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REGIONS OF AUTOCORRELATION COEFFICIENTS AND OF THEIR ESTIMATORS IN A STATIONARY TIME SERIES

TOSHINAO NAKATSUKA

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1. Summary and introduction

In many cases in time series analysis the region of estimators of parameters does not coincide with the region of those parameters. If the true values of the parameters are outside the region of estimators, then whatever observations we get, they are not given by those estimators. And unless the region of parameters contains all possible values of estimators, we may get the meaningless estimates outside it. Therefore it is important to know the relation between these regions.

In this paper we discuss about the regions of autocorrelations $(\rho_1, \cdots, \rho_p)$ and the regions of p variate serial correlations $(r_1, \cdots, r_p)$.

2. Notations

When $\{x_t\}$ is a real-valued wide sense stationary time series with zero mean and a spectral distribution function $F(\lambda)$ on $[0, \pi]$, its autocovariance γ_s at lag s and its autocorrelation ρ_s at lag s are represented such as

$$(2.1) \qquad \gamma_s = \mathrm{E}\, x_t x_{t-s} = \int_0^\pi \cos s\lambda dF(\lambda) ,$$

$$(2.2) \qquad \rho_s = \gamma_s/\gamma_0 , \qquad s=1, 2, \cdots .$$

Then, if we define $\mathcal{F}$ as the set of all probability distribution functions on $[0, \pi]$, $\gamma_0^{-1}F$ is contained in $\mathcal{F}$. Conversely it is well known that for any $G \in \mathcal{F}$ and for any positive number γ_0 there is a stationary process with the spectral distribution function $\gamma_0 G$.

Let $\mathcal{F}_1$ be the set of all absolutely continuous functions in $\mathcal{F}$, $\mathcal{F}_2$ be the set of all jump functions in $\mathcal{F}$ that are constant save for jumps at a finite or denumerable set of points, and $\mathcal{F}_3$ be the set of all continuous functions in $\mathcal{F}$ with a zero derivative almost everywhere. Then, for any F in $\mathcal{F}$, we may put

$$F=\alpha_1F_1+\alpha_2F_2+\alpha_3F_3\,, \qquad (F_j\in\mathcal{F}_j,\ j=1,2,3,\ \alpha_1+\alpha_2+\alpha_3=1)$$

by the Lebesgue's decomposition.

The process for which $F(\lambda)$ is absolutely continuous with the density

$$\frac{d}{d\lambda}F(\lambda)=\frac{\gamma_0}{\pi}\left\{1+2\sum_{s=1}^{p}\rho_s\cos s\lambda\right\}\geqq 0\,, \tag{2.3}$$

is called the pth order moving average process. This density function is rewritten by using real numbers $\theta_1,\cdots,\theta_p$, σ^2 such as

$$\frac{d}{d\lambda}F(\lambda)=\frac{\sigma^2}{\pi}\left|1-\sum_{s=1}^{p}\theta_s e^{is\lambda}\right|^2. \tag{2.4}$$

The autocorrelation ρ_s at lag s of this process is related with numbers θ_s such as

$$\rho_s=\begin{cases}\dfrac{-\theta_s+\theta_1\theta_{s+1}+\cdots+\theta_{p-s}\theta_p}{1+\theta_1^2+\cdots+\theta_p^2}\,, & s=1,\cdots,p\\ 0\,, & s>p\,.\end{cases} \tag{2.5}$$

Let $\mathcal{M}_p$ be the set in $\mathcal{F}_1$ which consists of all functions $\gamma_0^{-1}F$ defined via (2.3) and (2.1).

For an arbitrary set $\mathcal{U}$ in $\mathcal{F}$, we let

$$\boldsymbol{R}_0(\mathcal{U})=\left\{(\rho_1,\cdots,\rho_p):\ \rho_s=\int_0^\pi\cos s\lambda dG(\lambda),\ G\in\mathcal{U}\right\}.$$

For observations $x_1,\cdots,x_n$, we let $\bar{x}=\dfrac{1}{n}\sum_{t=1}^{n}x_t$ and consider the regions $\boldsymbol{U}_{p,n}$, $\boldsymbol{U}_{p,n}^*$, $\boldsymbol{V}_{p,n}$ and $\boldsymbol{V}_{p,n}^*$ such as

$$\boldsymbol{U}_{p,n}=\left\{(r_1,\cdots,r_p):\ r_s=\sum_{t=s+1}^{n}x_tx_{t-s}\Big/\sum_{t=1}^{n}x_t^2,\right.$$
$$\left.(x_1,\cdots,x_n)\in R^n,\ \sum_{t=1}^{n}x_t^2\neq 0\right\}, \tag{2.6}$$

$$\boldsymbol{U}_{p,n}^*=\left\{(r_1,\cdots,r_p):\ r_s=\sum_{t=s+1}^{n}(x_t-\bar{x})(x_{t-s}-\bar{x})\Big/\sum_{t=1}^{n}(x_t-\bar{x})^2,\right.$$
$$\left.(x_1,\cdots,x_n)\in R^n,\ \sum_{t=1}^{n}(x_t-\bar{x})^2\neq 0\right\}, \tag{2.7}$$

$$\boldsymbol{V}_{p,n}=\left\{(r_1,\cdots,r_p):\ r_s=\left(x_1x_{n-s+1}+x_2x_{n-s+2}+\cdots+x_sx_n\right.\right.$$
$$\left.\left.+\sum_{t=s+1}^{n}x_tx_{t-s}\right)\Big/\sum_{t=1}^{n}x_t^2,\ (x_1,\cdots,x_n)\in R^n,\ \sum_{t=1}^{n}x_t^2\neq 0\right\} \tag{2.8}$$

and

$$(2.9)\quad V^*_{p,n}=\left\{(r_1,\cdots,r_p):\ r_s=\left\{(x_1-\bar{x})(x_{n-s+1}-\bar{x})+\cdots+(x_s-\bar{x})(x_n-\bar{x})\right.\right.$$
$$\left.+\sum_{t=s+1}^{n}(x_t-\bar{x})(x_{t-s}-\bar{x})\right\}\Big/\sum_{t=1}^{n}(x_t-\bar{x})^2,$$
$$\left.(x_1,\cdots,x_n)\in R^n,\ \sum_{t=1}^{n}(x_t-\bar{x})^2\neq 0\right\}\ .$$

The r_s in (2.9) is called the circular serial correlation.

3. Regions of autocorrelations

In this section we discuss about $\boldsymbol{R}_p(\mathcal{F})$, $\boldsymbol{R}_p(\mathcal{F}_1)$, $\boldsymbol{R}_p(\mathcal{F}_2)$, $\boldsymbol{R}_p(\mathcal{F}_3)$ and $\boldsymbol{R}_p(\mathcal{M}_p)$. Let $\boldsymbol{p}(\lambda)=(\cos\lambda, \cos 2\lambda,\cdots,\cos p\lambda)$. Then the following theorem holds. For detail, see Karlin and Studden [3], p. 28.

THEOREM 3.1. *$\boldsymbol{R}_p(\mathcal{F})$ is the smallest closed convex set in R^p containing the curve $\{\boldsymbol{p}(\lambda):\ 0\leqq\lambda\leqq\pi\}$.*

For example, $\boldsymbol{R}_1(\mathcal{F})$ is the closed interval $[-1, 1]$ and $\boldsymbol{R}_2(\mathcal{F})$ is the two dimensional region surrounded by the line $\{\rho_2=1\}$ and the quadratic curve $\{\rho_2=2\rho_1^2-1\}$.

We use the notation ∂ to denote the set of all boundary points of a set in R^p. Then the following theorem holds.

THEOREM 3.2. *In R^p,*

$$\boldsymbol{R}_p(\mathcal{F}_1)=\boldsymbol{R}_p(\mathcal{F})-\partial\boldsymbol{R}_p(\mathcal{F})\ .$$

PROOF. If $\boldsymbol{\rho}=(\rho_1,\cdots,\rho_p)'\in\partial\boldsymbol{R}_p(\mathcal{F})$ and $\int\cos s\lambda dF=\rho_s$ $(s=1,\cdots,p)$, then there is a supporting hyperplane $\{\boldsymbol{y}\in R^p:\ \boldsymbol{c}'\boldsymbol{y}=a\}$ to $\boldsymbol{R}_p(\mathcal{F})$ at $\boldsymbol{\rho}$ such that $\boldsymbol{c}'\boldsymbol{y}\leqq a$ for any $\boldsymbol{y}$ in $\boldsymbol{R}_p(\mathcal{F})$. Therefore by Theorem 3.1, $\sum_{j=1}^{p}c_j\cdot\cos j\lambda\leqq a$ for any λ. On the other hand $\int\sum_{j=1}^{p}c_j\cos j\lambda dF=\boldsymbol{c}\boldsymbol{\rho}=a$. Hence F has masses only on the set $\left\{\lambda:\ \sum_{j=1}^{p}c_j\cos j\lambda=a\right\}$ which consists of at most p points. This means that $\boldsymbol{R}_p(\mathcal{F}_1)\subset\boldsymbol{R}_p(\mathcal{F})-\partial\boldsymbol{R}_p(\mathcal{F})$.

Next we prove the converse half. $\boldsymbol{R}_p(\mathcal{F}_1)$ is convex. Because $\nu F_1+(1-\nu)F_2\in\mathcal{F}_1$ for any $F_1, F_2\in\mathcal{F}_1$ and for any ν on the interval $[0, 1]$. Let F be an arbitrary function in $\mathcal{F}$, then there is a sequence in $\mathcal{F}_1$ which converges weakly to F. Therefore $\boldsymbol{R}_p(\mathcal{F}_1)$ is the dense convex set in $\boldsymbol{R}_p(\mathcal{F})$, so that $\boldsymbol{R}_p(\mathcal{F}_1)\supset\boldsymbol{R}_p(\mathcal{F})-\partial\boldsymbol{R}_p(\mathcal{F})$. Q.E.D.

It is easily proved that $\boldsymbol{R}_p(\mathcal{F}_2)$ is the convex set containing the curve $\{\boldsymbol{p}(\lambda):\ 0\leqq\lambda\leqq\pi\}$, so that $\boldsymbol{R}_p(\mathcal{F}_2)=\boldsymbol{R}_p(\mathcal{F})$. Particularly let $\mathcal{U}_{\lambda_1,\dots,\lambda_m}$ be the set of all jump functions in $\mathscr{F}_2$ with jumps only on $\lambda_1, \cdots, \lambda_m$. Let $f_j = F(\lambda_j) - F(\lambda_j -)$ for F in $\mathcal{U}_{\lambda_1,\cdots,\lambda_m}$. Then

$$\rho_s = \sum_{j=1}^{m} f_j \cos s\lambda_j \ , \qquad \sum_{j=1}^{m} f_j = 1 \ .$$

Hence, $\boldsymbol{R}_p(\mathcal{U}_{\lambda_1,\cdots,\lambda_m})$ is the convex set spanned by m points $\boldsymbol{p}(\lambda_1), \cdots, \boldsymbol{p}(\lambda_m)$.

THEOREM 3.3. *In* R^p,

$$\boldsymbol{R}_p(\mathscr{F}_3) = \boldsymbol{R}_p(\mathscr{F}) - \partial \boldsymbol{R}_p(\mathscr{F}) \ .$$

PROOF. By the same method as Theorem 3.2, we find that $\boldsymbol{R}_p(\mathscr{F}_3) \subset \boldsymbol{R}_p(\mathscr{F}) - \partial \boldsymbol{R}_p(\mathscr{F})$ and that $\boldsymbol{R}_p(\mathscr{F}_3)$ is convex. Let $\varphi(\lambda)$ be the Cantor's function on $[0, 1]$ and let

$$\psi(\lambda \,|\, a, b, \xi, \eta) = \begin{cases} \xi + \varphi\left(\dfrac{\lambda - a}{b - a}\right)(\eta - \xi) \ , & a \leqq \lambda \leqq b \\ 0 \ , & \text{otherwise} \end{cases}$$

where $0 \leqq a < b \leqq \pi$. Then for any $F \in \mathscr{F}_1$,

$$F_n(\lambda) = \sum_{j=0}^{n-1} \psi\left(\lambda \,\middle|\, \frac{j}{n}\pi, \frac{j+1}{n}\pi, F\left(\frac{j}{n}\pi\right), F\left(\frac{j+1}{n}\pi\right)\right) \in \mathscr{F}_3$$

and $F(\lambda) = \lim_{n\to\infty} F_n(\lambda)$. Therefore, $\boldsymbol{R}_p(\mathscr{F}_3)$ is the dense convex set in $\boldsymbol{R}_p(\mathscr{F})$, so that $\boldsymbol{R}_p(\mathscr{F}_3) \supset \boldsymbol{R}_p(\mathscr{F}) - \partial \boldsymbol{R}_p(\mathscr{F})$. Q.E.D.

$\boldsymbol{R}_1(\mathcal{M}_1) = [-1/2, 1/2]$. The form of $\boldsymbol{R}_2(\mathcal{M}_2)$ is given by Box and Jenkins [2], p. 72. The convexity of $\boldsymbol{R}_p(\mathcal{M}_p)$ was first proved by Anderson [1] by somewhat more tedious method than the following proof.

THEOREM 3.4. *$\boldsymbol{R}_p(\mathcal{M}_p)$ is the closed convex set in $\boldsymbol{R}_p(\mathscr{F}) - \partial \boldsymbol{R}_p(\mathscr{F})$.*

PROOF. From Theorem 3.2, $\boldsymbol{R}_p(\mathcal{M}_p) \subset \boldsymbol{R}_p(\mathscr{F}) - \partial \boldsymbol{R}_p(\mathscr{F})$. $\boldsymbol{R}_p(\mathcal{M}_p)$ is the set of all vectors $(\rho_1, \cdots, \rho_p)$ satisfying the inequality (2.3), so that the convexity follows. The function of $(y_1, \cdots, y_n)$

$$\min_{0 \leqq \lambda \leqq \pi} \left(1 + 2 \sum_{s=1}^{p} y_s \cos s\lambda\right)$$

is continuous, so that the closedness follows. Q.E.D.

4. $\boldsymbol{U}_{p,n}$ and $\boldsymbol{U}^*_{p,n}$

The following theorem shows that $\boldsymbol{U}_{p,n}$ is closely related with $\boldsymbol{R}_p(\mathcal{M}_p)$ and $\boldsymbol{R}_p(\mathscr{F})$. From this theorem we find that the relation $\boldsymbol{R}_p(\mathcal{M}_p) \subset \boldsymbol{U}_{p,n} \subset \boldsymbol{R}_p(\mathscr{F}) - \partial \boldsymbol{R}_p(\mathscr{F})$ holds for $n \geqq p+1$,

THEOREM 4.1.
(i) $U_{p,p+1}=R_p(\mathcal{M}_p)$.
(ii) $U_{p,n}$ *is the closed convex set in* $R_p(\mathcal{F})-\partial R_p(\mathcal{F})$.
(iii) $U_{p,n}\subset U_{p,n+1}$ *for all* $n\geqq p+1$.
(iv) $\bigcup_{n=p+1}^{\infty} U_{p,n}=R_p(\mathcal{F})-\partial R_p(\mathcal{F})$.

PROOF OF (i). If a vector $(\rho_1,\cdots,\rho_p)$ is contained in $R_p(\mathcal{M}_p)$, then there are numbers $\theta_1,\cdots,\theta_p$ satisfying the relation (2.5). By putting $x_1=-1$ and $x_t=\theta_{t-1}$ $(t=2,\cdots,p+1)$, we find that r_s of (2.6) becomes ρ_s. Therefore $U_{p,p+1}\supset R_p(\mathcal{M}_p)$. Similarly we can easily prove the converse part.

PROOF OF (ii). When $n\geqq p$, $R_p(\mathcal{F})$ and $U_{p,n}$ can be considered as the projections of $R_{n-1}(\mathcal{F})$ and $R_{n-1}(\mathcal{M}_{n-1})$, respectively, into the plane spanned by the first p coordinates. Therefore, (ii) follows from Theorem 3.4.

PROOF OF (iii). We find this easily by putting $x_{n+1}=0$.

PROOF OF (iv). Let $F(\lambda)$ be an arbitrary function in $\mathcal{F}_1$ and $f(\lambda)$ be its density function. Let $\sum_{j=-\infty}^{\infty} c_j e^{ij\lambda}$ be the Fourier series of $\sqrt{f}$. Then $\sqrt{f}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=-N}^{N} c_j e^{ij\lambda}$ in $L^2(-\pi,\pi)$. Therefore if we let $G_N(\lambda)=\int_{-\pi}^{\lambda}\left|\sum_{j=-N}^{N} c_j e^{ij\lambda}\right|^2 d\lambda$, then by the triangular inequality

$$|F(\lambda)-G_N(\lambda)|\leqq\int_{-\pi}^{\lambda}\left|\sqrt{f(\mu)}-\sum_{j=-N}^{N} c_j e^{ij\mu}\right|^2 d\mu\xrightarrow{N\to\infty}0\ .$$

Hence G_N converges weakly to F. Since (constant)$\times G_N$ is contained in $\mathcal{M}_{2N}$, the set $\bigcup_{n=p+1}^{\infty} U_{p,n}$ is the dense convex set in $R_p(\mathcal{F})-\partial R_p(\mathcal{F})$, so that (iv) holds. Q.E.D.

THEOREM 4.2.
(i) $U^*_{p,n}$ *is the closed convex set in* $U_{p,n}$.
(ii) $U^*_{p,n}\subset U^*_{p,n+1}$.
(iii) $U_{p,n}\subset U^*_{p,n+p+1}$.
(iv) $\bigcup_{n=p+1}^{\infty} U^*_{p,n}=R_p(\mathcal{F})-\partial R_p(\mathcal{F})$.

PROOF. Let r_s and r^*_s be the serial correlations defined by (2.6) and (2.7), respectively.

PROOF OF (i). Since $\sum_{t=1}^{n}(x_t-\bar{x})=0$, $U^*_{p,n}$ is the region of $(r_1,\cdots,r_p)$

in the case where $\sum_{t=1}^{n} x_t=0$.

Now, the equation

$$1+2\sum_{s=1}^{n-1} r_s \cos s\lambda=\left(\sum_{t=1}^{n} x_t^2\right)^{-1}\left|\sum_{s=0}^{n-1} x_{s+1}e^{-is\lambda}\right|^2$$

holds, and if we put $\lambda=0$ in this equation, we find that $\sum_{t=1}^{n} x_t=0$ is equivalent to $1+2\sum_{s=1}^{n-1} r_s=0$. Therefore, $U_{p,n}^*$ is the counterpart of $U_{p,n}$ and the hyperplane $\left\{(r_1,\cdots,r_p)\colon \sum_{s=1}^{n-1} r_s=-1/2\right\}$. Hence (i) follows.

PROOF OF (ii). For a vector $(x_1,\cdots,x_n)$, we put $x_{n+1}=\frac{1}{n}\sum_{t=1}^{n} x_t$. Then the r_s^* corresponding to $(x_1,\cdots,x_n)$ is equal to the r_s^* corresponding to $(x_1,\cdots,x_n,x_{n+1})$, so that (ii) follows.

PROOF OF (iii). For a vector $(x_1,\cdots,x_n)$, we put

$$x_t=\begin{cases} 0\,, & n+1\leqq t\leqq n+p \\ -\sum_{t=1}^{n} x_t\,, & t=n+p+1\,. \end{cases}$$

Then, $\sum_{t=1}^{n+p+1} x_t=0$, and the r_s corresponding to $(x_1,\cdots,x_n)$ is equal to the r_s^* corresponding to $(x_1,\cdots,x_{n+p+1})$, so that (iii) follows.

PROOF OF (iv). This is obtained from (iii) and Theorem 4.1.

Q.E.D.

5. $V_{p,n}$ and $V_{p,n}^*$

The purpose of this section is to prove that $V_{p,n}$ is the convex set spanned by the $[n/2]+1$ points $p(2\pi k/n)$, $(k=0,\cdots,[n/2])$ and that $V_{p,n}^*$ is the convex set spanned by the $[n/2]$ points $p(2\pi k/n)$, $(k=1,\cdots,[n/2])$, where $[n/2]$ is $n/2$ for even n and $(n-1)/2$ for odd n. From this conclusion we can obtain the relation $V_{p,n}^*\subset V_{p,n}\subset R_p(\mathscr{F})$.

Let r_s be the serial correlation defined in (2.8) and let $r(x)=(r_1,\cdots,r_n)'$, $x=(x_1,\cdots,x_n)'\neq 0$. Let $\mathrm{Toepl}_n\,[y_1,\cdots,y_n]$ be the $n\times n$ Toeplitz matrix with y_1 on the main diagonal, y_2 on the neighbouring diagonal, etc. Then r_s can be written such as

$$r_s=x'\,\mathrm{Toepl}_n\,[\overbrace{0,\cdots,0}^{s},1/2,0,\cdots,0,1/2,\overbrace{0,\cdots,0}^{s-1}]x/x'x\,.$$

It is well known that the eigenvalues of the circular symmetric matrix $\mathrm{Toepl}_n\,[0, 1/2, 0, \cdots, 0, 1/2]$ are

$$\omega_k = \cos\frac{2\pi k}{n}, \qquad k=0, \cdots, k_0$$

where $k_0=[n/2]$. Let H_k be the eigenspace corresponding to the eigenvalue ω_k. These spaces are orthogonal each other and span the whole space. Particularly, H_0 is the one dimensional space spanned by the vector whose elements are all one.

We select some different numbers $\delta_1, \cdots, \delta_m$ from $0, 1, \cdots, k_0$ and put $\delta=(\delta_1, \cdots, \delta_m)$. Let $B(\delta)$ be the set in R^p such as

$$B(\delta)=\left\{\boldsymbol{r}\left(\sum_{i=1}^m \boldsymbol{x}_i\right):\ \boldsymbol{x}_i \in H_{\delta_i}\right\} .$$

If $m=k_0+1$, $B(\delta)=\boldsymbol{V}_{p,n}$. If we put $\delta=(1, \cdots, k_0)$, $B(\delta)=\boldsymbol{V}^*_{p,n}$. Because, if we put $z_t=x_t-\bar{x}$ and $\boldsymbol{z}=(z_1, \cdots, z_n)'$, then the region of $\boldsymbol{z}$ is the space spanned by $H_1, \cdots, H_{k_0}$.

Let $\Delta(\delta)$ be the convex set in R^p spanned by m points $\boldsymbol{p}((2\pi/n)\delta_1)$, $\cdots, \boldsymbol{p}((2\pi/n)\delta_m)$, that is,

$$\Delta(\delta)=\left\{\sum_{i=1}^m \nu_i \boldsymbol{p}\left(\frac{2\pi}{n}\delta_i\right);\ 0\leqq\nu_i\leqq 1,\ \sum_{i=1}^m \nu_i=1\right\} .$$

Then, the following theorem holds and from this we can derive our purpose.

THEOREM 5.1. *For any* δ

$$B(\delta)=\Delta(\delta) .$$

PROOF. First we prove $B(\delta)\subset\Delta(\delta)$. Assume that there is a vector $\boldsymbol{x}=\sum_{i=1}^m \boldsymbol{x}_i$ $(\boldsymbol{x}_i \in H_{\delta_i})$ such as $\boldsymbol{r}(\boldsymbol{x})=(r_1, \cdots, r_p)' \notin \Delta(\delta)$. Then there is a separating hyperplane $\left\{(y_1, \cdots, y_p):\ \sum_{s=1}^p b_s y_s=c\right\}$ such that $\sum_{s=1}^p b_s y_s<c$ for any $(y_1, \cdots, y_p)$ in $\Delta(\delta)$ and $\sum_{s=1}^p b_s r_s=c$. For these numbers $c, b_1, \cdots, b_p$, we put

$$A=\mathrm{Toepl}_n\left[c, -\frac{1}{2}b_1, \cdots, -\frac{1}{2}b_p, 0, \cdots, 0, -\frac{1}{2}b_p, \cdots, -\frac{1}{2}b_1\right] . \tag{5.1}$$

The eigenvalues of this circular symmetric matrix A are

$$\psi_k=c-\sum_{s=1}^p b_s \cos\frac{2\pi sk}{n}, \qquad k=0, \cdots, k_0 ,$$

and H_k is the eigenspace of A corresponding to ψ_k. Since $\boldsymbol{p}((2\pi/n)\delta_i)$

$\in \Delta(\delta)$, $\psi_{\delta_i} = c - \sum_{s=1}^{p} b_s \cos \frac{2\pi s}{n} \delta_i > 0$. Therefore $c - \sum_{s=1}^{p} b_s r_s = x'Ax/x'x = \left(\sum_{i=1}^{m} \psi_{\delta_i} x_i' x_i\right) \Big/ x'x > 0$. This contradicts the equation $\sum_{s=1}^{p} b_s r_s = c$.

Next, we will prove $B(\delta) \supset \Delta(\delta)$ by the mathematical induction method. When $m=1$, $\Delta(\delta)$ consists of only one point, so that $B(\delta) \subset \Delta(\delta)$ means $B(\delta) = \Delta(\delta)$. Assume that if $m \leqq u-1$, then $B(\delta) \supset \Delta(\delta)$. We will prove the case when $m=u$.

For an arbitrary vector $x_0 = \sum_{i=1}^{u} \nu_i p\left(\frac{2\pi}{n} \delta_i\right)$ in $\Delta(\delta)$, from the assumption there are vectors $x_i \in H_{\delta_i}$ such that

$$x_0 = \nu_u r(x_u) + (1-\nu_u) r\left(\sum_{i=1}^{u-1} x_i\right) .$$

Let $l = \left\{\sum_{s=1}^{p} b_s y_s = c\right\}$ be a hyperplane such as $r(x_u) \in l$ and $r\left(\sum_{i=1}^{u-1} x_i\right) \in l$. From these numbers $c, b_1, \cdots, b_p$, we define A such as (5.1). Then $r(x) \in l$ is equivalent to $x'Ax = 0$, $(x \neq 0)$. Therefore

$$\left(\sum_{i=1}^{u-1} x_i\right)' A \left(\sum_{i=1}^{u-1} x_i\right) = 0 , \qquad x_u' A x_u = 0 ,$$

and

$$\left(\sum_{i=1}^{u-1} x_i\right)' A x_u = \psi_{\delta_u} \left(\sum_{i=1}^{u-1} x_i\right)' x_u = 0 .$$

Hence for any real number τ, $r(\tau) \equiv r\left(\tau x_u + (1-\tau) \sum_{i=1}^{u-1} x_i\right) \in l$, because

$$\left\{\tau x_u + (1-\tau) \sum_{i=1}^{u-1} x_i\right\}' A \left\{\tau x_u + (1-\tau) \sum_{i=1}^{u-1} x_i\right\} = 0 .$$

This means that $r(\tau)$ exists on the line

$$\left\{y \in R^p :\ y = \lambda r(x_u) + (1-\lambda) r\left(\sum_{i=1}^{u-1} x_i\right),\ -\infty < \lambda < \infty\right\} .$$

Since $r(\tau)$ is the continuous function of τ, there is τ such as $r(\tau) = x_0$. Since $r(\tau) \in B(\delta)$, the theorem follows. Q.E.D.

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