2(f_K(\lambda)-f(\lambda))\,\mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f^{(K)}(\lambda))$$

by comparing with the other terms. And we know

$$\begin{aligned}
&|(f^{(K)}(\lambda)-f_K(\lambda))^2+(f_K(\lambda)-f(\lambda))^2+2(f^{(K)}(\lambda)-f_K(\lambda))(f_K(\lambda)-f(\lambda))| \\
&\quad \leqq C_5(K\alpha^K)^2 ,
\end{aligned}$$

where C_5 is a constant, independent of K. When we consider

$$\lim_{N\to\infty} \frac{N}{K}\,\mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f(\lambda))^2 ,$$

we have

$$\lim_{N\to\infty} \frac{N}{K} C_5 (K\alpha^K)^2 = 0$$

by Assumption 2. Using the result of Berk [3], we can see that the distribution function of $\sqrt{N/K}\,(\hat{f}^{(K)}(\lambda)-f^{(K)}(\lambda))$ or $\sqrt{N/K}\,(\hat{f}^{(K)}(\lambda)-f(\lambda))$ tends to the normal distribution function with mean 0 and variance $2f(\lambda)^2$ or $4f(\lambda)^2$ corresponding to when $0<\lambda<1/2$ or when $\lambda=0$ or $1/2$, respectively.

Now let us define an autoregressive process of order K. Let $X_K(t)$ be a stationary normal process with mean 0 and satisfy the relation

$$\sum_{k=0}^{K} a_k X_K(t-k) = \xi(t) ,$$

where $\{a_k\}$ and $\xi(t)$ are the same as those in (1) and $a_0=1$. We shall put $R_K(h)=\mathrm{E}\, X_K(t+h)X_K(t)$ and $\rho_h^{(K)}=R_K(h)/R_K(0)$. Then we can show the following results (see Huzii [6]):

$$|\rho_h^{(K)}| \leqq C(2\alpha)^h , \qquad \sum_{h=1}^{\infty} |\rho_h^{(K)}-\rho_h| \leqq C_0 K(2\alpha)^K ,$$

where C and C_0 are constants, independent of K.

Let samples be $X_K(1), X_K(2), \cdots, X_K(N)$. We shall construct $\{\hat{a}_k^{(K)}\}$, $\hat{\sigma}_\xi^2(K)$ and $\hat{f}^{(K)}(\lambda)$ by using $X_K(t)$ instead of $X(t)$. We shall denote such $\hat{f}^{(K)}(\lambda)$ by $\hat{f}_K^{(K)}(\lambda)$. Using $X_K(t)$ instead of $X(t)$, we can show the same results as Theorems 5 and 6 in Berk's paper [3]. So we can obtain that the distribution function of $\sqrt{N/K}\,(\hat{f}_K^{(K)}(\lambda)-f_K(\lambda))$ tends to the normal distribution function with mean 0 and variance $2f(\lambda)^2$ when $0<\lambda<1/2$ or $4f(\lambda)^2$ when $\lambda=0$ or $1/2$, if $\sqrt{NK}\,\alpha^K$ tends to 0 when N tends to infinity.

Now, for $X_K(t)$, we have the following lemma.

LEMMA 3. *There exists a solution $\tilde{f}_K^{(K)}(\lambda)$ for $f_K(\lambda)$, obtained by solving the likelihood equations, such that*

$$g_N^{(K)}(\lambda) = \hat{f}_K^{(K)}(\lambda) - \tilde{f}_K^{(K)}(\lambda)$$

converges in probability to zero as N tends to infinity.

PROOF. Let us consider the joint estimation of $(\sigma_\xi^2, a_1, a_2, \cdots, a_K)$. Let $\phi_N(\boldsymbol{X}_N^{(K)})$ and $\phi_K(\boldsymbol{X}_K^{(K)})$ be the density functions of $\boldsymbol{X}_N^{(K)}=(X_K(1), X_K(2), \cdots, X_K(N))'$ and $\boldsymbol{X}_K^{(K)}=(X_K(1), X_K(2), \cdots, X_K(K))'$, respectively. Then we have

$$\phi_N(\boldsymbol{X}_N^{(K)}) = \phi_K(\boldsymbol{X}_K^{(K)}) \frac{1}{(2\pi)^{(N-K)/2}(\sigma_\xi^2)^{(N-K)/2}}$$

$$\times \exp\left[-\frac{1}{2\sigma_\xi^2}\sum_{t=K+1}^{N}(X_K(t)+a_1X_K(t-1)+\cdots+a_KX_K(t-K))^2\right].$$

To prove the assertion of this lemma, it is enough to show that

$$\frac{\partial \log \phi_K(\boldsymbol{X}_K^{(K)})}{\partial \sigma_\xi^2}$$

and

$$\frac{\partial \log \phi_K(\boldsymbol{X}_K^{(K)})}{\partial a_k} \qquad \text{for } k=1, 2, \cdots, K,$$

do not affect the solution of the likelihood equations.

Let $\boldsymbol{Q}_K^{(K)}$ be the $K\times K$ matrix whose (k, l) element is $\rho_{k-l}^{(K)}$ and $q_{k,l}^{(K)-1}$ be the (k, l) element of $\boldsymbol{Q}_K^{(K)-1}$. Then we have

$$\begin{aligned}\frac{1}{N}\frac{\partial \log \phi_K(\boldsymbol{X}_K^{(K)})}{\partial a_k} &= -\frac{1}{2}\frac{\rho_k^{(K)}}{\sigma_\xi^2}\frac{1}{N}\sum_{t=1}^{K}\sum_{s=1}^{K} q_{t,s}^{(K)-1}X_K(t)X_K(s)\\ &\quad -\frac{1}{2}\frac{\left(\sum_{k=0}^{K} a_k\rho_k^{(K)}\right)}{\sigma_\xi^2}\frac{1}{N}\sum_{t=1}^{K}\sum_{s=1}^{K}\frac{\partial q_{t,s}^{(K)-1}}{\partial a_k}X_K(t)X_K(s)\\ &\quad +\frac{1}{2}\frac{\rho_k^{(K)}}{\left(\sum_{j=0}^{K} a_j\rho_j^{(K)}\right)}-\frac{1}{2}\frac{1}{N}\frac{\partial}{\partial a_k}\log|\boldsymbol{Q}_K^{(K)}|.\end{aligned}$$

But we can show

$$\mathrm{V}\left(\frac{1}{N}\sum_{t=1}^{K}\sum_{s=1}^{K} q_{t,s}^{(K)-1}X_K(t)X_K(s)\right)=O\left(\frac{K^2}{N^2}\right).$$

Now we have, for $1\leqq j\leqq K-1$,

$$\left|\frac{\partial q_{t,s}^{(K)-1}}{\partial \rho_j^{(K)}}\right|\leqq C_5K, \qquad \left|\frac{\partial \rho_j^{(K)}}{\partial a_k}\right|\leqq C_6,$$

where C_5 and C_6 are constants, independent of K. So we can obtain

$$\left|\frac{\partial q_{t,s}^{(K)-1}}{\partial a_k}\right|=\left|\sum_{j=1}^{K-1}\frac{\partial q_{t,s}^{(K)-1}}{\partial \rho_j^{(K)}}\frac{\partial \rho_j^{(K)}}{\partial a_k}\right|\leqq C_7K^2,$$

where C_7 is a constant, independent of t, s and K. Therefore we can get

$$\mathrm{V}\left(\frac{1}{N}\sum_{t=1}^{K}\sum_{s=1}^{K}\frac{\partial q_{t,s}^{(K)-1}}{\partial a_k}X_K(t)X_K(s)\right)=O\left(\frac{K^6}{N^2}\right).$$

We know

$$\frac{1}{2}\frac{1}{N}\frac{\partial}{\partial a_k}\log|\boldsymbol{Q}_K^{(K)}|-\frac{1}{2}\frac{\rho_k^{(K)}}{\left(\sum_{j=0}^{K}a_j\rho_j^{(K)}\right)}$$
$$=\mathrm{E}\left(-\frac{1}{2}\frac{\rho_k^{(K)}}{\sigma_\xi^2}\frac{1}{N}\sum_{t=1}^{K}\sum_{s=1}^{K}q_{t,s}^{(K)-1}X_K(t)X_K(s)\right)$$
$$+\mathrm{E}\left(-\frac{1}{2}\frac{\left(\sum_{k=0}^{K}a_k\rho_k^{(K)}\right)}{\sigma_\xi^2}\frac{1}{N}\sum_{t=1}^{K}\sum_{s=1}^{K}\frac{\partial q_{t,s}^{(K)-1}}{\partial a_k}X_K(t)X_K(s)\right).$$

So we obtain

$$\left|\frac{1}{2}\frac{1}{N}\frac{\partial}{\partial a_k}\log|\boldsymbol{Q}_K^{(K)}|\right|=O\left(\frac{K^3}{N}\right).$$

Combining the above results, we can get

$$\mathrm{V}\left(\frac{1}{N}\frac{\partial\log\phi_K(\boldsymbol{X}_K^{(K)})}{\partial a_k}\right)=O\left(\frac{K^6}{N^2}\right).$$

And we also have

$$\mathrm{V}\left(\frac{1}{N}\frac{\partial\log\phi_K(\boldsymbol{X}_K^{(K)})}{\partial\sigma_\xi^2}\right)=O\left(\frac{K^2}{N^2}\right).$$

Let $\hat{\boldsymbol{R}}^{(K)}$ (or $\boldsymbol{R}^{(K)}$) be the matrix which is constructed by $X_K(t)$ instead of $X(t)$ in $\hat{\boldsymbol{R}}$ (or $\boldsymbol{R}$), and let us denote

$$\hat{\boldsymbol{\eta}}^{(K)}=\begin{pmatrix}\hat{\eta}_1^{(K)}\\ \hat{\eta}_2^{(K)}\\ \vdots\\ \hat{\eta}_K^{(K)}\end{pmatrix}=\hat{\boldsymbol{R}}^{(K)-1}\boldsymbol{l},$$

where

$$\boldsymbol{l}=(l_1,l_2,\cdots,l_K)'$$
$$=\left(\frac{1}{N}\frac{\partial\log\phi_K(\boldsymbol{X}_K^{(K)})}{\partial a_1},\frac{1}{N}\frac{\partial\log\phi_K(\boldsymbol{X}_K^{(K)})}{\partial a_2},\cdots,\frac{1}{N}\frac{\partial\log\phi_K(\boldsymbol{X}_K^{(K)})}{\partial a_K}\right).$$

Then we have

$$\sqrt{K}\,\|\hat{\boldsymbol{\eta}}^{(K)}\|\leqq\sqrt{K}\,\|\hat{\boldsymbol{R}}^{(K)-1}-\boldsymbol{R}^{-1}\|\,\|\boldsymbol{l}\|+\sqrt{K}\,\|\boldsymbol{R}^{-1}\|\,\|\boldsymbol{l}\|,$$

where

$$\|\boldsymbol{Y}\|^2=\sum_{k=1}^{K}y_k^2,\qquad \boldsymbol{Y}=(y_1,y_2,\cdots,y_k)',$$

and, for a $K\times K$ matrix $\boldsymbol{B}$, $\|\boldsymbol{B}\|=\sup\|\boldsymbol{BY}\|$ for $\|\boldsymbol{Y}\|\leqq 1$. We can show in the same way as that of Berk [3] that $\|\boldsymbol{R}^{-1}\|$ is bounded and $\sqrt{K}$

$\times\|\hat{\boldsymbol{R}}^{(K)-1}-\boldsymbol{R}^{-1}\|$ converges in probability to zero as N tends to infinity (see Berk [3]). Also we can show $\sqrt{K}\|\boldsymbol{l}\|$ converges in probability to zero if K^4/N tends to zero when N tends to infinity. Therefore, under Assumption 2, $\sqrt{K}\|\hat{\boldsymbol{\eta}}^{(K)}\|$ converges in probability to zero. As we have

$$\left|\sum_{k=1}^{K}\hat{\eta}_k^{(K)}\cos 2\pi k\lambda\right|\leqq\sqrt{K}\,\|\hat{\boldsymbol{\eta}}^{(K)}\|\,,\qquad\left|\sum_{k=1}^{K}\hat{\eta}_k^{(K)}\sin 2\pi k\lambda\right|\leqq\sqrt{K}\,\|\hat{\boldsymbol{\eta}}^{(K)}\|\,,$$

$\sum_{k=1}^{K}\hat{\eta}_k^{(K)}\cos 2\pi k\lambda$ and $\sum_{k=1}^{K}\hat{\eta}_k^{(K)}\sin 2\pi k\lambda$ converge in probability to zero under Assumption 2. Furthermore,

$$\begin{aligned}\left|\sum_{k=1}^{K}\hat{\eta}_k^{(K)}\hat{R}(0,k)\right|&\leqq\left|\sum_{k=1}^{K}\hat{\eta}_k^{(K)}(\hat{R}(0,k)-R_K(k))\right|+\left|\sum_{k=1}^{K}\hat{\eta}_k^{(K)}R_K(k)\right|\\&\leqq\|\hat{\boldsymbol{\eta}}^{(K)}\|\sqrt{\sum_{k=1}^{K}(\hat{R}(0,k)-R_K(k))^2}+\|\hat{\boldsymbol{\eta}}^{(K)}\|CR(0)/(1-2\alpha)\,.\end{aligned}$$

Under Assumption 2, we can show $\|\hat{\boldsymbol{\eta}}^{(K)}\|$ and $\sum_{k=1}^{K}(\hat{R}(0,k)-R_K(k))^2$ converge in probability to zero. So under Assumption 2, $\left|\sum_{k=1}^{K}\hat{\eta}_k^{(K)}\hat{R}(0,k)\right|$ converges in probability to zero.

From $\partial\log\psi_N(\boldsymbol{X}_N^{(K)})/\partial a_k=0$ for $k=1,2,\cdots,K$, we have

$$\sum_{j=1}^{K}a_j\hat{R}(0,j)=-\hat{R}(0,k)-2\sigma_\xi^2 l_k$$

for $k=1,2,\cdots,K$, and from $\partial\log\psi(\boldsymbol{X}_N^{(K)})/\partial\sigma_\xi^2=0$, we have

$$\sigma_\xi^2=\hat{R}(0,0)+\sum_{k=1}^{K}a_k\hat{R}(0,k)+\frac{2\sigma_\xi^4}{N-K}\frac{\partial\log\psi_K(\boldsymbol{X}_K^{(K)})}{\partial\sigma_\xi^2}\,.$$

Using the above results, we can obtain the assertion of this lemma.

In the following, let us put $\theta_0=\sigma_\xi^2$, $\theta_1=a_1$, $\theta_2=a_2,\cdots,\theta_K=a_K$ and $\hat{\theta}_0^{(K)}=\hat{\sigma}_\xi^2(K)$, $\hat{\theta}_1^{(K)}=\hat{a}_1^{(K)}$, $\hat{\theta}_2^{(K)}=\hat{a}_2^{(K)},\cdots,\hat{\theta}_K^{(K)}=\hat{a}_K^{(K)}$, for simplicity. We shall denote

$$\boldsymbol{\theta}^{(K)}=(\theta_0,\theta_1,\cdots,\theta_K)'\,,$$

and

$$\hat{\boldsymbol{\theta}}^{(K)}=(\hat{\theta}_0^{(K)},\hat{\theta}_1^{(K)},\cdots,\hat{\theta}_K^{(K)})'\,.$$

Let $\boldsymbol{U}^{(K)}(X)$ and $\boldsymbol{U}^{(K)}(X_K)$ be the $(K+1)\times(K+1)$ matrices whose (i,j) elements are $\mathrm{E}_X(\hat{\theta}_{i-1}^{(K)}-\theta_{i-1})(\hat{\theta}_{j-1}^{(K)}-\theta_{j-1})$ and $\mathrm{E}_{X_K}(\hat{\theta}_{i-1}^{(K)}-\theta_{i-1})(\hat{\theta}_{j-1}^{(K)}-\theta_{j-1})$, respectively, where, for example, $\mathrm{E}_X(\gamma)$ means the mean of the statistic γ for the process $X(t)$. Let $\boldsymbol{W}^{(K)}(X)^{-1}$ and $\boldsymbol{W}^{(K)}(X_K)^{-1}$ be the $(K+1)\times(K+1)$ matrices whose (i,j) elements are

$$\mathrm{E}_X\left(\frac{\partial \log \phi_N(\boldsymbol{X}_N)}{\partial \theta_{i-1}} \frac{\partial \log \phi_N(\boldsymbol{X}_N)}{\partial \theta_{j-1}}\right)$$

and

$$\mathrm{E}_{X_K}\left(\frac{\partial \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial \theta_{i-1}} \frac{\partial \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial \theta_{j-1}}\right),$$

respectively. Let us denote the (i, j) element of $\boldsymbol{W}^{(K)}(X)$ (or $W^{(K)}(X_K)$) as $w_{ij}^{(K)}(X)$ (or $w_{ij}^{(K)}(X_K)$).

Then we have the following lemma.

LEMMA 4. *It holds*

$$\lim_{N\to\infty} \frac{\boldsymbol{y}_K' \boldsymbol{U}^{(K)}(X_K)\boldsymbol{y}_K}{\boldsymbol{y}_K' \boldsymbol{W}^{(K)}(X_K)\boldsymbol{y}_K}=1$$

for any sequence $\{\boldsymbol{y}_K\}$ *of real vectors* $\boldsymbol{y}_K=(y_1^{(K)}, y_2^{(K)},\cdots, y_K^{(K)})'$ *such that* $\boldsymbol{y}_K\neq(0, 0,\cdots, 0)'$.

PROOF. This result can be shown by the same method as the case when K is fixed.

$$\left.\frac{\partial \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial \theta_i}\right|_{\hat{\boldsymbol{\theta}}} \approx 0$$

means

$$\frac{1}{N}\left.\frac{\partial \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial \theta_i}\right|_{\boldsymbol{\theta}}+\frac{1}{N}\sum_{j=0}^{K}(\hat{\theta}_j-\theta_j)\left\{\left.\frac{\partial^2 \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial \theta_j \partial \theta_i}\right|_{\boldsymbol{\theta}}\right.$$
$$\left.+\frac{1}{2}\frac{1}{N}\sum_{l=0}^{K}(\hat{\theta}_l-\theta_l)\left.\frac{\partial^3 \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial \theta_l \partial \theta_j \partial \theta_i}\right|_{\boldsymbol{\theta}+\mu(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})}\right\}=0,$$

where $0<\mu<1$. Let $\hat{\boldsymbol{B}}^{(K)}$ be the $(K+1)\times(K+1)$ matrix whose $(i+1, j+1)$ element $\hat{b}_{i+1,j+1}^{(K)}$ is

$$\hat{b}_{i+1,j+1}^{(K)}=\frac{1}{N}\left.\frac{\partial^2 \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial \theta_j \partial \theta_i}\right|_{\boldsymbol{\theta}}$$
$$+\frac{1}{2}\frac{1}{N}\sum_{l=0}^{K}(\hat{\theta}_l-\theta_l)\left.\frac{\partial^3 \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial \theta_l \partial \theta_j \partial \theta_i}\right|_{\boldsymbol{\theta}+\mu(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})}$$

and $\boldsymbol{B}^{(K)}$ be the $(K+1)\times(K+1)$ matrix whose $(i+1, j+1)$ element $b_{i+1,j+1}$ is

$$b_{i+1,j+1}=\begin{cases} 1/2(\sigma_\xi^2)^2 & i=0,\ j=0 \\ 0 & i=0,\ j\geqq 1 \text{ or } i\geqq 1,\ j=0, \\ R(i\dot{-}j) & i\geqq 1,\ j\geqq 1, \end{cases}$$

Then we can show, by the same method as that of Berk [3], that $\sqrt{K}\|\hat{\boldsymbol{B}}^{(K)-1}-\boldsymbol{B}^{(K)-1}\|$ converges in probability to zero under Assumption 2. Let us put

$$\boldsymbol{l}_N^{(K)}=\left(\frac{1}{N}\frac{\partial \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial\theta_0}, \frac{1}{N}\frac{\partial \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial\theta_1}, \cdots, \frac{1}{N}\frac{\partial \log \phi_N(\boldsymbol{X}_N^{(K)})}{\partial\theta_K}\right)'.$$

Then we have

$$\hat{\boldsymbol{\theta}}^{(K)}-\boldsymbol{\theta}^{(K)}=-\hat{\boldsymbol{B}}^{(K)-1}\boldsymbol{l}_N^{(K)}.$$

And we can show

$$-\sqrt{\frac{N}{K}}\frac{1}{\|\boldsymbol{y}_K\|}\boldsymbol{y}_K'\hat{\boldsymbol{B}}^{(K)-1}\boldsymbol{l}_N^{(K)}-\left(-\sqrt{\frac{N}{K}}\frac{1}{\|\boldsymbol{y}_K\|}\boldsymbol{y}_K'\boldsymbol{B}^{(K)-1}\boldsymbol{l}_N^{(K)}\right)$$

converges in probability to zero when N tends to infinity. So we can obtain

$$\begin{aligned}&\lim_{N\to\infty}\frac{N}{K}\frac{1}{\|\boldsymbol{y}_K\|^2}\boldsymbol{y}_K'\boldsymbol{U}^{(K)}(X_K)\boldsymbol{y}_K\\ &\quad=\lim_{N\to\infty}\frac{N}{K}\frac{1}{\|\boldsymbol{y}_K\|^2}\boldsymbol{y}_K'\boldsymbol{B}^{(K)-1}(\mathrm{E}\,\boldsymbol{l}_N^{(K)}\boldsymbol{l}_N^{(K)\prime})\boldsymbol{B}^{(K)-1}\boldsymbol{y}_K.\end{aligned}$$

Therefore, we can get

$$\lim_{N\to\infty}\frac{\boldsymbol{y}_K'\boldsymbol{U}^{(K)}(X_K)\boldsymbol{y}_K}{\boldsymbol{y}_K'\boldsymbol{W}^{(K)}(X_K)\boldsymbol{y}_K}=\lim_{N\to\infty}\frac{\frac{N}{K}\frac{1}{\|\boldsymbol{y}_K\|^2}\boldsymbol{y}_K'\boldsymbol{B}^{(K)-1}(\mathrm{E}\,\boldsymbol{l}_N^{(K)}\boldsymbol{l}_N^{(K)\prime})\boldsymbol{B}^{(K)-1}\boldsymbol{y}_K}{\frac{N}{K}\frac{1}{\|\boldsymbol{y}_K\|^2}\boldsymbol{y}_K'\boldsymbol{W}^{(K)}(X_K)\boldsymbol{y}_K}=1$$

by using Assumption 2.

In the following, we shall consider the difference between $\boldsymbol{W}^{(K)}(X)$ and $\boldsymbol{W}^{(K)}(X_K)$. This means we have to evaluate

$$\boldsymbol{y}_K'\boldsymbol{W}^{(K)}(X)\boldsymbol{y}_K-\boldsymbol{y}_K'\boldsymbol{W}^{(K)}(X_K)\boldsymbol{y}_K$$

for any sequence $\{\boldsymbol{y}_K\}$ of real vectors $\boldsymbol{y}_K=(y_1^{(K)}, y_2^{(K)},\cdots, y_K^{(K)})'$. But for this purpose, we shall compare $\boldsymbol{W}^{(K)}(X)^{-1}$ with $\boldsymbol{W}^{(K)}(X_K)^{-1}$. In the first place, we shall compare $\mathrm{E}_X(\partial\log\phi_N(\boldsymbol{X}_N)/\partial\theta_{i-1})(\partial\log\phi_N(\boldsymbol{X}_N)/\partial\theta_{j-1})$ with $\mathrm{E}_{X_K}(\partial\log\phi_N(\boldsymbol{X}_N^{(K)})/\partial\theta_{i-1})(\partial\log\phi_N(\boldsymbol{X}_N^{(K)})/\partial\theta_{j-1})$. Let us put $\boldsymbol{\theta}=(\theta_0,\theta_1,\theta_2,\cdots)'$ $=(\sigma_\xi^2, a_1, a_2,\cdots)'$ and $F_{ij}(\boldsymbol{\theta})=\mathrm{E}_X(\partial\log\phi_N(\boldsymbol{X}_N)/\partial\theta_{i-1})(\partial\log\phi_N(\boldsymbol{X}_N)/\partial\theta_{j-1})$. Then we have

$$F_{ij}(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_K}=\mathrm{E}_{X_K}(\partial\log\phi_N(\boldsymbol{X}_N^{(K)})/\partial\theta_{i-1})(\partial\log\phi_N(\boldsymbol{X}_N^{(K)})/\partial\theta_{j-1}),$$

where $\boldsymbol{\theta}_K=(\sigma_\xi^2, a_1, a_2,\cdots, a_K, 0, 0,\cdots)$, and

$$F_{ij}(\boldsymbol{\theta})=F_{ij}(\boldsymbol{\theta})\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_K}+\sum_{k=K+1}^{\infty}a_k\frac{\partial F_{ij}(\boldsymbol{\theta})}{\partial\theta_k}\Big|_{\boldsymbol{\theta}_K+\tau(\boldsymbol{\theta}-\boldsymbol{\theta}_K)},$$

where $0<\tau<1$. By differentiating both sides of the equations

$$\sum_{j=1}^{N}q_{ij}^{-1}\rho_{i'-j}=\begin{cases}1 & i=i'\\ 0 & i\neq i'\end{cases}$$

with respect to ρ_h, we can obtain

$$\left|\frac{\partial q_{ij}^{-1}}{\partial\rho_h}\right|\leqq C_8N\,,\qquad\left|\frac{\partial^2 q_{ij}^{-1}}{\partial\rho_{h'}\partial\rho_h}\right|\leqq C_8'N^2\,,\qquad 1\leqq h,\ h'\leqq N-1$$

where C_8 and C_8' are constants, independent of i, j, h, h' and N. And we have

$$\sum_{h=1}^{\infty}\left|\frac{\partial\rho_h}{\partial a_k}\right|\leqq C_9\,,\qquad\sum_{h=1}^{\infty}\left|\frac{\partial^2\rho_h}{\partial a_{K+l}\partial a_k}\right|<C_9'\,,$$

where C_9 and C_9' are constants, independent of l, h, k and K. So we can get

$$\left|\frac{\partial q_{ij}^{-1}}{\partial a_{K+l}}\right|=\left|\sum_{h=1}^{N-1}\frac{\partial q_{ij}^{-1}}{\partial\rho_h}\frac{\partial\rho_h}{\partial a_{K+l}}\right|\leqq C_{10}N^2\,,\qquad\text{for } 1\leqq l\,,$$

where C_{10} is a constant, independent of i, j, K and l. And we also have

$$\begin{aligned}\left|\frac{\partial^2 q_{ij}^{-1}}{\partial a_{K+l}\partial a_m}\right|&=\left|\sum_{h=1}^{N-1}\left(\frac{\partial^2 q_{ij}^{-1}}{\partial a_{K+l}\partial\rho_h}\frac{\partial\rho_h}{\partial a_m}+\frac{\partial q_{ij}^{-1}}{\partial\rho_h}\frac{\partial^2\rho_h}{\partial a_{K+l}\partial a_m}\right)\right|\\&=\left|\sum_{h=1}^{N-1}\left\{\left(\sum_{h'=1}^{N-1}\frac{\partial^2 q_{ij}^{-1}}{\partial\rho_{h'}\partial\rho_h}\frac{\partial\rho_{h'}}{\partial a_{K+l}}\right)\frac{\partial\rho_h}{\partial a_m}+\frac{\partial q_{ij}^{-1}}{\partial\rho_h}\frac{\partial^2\rho_h}{\partial a_{K+l}\partial a_m}\right\}\right|\\&\leqq C_{11}N^2\,,\end{aligned}$$

where C_{11} is a constant independent of m, l, i, j and N.

Using the above results, we can get

$$\left|\sum_{k=K+1}^{\infty}a_k\frac{\partial F_{ij}(\boldsymbol{\theta})}{\partial\theta_k}\Big|_{\boldsymbol{\theta}_K+\tau(\boldsymbol{\theta}-\boldsymbol{\theta}_K)}\right|\leqq C_{12}N^9\alpha^K\,,$$

where C_{12} is a constant, independent of i, j and N.

So we can obtain

$$\|\boldsymbol{W}^{(K)}(X)^{-1}-\boldsymbol{W}^{(K)}(X_K)^{-1}\|\leqq\sqrt{C_{12}N^9K^2\alpha^K}\,,$$

the right-hand side of which tends to zero as N tends to infinity. Now we have

$$\begin{aligned}\|\boldsymbol{W}^{(K)}(X)-\boldsymbol{W}^{(K)}(X_K)\| &= \|\boldsymbol{W}^{(K)}(X)(\boldsymbol{W}^{(K)}(X_K)^{-1}-\boldsymbol{W}^{(K)}(X)^{-1})\boldsymbol{W}^{(K)}(X_K)\| \\ &\leqq \|\boldsymbol{W}^{(K)}(X)\|\,\|\boldsymbol{W}^{(K)}(X_K)^{-1}-\boldsymbol{W}^{(K)}(X)^{-1}\| \\ &\quad\times \|\boldsymbol{W}^{(K)}(X_K)\| .\end{aligned}$$

We know

$$\boldsymbol{W}^{(K)}(X_K)^2 \leqq \left\{\frac{\sigma_\xi^2}{N-K}\begin{pmatrix} 2\sigma_\xi^2 & 0 \cdots 0 \\ 0 & \\ \vdots & \boldsymbol{R}^{(K)-1} \\ 0 & \end{pmatrix}\right\}^2 .$$

So we can get

$$\|\boldsymbol{W}^{(K)}(X_K)\| \leqq \sqrt{\frac{C_{13}K^3}{(N-K)^2}} ,$$

where C_{13} is a constant, independent of N. Using the above facts, we can obtain

$$\|\boldsymbol{W}^{(K)}(X)-\boldsymbol{W}^{(K)}(X_K)\| \leqq \frac{\|\boldsymbol{W}^{(K)}(X_K)\|^2\|\boldsymbol{W}^{(K)}(X_K)^{-1}-\boldsymbol{W}^{(K)}(X)^{-1}\|}{1-\|\boldsymbol{W}^{(K)}(X_K)^{-1}-\boldsymbol{W}^{(K)}(X)^{-1}\|\,\|\boldsymbol{W}^{(K)}(X_K)\|} ,$$

the right-hand side of which converges to zero as N tends to infinity. Therefore we have

$$\begin{aligned}&\lim_{N\to\infty} |(\boldsymbol{y}_K/\|\boldsymbol{y}_K\|)'(\boldsymbol{W}^{(K)}(X)-\boldsymbol{W}^{(K)}(X_K))(\boldsymbol{y}_K/\|\boldsymbol{y}_K\|)| \\ &\quad\leqq \lim_{N\to\infty} \|\boldsymbol{W}^{(K)}(X)-\boldsymbol{W}^{(K)}(X_K)\| = 0 .\end{aligned}$$

Consequently, we can obtain

$$\begin{aligned}\lim_{N\to\infty}\frac{\boldsymbol{y}_K'\boldsymbol{W}^{(K)}(X_K)\boldsymbol{y}_K}{\boldsymbol{y}_K'\boldsymbol{W}^{(K)}(X)\boldsymbol{y}_K} &= \lim_{N\to\infty}\frac{(\boldsymbol{y}_K/\|\boldsymbol{y}_K\|)'\boldsymbol{W}^{(K)}(X_K)(\boldsymbol{y}_K/\|\boldsymbol{y}_K\|)}{(\boldsymbol{y}_K/\|\boldsymbol{y}_K\|)'\boldsymbol{W}^{(K)}(X)(\boldsymbol{y}_K/\|\boldsymbol{y}_K\|)} \\ &= \lim_{N\to\infty}\frac{\boldsymbol{Z}_K'\boldsymbol{W}^{(K)}(X_K)\boldsymbol{Z}_K}{\boldsymbol{Z}_K'\boldsymbol{W}^{(K)}(X)\boldsymbol{Z}_K} \\ &= \lim_{N\to\infty}\frac{\boldsymbol{Z}_K'\boldsymbol{W}^{(K)}(X)\boldsymbol{Z}_K+\boldsymbol{Z}_K'(\boldsymbol{W}^{(K)}(X_K)-\boldsymbol{W}^{(K)}(X))\boldsymbol{Z}_K}{\boldsymbol{Z}_K'\boldsymbol{W}^{(K)}(X)\boldsymbol{Z}_K} \\ &= 1 ,\end{aligned}$$

where $\boldsymbol{Z}_K=\boldsymbol{y}_K/\|\boldsymbol{y}_K\|$.

Summarizing the above results, we have the following lemma.

LEMMA 5. *Let* $\{\boldsymbol{y}_K\}$ *be any sequence of real vectors* $\boldsymbol{y}_K=(y_1^{(K)}, y_2^{(K)}, \cdots, y_K^{(K)})'$ *such that* $\boldsymbol{y}_K\neq(0, 0,\cdots, 0)'$. *Then*

$$\lim_{N\to\infty}\frac{\boldsymbol{y}_K'\boldsymbol{W}^{(K)}(X_K)\boldsymbol{y}_K}{\boldsymbol{y}_K'\boldsymbol{W}^{(K)}(X)\boldsymbol{y}_K}=1 .$$

We shall take $\boldsymbol{f}^{(K)}$ as $\boldsymbol{y}_K$ in Lemma 5. Then the above discussion means, for sufficiently large N,

$$\frac{N}{K}\,\mathrm{E}_X\,(\hat{f}^{(K)}(\lambda)-f(\lambda))^2 \approx \frac{N}{K}\,\mathrm{E}_{X_K}\,(\hat{f}_K^{(K)}(\lambda)-f(\lambda))^2$$
$$\approx \frac{N}{K}\boldsymbol{f}^{(K)\prime}\boldsymbol{W}^{(K)}(X_K)\boldsymbol{f}^{(K)}$$
$$\approx \frac{N}{K}\boldsymbol{f}^{(K)\prime}\boldsymbol{W}^{(K)}(X)\boldsymbol{f}^{(K)}\,,$$

where "$A\approx B$" means $\lim_{N\to\infty}(A/B)=1$. So, when N is sufficiently large, we have

$$\frac{N}{K}\,\mathrm{E}_X\,(\hat{f}^{(K)}(\lambda)-f(\lambda))^2 \approx \frac{N}{K}\boldsymbol{f}^{(K)\prime}\boldsymbol{W}^{(K)}(X)\boldsymbol{f}^{(K)}\,.$$

Summarizing the above results, we have the following theorem.

THEOREM 2. *Let $X(t)$ be a stationary normal process with mean zero and satisfy Assumption* 1, *and let N and K satisfy Assumption* 2. *Then we have*

$$\lim_{N\to\infty}\frac{(N/K)\,\mathrm{E}\,(\hat{f}^{(K)}(\lambda)-f(\lambda))^2}{(N/K)\boldsymbol{f}^{(K)\prime}\boldsymbol{W}^{(K)}(X)\boldsymbol{f}^{(K)}}=1\,.$$

In general, there would be many ways to define the joint efficiency of estimators of infinite dimensional unknown parameters. In the above discussion, we have defined the efficiency as the limit of the sequence of efficiencies of joint estimators of $(K+1)$ dimensional unknown parameters $(\sigma_\xi^2, a_1, a_2, \cdots, a_K)$ for $K=1, 2, 3, \cdots$. We have shown $\hat{f}^{(K)}(\lambda)$ has the efficiency in this sense for the case we have treated.

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TOKYO INSTITUTE OF TECHNOLOGY AND STANFORD UNIVERSITY

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CORRECTIONS TO
"ON A SPECTRAL ESTIMATE OBTAINED BY AN AUTOREGRESSIVE MODEL FITTING"

MITUAKI HUZII

In the above titled paper (this Annals 29(1977), 415–431), the following corrections should be made:

On page 420, line 15:

$K^4/N \longrightarrow K^7/N$

On page 429, line 10:

$$\| \boldsymbol{W}^{(K)}(X) - \boldsymbol{W}^{(K)}(X_K) \| \leqq \frac{\| \boldsymbol{W}^{(K)}(X_K) \|^2 \| \boldsymbol{W}^{(K)}(X_K)^{-1} - \boldsymbol{W}^{(K)}(X)^{-1} \|}{1 - \| \boldsymbol{W}^{(K)}(X_K)^{-1} - \boldsymbol{W}^{(K)}(X)^{-1} \| \| \boldsymbol{W}^{(K)}(X_K) \|}$$

$$\longrightarrow \| \boldsymbol{W}^{(K)}(X) - \boldsymbol{W}^{(K)}(X_K) \|$$

$$\leqq \frac{\sqrt{2} \| \boldsymbol{W}^{(K)}(X_K) \|^2 \| \boldsymbol{W}^{(K)}(X_K)^{-1} - \boldsymbol{W}^{(K)}(X)^{-1} \|}{1 - \sqrt{2} \| \boldsymbol{W}^{(K)}(X_K)^{-1} \| \| \boldsymbol{W}^{(K)}(X_K) \|^2 \| \boldsymbol{W}^{(K)}(X_K)^{-1} - \boldsymbol{W}^{(K)}(X)^{-1} \|} .$$