REGIONS OF AUTOCORRELATION COEFFICIENTS AND OF THEIR ESTIMATORS IN A STATIONARY TIME SERIES

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1. Summary and introduction

In many cases in time series analysis the region of estimators of parameters does not coincide with the region of those parameters. If the true values of the parameters are outside the region of estimators, then whatever observations we get, they are not given by those estimators. And unless the region of parameters contains all possible values of estimators, we may get the meaningless estimates outside it. Therefore it is important to know the relation between these regions.

In this paper we discuss about the regions of autocorrelations \((\rho_1, \ldots, \rho_p)\) and the regions of \(p\) variate serial correlations \((r_1, \ldots, r_p)\).

2. Notations

When \(\{x_t\}\) is a real-valued wide sense stationary time series with zero mean and a spectral distribution function \(F(\lambda)\) on \([0, \pi]\), its autocovariance \(\gamma_s\) at lag \(s\) and its autocorrelation \(\rho_s\) at lag \(s\) are represented such as

\[
\gamma_s = \mathbb{E} x_{s}x_{s-1} = \int_0^\pi \cos s\lambda dF(\lambda),
\]

\[
\rho_s = \gamma_s / \gamma_0, \quad s = 1, 2, \ldots.
\]

Then, if we define \(\mathcal{F}\) as the set of all probability distribution functions on \([0, \pi]\), \(\gamma_0^{-1}F\) is contained in \(\mathcal{F}\). Conversely it is well known that for any \(G \in \mathcal{F}\) and for any positive number \(\gamma_0\) there is a stationary process with the spectral distribution function \(\gamma_0G\).

Let \(\mathcal{F}_1\) be the set of all absolutely continuous functions in \(\mathcal{F}\), \(\mathcal{F}_2\) be the set of all jump functions in \(\mathcal{F}\) that are constant save for jumps at a finite or denumerable set of points, and \(\mathcal{F}_3\) be the set of all continuous functions in \(\mathcal{F}\) with a zero derivative almost everywhere. Then, for any \(F\) in \(\mathcal{F}\), we may put
by the Lebesgue's decomposition.

The process for which \( F(\lambda) \) is absolutely continuous with the density

\[
\frac{d}{d\lambda} F(\lambda) = \frac{1}{\pi} \left[ 1 + 2 \sum_{i=1}^{p} \rho_i \cos s\lambda \right] \geq 0,
\]

is called the \( p \)th order moving average process. This density function is rewritten by using real numbers \( \theta_1, \ldots, \theta_p, \sigma^2 \) such as

\[
\frac{d}{d\lambda} F(\lambda) = \frac{\sigma^2}{\pi} \left[ 1 - \sum_{i=1}^{p} \theta_i e^{is\lambda} \right]^2.
\]

The autocorrelation \( \rho_s \) at lag \( s \) of this process is related with numbers \( \theta_s \) such as

\[
\rho_s = \begin{cases} 
-\theta_1 + \theta_1 \theta_{s+1} + \cdots + \theta_{s-1} \theta_p, & s = 1, \ldots, p \\
1 + \theta_1^2 + \cdots + \theta_p^2, & s > p
\end{cases}
\]

Let \( \mathcal{M}_p \) be the set in \( \mathcal{F}_1 \) which consists of all functions \( \gamma^{-1}_{\lambda} F \) defined via (2.3) and (2.1).

For an arbitrary set \( \mathcal{U} \) in \( \mathcal{F} \), we let

\[
\mathcal{R}_\mathcal{U}(\mathcal{U}) = \left\{ (\rho_1, \ldots, \rho_p) : \rho_s = \int_0^\infty \cos s\lambda dG(\lambda), \ G \in \mathcal{U} \right\}.
\]

For observations \( x_1, \ldots, x_n \), we let \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \) and consider the regions \( U_{p,n}, U_{p,n}^*, V_{p,n} \) and \( V_{p,n}^* \) such as

\[
U_{p,n} = \left\{ (r_1, \ldots, r_p) : r_s = \sum_{i=s+1}^{n} x_i x_{i-s}/\sum_{i=1}^{n} x_i^2, \right. \\
\left. (x_1, \ldots, x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^2 \neq 0 \right\},
\]

\[
U_{p,n}^* = \left\{ (r_1, \ldots, r_p) : r_s = \sum_{i=s+1}^{n} (x_i - \bar{x})(x_{i-s} - \bar{x})/\sum_{i=1}^{n} (x_i - \bar{x})^2, \right. \\
\left. (x_1, \ldots, x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} (x_i - \bar{x})^2 \neq 0 \right\},
\]

\[
V_{p,n} = \left\{ (r_1, \ldots, r_p) : r_s = \frac{x_1 x_{n-s+1} + x_2 x_{n-s+2} + \cdots + x_s x_n}{\sum_{i=s+1}^{n} x_i^2}, \frac{x_1, \ldots, x_n}{\mathbb{R}^n, \sum_{i=1}^{n} x_i^2 \neq 0} \right\}
\]

and
(2.9) \( V_{p,n}^* = \{ (r_1, \ldots, r_p) : r_s = (x_1 - \bar{x})(x_{n-s+1} - \bar{x}) + \cdots + (x_s - \bar{x})(x_n - \bar{x}) \)
\( + \sum_{i=s+1}^{n} (x_{i} - \bar{x})(x_{i-s} - \bar{x}) / \sum_{i=1}^{n} (x_i - \bar{x})^2, \)
\( (x_1, \ldots, x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} (x_i - \bar{x})^2 \neq 0 \). \)

The \( r_s \) in (2.9) is called the circular serial correlation.

3. Regions of autocorrelations

In this section we discuss about \( R_p(\mathcal{F}) \), \( R_p(\mathcal{F}_1) \), \( R_p(\mathcal{F}_2) \), \( R_p(\mathcal{F}_3) \) and \( R_p(\mathcal{M}_p) \). Let \( p(\lambda) = (\cos \lambda, \cos 2\lambda, \ldots, \cos p\lambda) \). Then the following theorem holds. For detail, see Karlin and Studden [3], p. 28.

**Theorem 3.1.** \( R_p(\mathcal{F}) \) is the smallest closed convex set in \( \mathbb{R}^p \) containing the curve \( \{ p(\lambda) : 0 \leq \lambda \leq \pi \} \).

For example, \( R_1(\mathcal{F}) \) is the closed interval \([-1, 1]\) and \( R_2(\mathcal{F}) \) is the two dimensional region surrounded by the line \( \{ \rho_2 = 1 \} \) and the quadratic curve \( \{ \rho_1 = 2\rho_2^2 - 1 \} \).

We use the notation \( \partial \) to denote the set of all boundary points of a set in \( \mathbb{R}^n \). Then the following theorem holds.

**Theorem 3.2.** In \( \mathbb{R}^n \),
\[ R_p(\mathcal{F}_1) = R_p(\mathcal{F}) - \partial R_p(\mathcal{F}). \]

**Proof.** If \( \rho = (\rho_1, \ldots, \rho_p) \in \partial R_p(\mathcal{F}) \) and \( \int \cos s\lambda dF = \rho_s \) \( (s=1, \ldots, p) \), then there is a supporting hyperplane \( \{ y \in \mathbb{R}^p : c'y = a \} \) to \( R_p(\mathcal{F}) \) at \( \rho \) such that \( c'y \leq a \) for any \( y \) in \( R_p(\mathcal{F}) \). Therefore by Theorem 3.1, \( \sum_{j=1}^{p} c_j \cos j\lambda \leq a \) for any \( \lambda \). On the other hand \( \int \sum_{j=1}^{p} c_j \cos j\lambda dF = cp = a \). Hence \( F \) has masses only on the set \( \{ \lambda : \sum_{j=1}^{p} c_j \cos j\lambda = a \} \) which consists of at most \( p \) points. This means that \( R_p(\mathcal{F}_1) \subset R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) \).

Next we prove the converse half. \( R_p(\mathcal{F}_1) \) is convex. Because \( \nu F_1 + (1-\nu)F_2 \in \mathcal{F}_1 \) for any \( F_1, F_2 \in \mathcal{F} \) and for any \( \nu \) on the interval \([0, 1]\). Let \( F \) be an arbitrary function in \( \mathcal{F} \), then there is a sequence in \( \mathcal{F}_1 \) which converges weakly to \( F \). Therefore \( R_p(\mathcal{F}_1) \) is the dense convex set in \( R_p(\mathcal{F}) \), so that \( R_p(\mathcal{F}_1) \supset R_p(\mathcal{F}) - \partial R_p(\mathcal{F}) \). Q.E.D.

It is easily proved that \( R_p(\mathcal{F}_2) \) is the convex set containing the curve \( \{ p(\lambda) : 0 \leq \lambda \leq \pi \} \), so that \( R_p(\mathcal{F}_2) = R_p(\mathcal{F}) \). Particularly let \( \mathcal{U}_{1, \ldots, m} \) be the
set of all jump functions in $F$ with jumps only on $\lambda_1, \ldots, \lambda_m$. Let $f_j = F(\lambda_j) - F(\lambda_j -)$ for $F$ in $\cup_{i_1, \ldots, i_m}$. Then

$$\rho = \sum_{j=1}^m f_j \cos \lambda_j, \quad \sum_{j=1}^m f_j = 1.$$ 

Hence, $R_p(\cup_{i_1, \ldots, i_m})$ is the convex set spanned by $m$ points $p(\lambda_1), \ldots, p(\lambda_m)$.

**Theorem 3.3.** In $R^p$,

$$R_p(F) = R_p(F) - \partial R_p(F).$$

**Proof.** By the same method as Theorem 3.2, we find that $R_p(F)$

$$\subset R_p(F) - \partial R_p(F)$$

and that $R_p(F)$ is convex. Let $\phi(\lambda)$ be the Cantor's function on $[0, 1]$ and let

$$\phi(\lambda | a, b, \xi, \eta) = \begin{cases} \xi + \phi\left(\frac{\lambda - a}{b - a}\right)(\eta - \xi), & a \leq \lambda \leq b \\ 0, & \text{otherwise} \end{cases}$$

where $0 \leq a < b \leq \pi$. Then for any $F \in F$, \n
$$F_n(\lambda) = \sum_{j=0}^{n-1} \phi\left(\lambda | \frac{j}{n}, \frac{j+1}{n}, F\left(\frac{j}{n}, \frac{j+1}{n}ight) \right) \in F,$$

and $F(\lambda) = \lim_{n \to \infty} F_n(\lambda)$. Therefore, $R_p(F)$ is the dense convex set in $R_p(F)$, so that $R_p(F) \supset R_p(F) - \partial R_p(F)$. Q.E.D.

$R_p(M_0) = [-1/2, 1/2]$. The form of $R_p(M_0)$ is given by Box and Jenkins [2], p. 72. The convexity of $R_p(M_0)$ was first proved by Anderson [1] by somewhat more tedious method than the following proof.

**Theorem 3.4.** $R_p(M_0)$ is the closed convex set in $R_p(F) - \partial R_p(F)$.

**Proof.** From Theorem 3.2, $R_p(M_0) \subset R_p(F) - \partial R_p(F)$. $R_p(M_0)$ is the set of all vectors $(\rho_1, \ldots, \rho_p)$ satisfying the inequality (2.3), so that the convexity follows. The function of $(y_1, \ldots, y_n)$

$$\min_{0 \leq i \leq n} \left(1 + 2 \sum_{i=1}^n y_i \cos \lambda_i \right)$$

is continuous, so that the closedness follows. Q.E.D.

4. $U_{p,n}$ and $U^*_p$

The following theorem shows that $U_{p,n}$ is closely related with $R_p(M_p)$ and $R_p(F)$. From this theorem we find that the relation $R_p(M_p) \subset U_{p,n} \subset R_p(F) - \partial R_p(F)$ holds for $n \geq p + 1$. 


THEOREM 4.1.
(i) \( U_{p,p+1} = R_p(\mathcal{M}_p) \).
(ii) \( U_{p,n} \) is the closed convex set in \( R_p(\mathbb{F}) - \partial R_p(\mathbb{F}) \).
(iii) \( U_{p,n} \subset U_{p,n+1} \) for all \( n \geq p + 1 \).
(iv) \( \bigcup_{n=p+1}^{\infty} U_{p,n} = R_p(\mathbb{F}) - \partial R_p(\mathbb{F}) \).

PROOF OF (i). If a vector \((\rho_1, \ldots, \rho_p)\) is contained in \( R_p(\mathcal{M}_p) \), then there are numbers \( \theta_1, \ldots, \theta_p \) satisfying the relation (2.5). By putting \( x_i = -1 \) and \( x_i = \theta_i - 1 \) (\( i = 2, \ldots, p + 1 \)), we find that \( r_s \) of (2.6) becomes \( \rho_s \). Therefore \( U_{p,p+1} \supset R_p(\mathcal{M}_p) \). Similarly we can easily prove the converse part.

PROOF OF (ii). When \( n \geq p \), \( R_p(\mathbb{F}) \) and \( U_{p,n} \) can be considered as the projections of \( R_{n-1}(\mathbb{F}) \) and \( R_{n-1}(\mathcal{M}_{n-1}) \), respectively, into the plane spanned by the first \( p \) coordinates. Therefore, (ii) follows from Theorem 3.4.

PROOF OF (iii). We find this easily by putting \( x_{n+1} = 0 \).

PROOF OF (iv). Let \( F(\lambda) \) be an arbitrary function in \( \mathcal{F}_1 \) and \( f(\lambda) \) be its density function. Let \( \sum_{j=-\infty}^{N} c_j e^{ij\lambda} \) be the Fourier series of \( \sqrt{f} \).

Then \( \sqrt{f} = \text{l.i.m.} \sum_{j=-N}^{N} c_j e^{ij\lambda} \) in \( L^1(-\pi, \pi) \). Therefore if we let \( G_{N}(\lambda) = \int_{-\pi}^{\lambda} \left| \sum_{j=-N}^{N} c_j e^{ij\mu} \right|^2 d\mu \), then by the triangular inequality

\[
|F(\lambda) - G_{N}(\lambda)| \leq \int_{-\pi}^{\lambda} \sqrt{f(\mu)} - \sum_{j=-N}^{N} c_j e^{ij\mu} \right|^2 d\mu \xrightarrow{N \to \infty} 0.
\]

Hence \( G_{N} \) converges weakly to \( F \). Since \((\text{constant}) \times G_{N} \) is contained in \( \mathcal{M}_{2N} \), the set \( \bigcup_{n=p+1}^{\infty} U_{p,n} \) is the dense convex set in \( R_p(\mathbb{F}) - \partial R_p(\mathbb{F}) \), so that (iv) holds.

Q.E.D.

THEOREM 4.2.
(i) \( U_{p,n}^* \) is the closed convex set in \( U_{p,n} \).
(ii) \( U_{p,n}^* \subset U_{p,n+1}^* \).
(iii) \( U_{p,n} \subset U_{p,n+p+1}^* \).
(iv) \( \bigcup_{n=p+1}^{\infty} U_{p,n}^* = R_p(\mathbb{F}) - \partial R_p(\mathbb{F}) \).

PROOF. Let \( r_s \) and \( r_s^* \) be the serial correlations defined by (2.6) and (2.7), respectively.

PROOF OF (i). Since \( \sum_{i=1}^{n} (x_i - \bar{x}) = 0 \), \( U_{p,n}^* \) is the region of \( (r_1, \ldots, r_p) \)
in the case where $\sum_{i=1}^n x_i = 0$.

Now, the equation

$$1 + 2 \sum_{i=1}^{n-1} r_i \cos s_i = \left( \sum_{i=1}^n x_i^2 \right)^{1/2} | \sum_{i=0}^{n-1} x_{i+1} e^{-i\alpha i} |^2$$

holds, and if we put $\lambda = 0$ in this equation, we find that $\sum_{i=1}^n x_i = 0$ is equivalent to $1 + 2 \sum_{i=1}^{n-1} r_i = 0$. Therefore, $U_{p,n}^*$ is the counterpart of $U_{p,n}$ and the hyperplane $\left\{ (r_1, \cdots, r_p) : \sum_{i=1}^{n-1} r_i = -1/2 \right\}$. Hence (i) follows.

**Proof of (ii).** For a vector $(x_1, \cdots, x_n)$, we put $x_{n+1} = \frac{1}{n} \sum_{i=1}^n x_i$. Then the $r_i^*$ corresponding to $(x_1, \cdots, x_n)$ is equal to the $r_i^*$ corresponding to $(x_1, \cdots, x_n, x_{n+1})$, so that (ii) follows.

**Proof of (iii).** For a vector $(x_1, \cdots, x_n)$, we put

$$x_i = \begin{cases} 0 , & n+1 \leq t \leq n+p \\ -\sum_{i=1}^n x_i , & t = n+p+1 . \end{cases}$$

Then, $\sum_{i=1}^{n+p+1} x_i = 0$, and the $r_i$ corresponding to $(x_1, \cdots, x_n)$ is equal to the $r_i^*$ corresponding to $(x_1, \cdots, x_{n+1})$, so that (iii) follows.

**Proof of (iv).** This is obtained from (iii) and Theorem 4.1.

Q.E.D.

5. $V_{p,n}$ and $V_{p,n}^*$

The purpose of this section is to prove that $V_{p,n}$ is the convex set spanned by the $[n/2]+1$ points $p(2\pi k/n)$, $(k=0, \cdots, [n/2])$ and that $V_{p,n}^*$ is the convex set spanned by the $[n/2]$ points $p(2\pi k/n)$, $(k=1, \cdots, [n/2])$, where $[n/2]$ is $n/2$ for even $n$ and $(n-1)/2$ for odd $n$. From this conclusion we can obtain the relation $V_{p,n}^* \subset V_{p,n} \subset R_p(\mathcal{F})$.

Let $r_i$ be the serial correlation defined in (2.8) and let $r(x) = (r_1, \cdots, r_n)$, $x = (x_1, \cdots, x_n) \neq 0$. Let Toepl$_n[y_1, \cdots, y_n]$ be the $n \times n$ Toeplitz matrix with $y_i$ on the main diagonal, $y_t$ on the neighbouring diagonal, etc. Then $r_i$ can be written such as

$$r_i = x' \text{Toepl}_n \left[ 0, \cdots, 0, 1/2, 0, \cdots, 0, 1/2, 0, \cdots, 0 \right] x / x' x .$$

It is well known that the eigenvalues of the circular symmetric
matrix $\text{Toepl}_n [0, 1/2, 0, \cdots, 0, 1/2]$ are

$$\omega_k = \cos \frac{2\pi k}{n}, \quad k = 0, \cdots, k_0$$

where $k_0 = [n/2]$. Let $H_\delta$ be the eigenspace corresponding to the eigenvalue $\omega_\delta$. These spaces are orthogonal each other and span the whole space. Particularly, $H_\delta$ is the one dimensional space spanned by the vector whose elements are all one.

We select some different numbers $\delta_1, \cdots, \delta_m$ from 0, 1, \cdots, $k_0$ and put $\delta = (\delta_1, \cdots, \delta_m)$. Let $B(\delta)$ be the set in $\mathbb{R}^p$ such as

$$B(\delta) = \left\{ r\left( \sum_{i=1}^m x_i \right): x_i \in H_{\delta_i} \right\}.$$

If $m = k_0 + 1$, $B(\delta) = V_{\nu_n}$. If we put $\delta = (1, \cdots, k_0)$, $B(\delta) = V_{\nu_n}^*$. Because, if we put $z_i = x_i - \bar{x}$ and $z = (z_1, \cdots, z_n)'$, then the region of $z$ is the space spanned by $H_1, \cdots, H_{k_0}$.

Let $\Delta(\delta)$ be the convex set in $\mathbb{R}^p$ spanned by $m$ points $p((2\pi/n)\delta_i), \cdots, p((2\pi/n)\delta_m)$, that is,

$$\Delta(\delta) = \left\{ \sum_{i=1}^m v_i p\left( \frac{2\pi}{n} \delta_i \right): 0 \leq v_i \leq 1, \sum_{i=1}^m v_i = 1 \right\}.$$

Then, the following theorem holds and from this we can derive our purpose.

**THEOREM 5.1.** For any $\delta$

$$B(\delta) = \Delta(\delta).$$

**PROOF.** First we prove $B(\delta) \subset \Delta(\delta)$. Assume that there is a vector $x = \sum_{i=1}^m x_i$ ($x_i \in H_{\delta_i}$) such as $r(x) = (r_1, \cdots, r_p)' \notin \Delta(\delta)$. Then there is a separating hyperplane \( \left\{ (y_1, \cdots, y_p): \sum_{i=1}^p b_i y_i = c \right\} \) such that $\sum_{i=1}^p b_i y_i < c$ for any $(y_1, \cdots, y_p)$ in $\Delta(\delta)$ and $\sum_{i=1}^p b_i r_i = c$. For these numbers $c$, $b_1, \cdots, b_p$, we put

$$(5.1) \quad A = \text{Toepl}_n \left[ c, -\frac{1}{2} b_1, \cdots, -\frac{1}{2} b_p, 0, \cdots, 0, \frac{1}{2} b_p, \cdots, \frac{1}{2} b_1 \right].$$

The eigenvalues of this circular symmetric matrix $A$ are

$$\phi_k = c - \sum_{i=1}^p b_i \cos \frac{2\pi k}{n}, \quad k = 0, \cdots, k_0,$$

and $H_k$ is the eigenspace of $A$ corresponding to $\phi_k$. Since $p((2\pi/n)\delta_i)$
$\in A(\delta)$, $\phi_i = c - \sum_{i=1}^{p} b_i \cos \frac{2\pi s}{n} \delta_i > 0$. Therefore $c - \sum_{i=1}^{p} b_i r_i = x'Ax/x'x = \frac{1}{n} \sum_{i=1}^{m} \phi_i \langle x_i, x \rangle / x'x > 0$. This contradicts the equation $\sum_{i=1}^{p} b_i r_i = c$.

Next, we will prove $B(\delta) \supset A(\delta)$ by the mathematical induction method. When $m = 1$, $A(\delta)$ consists of only one point, so that $B(\delta) \subset A(\delta)$ means $B(\delta) = A(\delta)$. Assume that if $m \leq u - 1$, then $B(\delta) \supset A(\delta)$. We will prove the case when $m = u$.

For an arbitrary vector $x_0 = \sum_{i=1}^{u} \nu_i P\left(\frac{2\pi}{n} \delta_i\right)$ in $A(\delta)$, from the assumption there are vectors $x_i \in H_i$ such that

$$x_0 = \nu u r(x_0) + (1 - \nu u) r\left(\sum_{i=1}^{u-1} x_i\right).$$

Let $l = \left\{ \sum_{i=1}^{p} b_i y_i = c \right\}$ be a hyperplane such as $r(x_0) \in l$ and $r\left(\sum_{i=1}^{u-1} x_i\right) \in l$. From these numbers $c$, $b_1, \ldots, b_p$, we define $A$ such as (5.1). Then $r(x) \in l$ is equivalent to $x'Ax = 0$, $(x \neq 0)$. Therefore

$$\left(\sum_{i=1}^{u-1} x_i\right)'A\left(\sum_{i=1}^{u-1} x_i\right) = 0, \quad x_0'Ax_0 = 0,$$

and

$$\left(\sum_{i=1}^{u-1} x_i\right)'Ax_0 = \phi_0 \left(\sum_{i=1}^{u-1} x_i\right)'x_0 = 0.$$

Hence for any real number $\tau$, $r(\tau) \equiv r\left(\tau x_0 + (1 - \tau) \sum_{i=1}^{u-1} x_i\right) \in l$, because

$$\left(\tau x_0 + (1 - \tau) \sum_{i=1}^{u-1} x_i\right)'A\left(\tau x_0 + (1 - \tau) \sum_{i=1}^{u-1} x_i\right) = 0.$$

This means that $r(\tau)$ exists on the line

$$\left\{ y \in R^p: y = \lambda r(x_0) + (1 - \lambda) r\left(\sum_{i=1}^{u-1} x_i\right), \; -\infty < \lambda < \infty \right\}.$$

Since $r(\tau)$ is the continuous function of $\tau$, there is $\tau$ such as $r(\tau) = x_0$. Since $r(\tau) \in B(\delta)$, the theorem follows.

Q.E.D.

REFERENCES

