Ann. Inst. Statist. Math.
29 (1977), Part A, 389-396

ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF A FUNCTION OF LATENT ROOTS OF THE COVARIANCE MATRIX

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(Received Jan. 10, 1977)

1. Introduction and summary

Most of the statistics used in multivariate analysis based on the assumption of a multivariate normal distribution can be expressed as functions of the latent roots of random matrices. The problems of deriving the distributions have been individually considered by many authors. Several works have been done on the distribution problems associated with the statistics based on the latent roots of the sample covariance matrix, which will be discussed in this paper. Examples are found in T. W. Anderson [2], [3], Davis [4], James [7], Krishnaiah and Schuurmann [9], Krishnaiah and Waikar [10], [11], Sugiura [14], Sugiyama and Tong [16] and others for the test statistics, and in G. A. Anderson [1], Fujikoshi [5], James [6], Muirhead and Chikuse [12], Sugiura [13], Sugiyama [15] and others for the distributions of the latent roots. Some of the articles are concerned with the derivation of exact distributions, and others with that of asymptotic distributions. In many distribution problems, however, the exact distributions in the nonnull case are of no practical use, because of the complicated nature of the expressions. It is therefore of interest to derive the formula of asymptotic expansions for the distributions of statistics expressed, in general, as functions of latent roots.

In this paper we derive an asymptotic expansion for the distribution of a function of latent roots of the sample covariance matrix, when the corresponding latent roots of population covariance matrix are simple. The resulting expansion is given up to and including the term of order n^{-1}. As special cases of our result, asymptotic expansions for the distributions of various kinds of statistics can be derived, some of which have been obtained so far by several authors. In Section 2, as immediate consequences of our result, asymptotic expansions for the distributions of various ratios of latent roots are also given.

2. Main theorem

Let S be the matrix of the corrected sum of squares and sum of products of observations in a sample of size $n+1$ from a p-variate normal distribution with population covariance matrix Σ. Then S has the Wishart distribution $W_p(n, \Sigma)$. Let $l_1 > l_2 > \cdots > l_p > 0$ and $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_p > 0$ be the latent roots of S/m and Σ respectively, where $m = n - 2\Delta$ for a correction factor Δ. Also, let $f(l_1, l_2, \cdots, l_p)$ be a real-valued function defined on a domain D in p-dimensional Euclidean space. We assume that the function f can be expanded around $\Lambda^* = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ $(\in D)$ in Taylor's series.

The main result in this paper is the following theorem.

THEOREM 2.1. *Let $\Phi^{(j)}(x)$ be the jth derivatives of the standard normal distribution function $\Phi(x)$. If the latent roots of Σ are all simple, then the distribution function of*

$$f^* = \sqrt{m}\,\{f(l_1, l_2, \cdots, l_p) - f(\lambda_1, \lambda_2, \cdots, \lambda_p)\}/\tau$$

can be expanded for large m as

$$\text{(2.1)} \qquad \mathrm{P}(f^* < x) = \Phi(x) - \frac{1}{\sqrt{m}}\{g_1 \Phi^{(1)}(x)/\tau + g_3 \Phi^{(3)}(x)/\tau^3\} + \frac{1}{m}\sum_{j=1}^{3} h_{2j} \Phi^{(2j)}(x)/\tau^{2j} + O(m^{-3/2}),$$

where $\tau^2 = 2\sum_{\alpha=1}^{p} \lambda_\alpha^2 \left(\left. \frac{\partial f}{\partial l_\alpha} \right|_{L=\Lambda^*} \right)^2$ *for* $L = (l_1, l_2, \cdots, l_p)$ *and* $\Lambda^* = (\lambda_1, \lambda_2, \cdots, \lambda_p)$. *The coefficients g_1, g_3, h_2, h_4 and h_6 are*

$$g_1 = 2\Delta \sum_\alpha \lambda_\alpha f_\alpha + \sum_\alpha (a_\alpha f_\alpha + \lambda_\alpha^2 f_{\alpha\alpha}),$$

$$g_3 = \frac{4}{3}\sum_\alpha \lambda_\alpha^3 f_\alpha^3 + 2\sum_\alpha \sum_\beta \lambda_\alpha^2 \lambda_\beta^2 f_\alpha f_\beta f_{\alpha\beta},$$

(2.2)

$$h_2 = \frac{1}{2} g_1^2 + 2\Delta\{\sum_\alpha \lambda_\alpha^2 f_\alpha^2 + 2\sum_\alpha \sum_\beta \lambda_\alpha^2 \lambda_\beta f_\alpha f_{\alpha\beta}\} + \sum_\alpha \{(2\lambda_\alpha a_\alpha + b_\alpha) f_\alpha + 4\lambda_\alpha^3 f_{\alpha\alpha}\} f_\alpha + \sum_\alpha \sum_\beta \lambda_\alpha^2 \{2(a_\beta f_{\alpha\beta} + \lambda_\beta^2 f_{\alpha\beta\beta}) f_\alpha + \lambda_\beta^2 f_{\alpha\beta}^2\} + \sum_\alpha \sum_{\beta \neq \alpha} \lambda_\alpha^2 \lambda_\beta \lambda_{\alpha\beta}^2 (\lambda_\beta f_\beta - 2\lambda_\alpha f_\alpha) f_\alpha,$$

$$h_4 = g_1 g_3 - 4\Delta \sum_\alpha \lambda_\alpha f_\alpha \sum_\beta \lambda_\beta^4 f_\beta^2 f_{\beta\beta} + 2\sum_\alpha \lambda_\alpha^4 f_\alpha^4 + 8\sum_\alpha \sum_\beta \lambda_\alpha^3 \lambda_\beta^2 f_\alpha^2 f_\beta f_{\alpha\beta} + \frac{4}{3}\sum_\alpha \sum_\beta \sum_\gamma \lambda_\alpha^2 \lambda_\beta^2 \lambda_\gamma^2 f_\alpha f_\beta f_\gamma f_{\alpha\beta\gamma} + 8\sum_\alpha \sum_\beta \lambda_\alpha^2 \lambda_\beta^4 f_\alpha f_\beta f_{\beta\beta} f_{\alpha\beta}$$

$$-4\sum_{\alpha}\lambda_{\alpha}^{6}f_{\alpha}^{2}f_{\alpha\alpha}^{2}+4\sum_{\alpha<\beta}\sum_{\alpha'<\beta'}\lambda_{\alpha}^{2}\lambda_{\beta}^{2}\{\lambda_{\alpha'}^{2}(\delta_{\alpha\beta'}f_{\beta}+\delta_{\beta\beta'}f_{\alpha})f_{\alpha'}$$
$$+\lambda_{\beta'}^{2}(\delta_{\alpha\alpha'}f_{\beta}+\delta_{\alpha'\beta}f_{\alpha})f_{\beta'}\}f_{\alpha\beta}f_{\alpha'\beta'}\ ,$$

$$h_6=\frac{1}{2}g_3^2\ ,$$

where

$$a_{\alpha}=\lambda_{\alpha}\sum_{\beta\neq\alpha}^{p}\lambda_{\beta}\lambda_{\alpha\beta}\ ,\qquad b_{\alpha}=\lambda_{\alpha}^{2}\sum_{\beta\neq\alpha}^{p}\lambda_{\beta}^{2}\lambda_{\alpha\beta}^{2}\ ,$$

$$f_{\underset{q_\alpha}{\alpha\cdots\alpha}\underset{q_\beta}{\beta\cdots\beta}\underset{q_\gamma}{\gamma\cdots\gamma}}=\frac{\partial^{q_\alpha+q_\beta+q_\gamma}}{\partial l_{\alpha}^{q_\alpha}\partial l_{\beta}^{q_\beta}\partial l_{\gamma}^{q_\gamma}}f(l_1,l_2,\cdots,l_p)|_{L=\Lambda^*}\ ,$$

$\delta_{\alpha\beta}$ *is the Kronecker delta and summations* $\sum_{\alpha}$, $\sum_{\alpha}\sum_{\beta\neq\alpha}$ *and* $\sum_{\alpha<\beta}$ *mean* $\sum_{\alpha=1}^{p}$, $\sum_{\alpha=1}^{p}\sum_{\beta\neq\alpha}^{p}$ *and* $\sum_{\alpha=1}^{p}\sum_{\beta=\alpha+1}^{p}$ *respectively, throughout this paper.*

As special cases of this theorem, we have several interesting results, some of which can be found in literatures. An asymptotic expansion for the distribution of the αth latent root l_α of the sample covariance matrix, which has been studied by G. A. Anderson [1], Muirhead and Chikuse [12] and Sugiura [13], can be obtained by putting $f(l_1,\cdots,l_p)=l_\alpha$ in Theorem 2.1. We note that our formula (2.1) for the latent root l_α holds, if the corresponding latent root λ_α of Σ is simple. Putting $f(l_1,\cdots,l_p)=\sum_{\alpha=1}^{q}l_\alpha\Big/\sum_{\alpha=1}^{p}l_\alpha$ $(q<p)$ in Theorem 2.1 gives an asymptotic expansion for the distribution of the statistic $\sum_{\alpha=1}^{q}l_\alpha\Big/\sum_{\alpha=1}^{p}l_\alpha$, which has recently been obtained by Sugiyama and Tong [16].

Krishnaiah and Waikar [10], [11] have discussed tests of equality of the latent roots of certain matrices. These tests are based upon various ratios of the latent roots, including the ratios of the individual roots to the trace and the ratio of the largest root to the smallest root of the sample covariance matrix. The exact distributions of the above ratios in the null case have been studied by Davis [4], Krishnaiah and Schuurmann [9] and Krishnaiah and Waikar [10], [11]. However, for the nonnull case the exact distributions in the forms useful in practice have not as yet been obtained.

We now give, in special cases of Theorem 2.1, asymptotic expansions for the distributions of test statistics l_i/l_j $(i<j)$, $l_i\Big/\sum_{\alpha=1}^{p}l_\alpha$ for $i,j=1,2,\cdots,p$. Putting $f(l_1,\cdots,l_p)=l_i/l_j$ in Theorem 2.1, we have the following

COROLLARY 2.1. *The distribution function of* $\Lambda_{ij}=\sqrt{m}(l_i/l_j-\lambda_i/\lambda_j)/\tau_{ij}$

$(i<j)$, when the latent roots λ_i and λ_j of Σ are simple, can be expanded for large m as

$$\text{(2.3)}\qquad \mathrm{P}\,(\Lambda_{ij}<x)=\Phi(x)-\frac{1}{\sqrt{m}}\{g_1\Phi^{(1)}(x)/\tau_{ij}+g_3\Phi^{(3)}(x)/\tau_{ij}^3\}+\frac{1}{m}\sum_{k=1}^{3}h_{2k}\Phi^{(2k)}(x)/\tau_{ij}^{2k}+O(m^{-3/2})\,,$$

where $\tau_{ij}^2=4u_{ij}^2$ for $u_{ij}=\lambda_i/\lambda_j$ and the coefficients g_k, h_k are

$$\text{(2.4)}\qquad \begin{aligned} g_1&=u_{ij}(\sum_{\beta\neq i}\lambda_\beta\lambda_{i\beta}-\sum_{\beta\neq j}\lambda_\beta\lambda_{j\beta}+2)\,,\\ g_3&=8u_{ij}^3\,,\\ h_2&=\frac{1}{2}g_1^2-4\Delta u_{ij}^2+u_{ij}^2\{4(\sum_{\beta\neq i}\lambda_\beta\lambda_{i\beta}-\sum_{\beta\neq j}\lambda_\beta\lambda_{j\beta})+\sum_{\beta\neq i}\lambda_\beta^2\lambda_{i\beta}^2+\sum_{\beta\neq j}\lambda_\beta^2\lambda_{j\beta}^2\}\\ &\quad+14u_{ij}^2-6u_{ij}\lambda_i^2\lambda_{ij}^2\,,\\ h_4&=g_1g_3+44u_{ij}^4\,,\\ h_6&=\frac{1}{2}g_3^2\,. \end{aligned}$$

Now put $f(l_1,\cdots,l_p)=l_i\Big/\sum_{\alpha=1}^{p}l_\alpha$. Then the partial derivatives with respect to l_α at Λ^* are given by

$$\text{(2.5)}\qquad f_\alpha=c_\alpha\,,\qquad f_{\alpha\beta}=-(c_\alpha+c_\beta)/\bar{\lambda}\,,\qquad f_{\alpha\beta\gamma}=2(c_\alpha+c_\beta+c_\gamma)/\bar{\lambda}^2\quad \text{for } \alpha,\beta,\gamma=1,2,\cdots,p\,,$$

where $c_i=\sum_{\alpha\neq i}\lambda_\alpha/(\sum_\alpha\lambda_\alpha)^2$, $c_\alpha=-\lambda_i/(\sum_\alpha\lambda_\alpha)^2$ $(\alpha\neq i)$ and $\bar{\lambda}=\sum_\alpha\lambda_\alpha$. Substituting (2.5) in (2.1) and (2.2) gives the asymptotic expansion in

COROLLARY 2.2. *If the latent roots of Σ are all simple, then the distribution function of $\Lambda_i=\sqrt{m}\,(l_i/\sum_\alpha l_\alpha-\lambda_i/\sum_\alpha\lambda_\alpha)/\tau_i$ can be expanded for large m as*

$$\text{(2.6)}\qquad \mathrm{P}\,(\Lambda_i<x)=\Phi(x)-\frac{1}{\sqrt{m}}\{g_1\Phi^{(1)}(x)/\tau_i+g_3\Phi^{(3)}(x)/\tau_i^3\}+\frac{1}{m}\sum_{j=1}^{3}h_{2j}\Phi^{(2j)}(x)/\tau_i^{2j}+O(m^{-3/2})\,,$$

where $\tau_i^2=2\sum_\alpha\lambda_\alpha^2c_\alpha^2$ and the coefficients g_j, h_j are

$$g_1=\sum_\alpha a_\alpha c_\alpha-\frac{2}{\bar{\lambda}}\sum_\alpha\lambda_\alpha^2c_\alpha\,,$$

$$g_3=\frac{4}{3}\sum_\alpha \lambda_\alpha^3 c_\alpha^3-\frac{2}{\bar\lambda}\sum_\alpha\sum_\beta \lambda_\alpha^2\lambda_\beta^2 c_\alpha c_\beta(c_\alpha+c_\beta)\ ,$$

(2.7)

$$h_2=\frac{1}{2}g_1^2+2\Delta\left\{\sum_\alpha \lambda_\alpha^2 c_\alpha^2-\frac{2}{\bar\lambda}\sum_\alpha\sum_\beta \lambda_\alpha^2\lambda_\beta c_\alpha(c_\alpha+c_\beta)\right\}+\sum_\alpha\left\{(2\lambda_\alpha a_\alpha+b_\alpha)\right.$$
$$\left.-\frac{8}{\bar\lambda}\lambda_\alpha^3\right\}c_\alpha^2-\frac{1}{\bar\lambda}\sum_\alpha\sum_\beta \lambda_\alpha^2\{2a_\beta c_\alpha-\lambda_\beta^2(c_\alpha+c_\beta)/\bar\lambda\}(c_\alpha+c_\beta)$$
$$+\frac{4}{\bar\lambda^2}\sum_\alpha\sum_\beta \lambda_\alpha^2\lambda_\beta^2 c_\alpha(c_\alpha+2c_\beta)+\sum_\alpha\sum_{\beta\neq\alpha}\lambda_\alpha^2\lambda_\beta\lambda_{\alpha\beta}^2(\lambda_\beta c_\beta-2\lambda_\alpha c_\alpha)c_\alpha\ ,$$

$$h_4=g_1g_3+2\sum_\alpha \lambda_\alpha^4 c_\alpha^4-\frac{8}{\bar\lambda}\sum_\alpha\sum_\beta \lambda_\alpha^3\lambda_\beta^2 c_\alpha^2 c_\beta(c_\alpha+c_\beta)$$
$$+\frac{8}{3\bar\lambda^2}\sum_\alpha\sum_\beta\sum_\gamma \lambda_\alpha^2\lambda_\beta^2\lambda_\gamma^2 c_\alpha c_\beta c_\gamma(c_\alpha+c_\beta+c_\gamma)$$
$$+\frac{16}{\bar\lambda^2}\{\sum_\alpha\sum_\beta \lambda_\alpha^2\lambda_\beta^4 c_\alpha c_\beta^2(c_\alpha+c_\beta)-\sum_\alpha \lambda_\alpha^6 c_\alpha^4\}$$
$$+\frac{4}{\bar\lambda^2}\sum_{\alpha<\beta}\sum_{\alpha'<\beta'}\lambda_\alpha^2\lambda_\beta^2\{\lambda_{\alpha'}^2(\delta_{\alpha\beta'}c_\beta+\delta_{\beta\beta'}c_\alpha)c_{\alpha'}+\lambda_{\beta'}^2(\delta_{\alpha\alpha'}c_\beta+\delta_{\alpha'\beta}c_\alpha)c_{\beta'}\}$$
$$\cdot(c_\alpha+c_\beta)(c_{\alpha'}+c_{\beta'})\ ,$$

$$h_6=\frac{1}{2}g_3^2\ .$$

3. Proof of the main theorem

This section contains the proof of Theorem 2.1. The method adopted here is based on that discussed in Sugiura [13]. We consider first the function of the latent roots of a symmetric matrix M of order p having the following form

$$M=\Lambda+\varepsilon V^{(1)}+\varepsilon^2V^{(2)}+\varepsilon^3V^{(3)}+\cdots\ , \tag{3.1}$$

where $V^{(j)}=(v_{\alpha\beta}^{(j)})$ $(j=1,2,\cdots)$ are symmetric matrices of order p with $\Lambda=\mathrm{diag}\,[\lambda_1,\lambda_2,\cdots,\lambda_p]$ $(\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_p)$ and ε is a factor to be thought of as real-valued and small. If λ_α is simple, then the perturbation expansion (see for example Wigner [17], p. 40) for the αth largest root l_α of M can be expressed in the form

$$l_\alpha=\lambda_\alpha+\varepsilon l_\alpha^{(1)}+\varepsilon^2 l_\alpha^{(2)}+\varepsilon^3 l_\alpha^{(3)}+\cdots\ , \tag{3.2}$$

where

$$l_\alpha^{(1)}=v_{\alpha\alpha}^{(1)}\ ,$$

(3.3)
$$l_\alpha^{(2)}=v_{\alpha\alpha}^{(2)}+\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}v_{\alpha\beta}^{(1)^2}\ ,$$

$$l_\alpha^{(3)}=v_{\alpha\alpha}^{(3)}+2\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}v_{\alpha\beta}^{(1)}v_{\alpha\beta}^{(2)}-\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}^2 v_{\alpha\alpha}^{(1)}v_{\alpha\beta}^{(1)^2}+\sum_{\beta\neq\alpha}\sum_{\gamma\neq\alpha}\lambda_{\alpha\beta}\lambda_{\alpha\gamma}v_{\alpha\beta}^{(1)}v_{\beta\gamma}^{(1)}v_{\gamma\alpha}^{(1)}$$

and $\lambda_{\alpha\beta}=(\lambda_\alpha-\lambda_\beta)^{-1}$ $(\alpha\neq\beta)$.

The perturbation expansion in the case when the latent root λ_α has arbitrary multiplicity has been studied by Fujikoshi [5] and Konishi [8].

Expanding the real-valued function $f(l_1, l_2, \cdots, l_p)$ around $\Lambda^*=(\lambda_1, \lambda_2, \cdots, \lambda_p)$ in Taylor's series and substituting (3.2) in the resulting expansion gives

$$f(l_1,\cdots,l_p)=f(\lambda_1,\cdots,\lambda_p)+\varepsilon F_1+\varepsilon^2 F_2+\varepsilon^3 F_3+\cdots, \tag{3.4}$$

where

$$F_1=\sum_\alpha l_\alpha^{(1)}f_\alpha ,$$

$$F_2=\sum_\alpha l_\alpha^{(2)}f_\alpha+\frac{1}{2}\sum_\alpha\sum_\beta l_\alpha^{(1)}l_\beta^{(1)}f_{\alpha\beta} , \tag{3.5}$$

$$F_3=\sum_\alpha l_\alpha^{(3)}f_\alpha+\frac{1}{2}\sum_\alpha\sum_\beta (l_\alpha^{(1)}l_\beta^{(2)}+l_\alpha^{(2)}l_\beta^{(1)})f_{\alpha\beta}+\frac{1}{6}\sum_\alpha\sum_\beta\sum_\gamma l_\alpha^{(1)}l_\beta^{(1)}l_\gamma^{(1)}f_{\alpha\beta\gamma} .$$

Here f_α, $f_{\alpha\beta}$ and $f_{\alpha\beta\gamma}$ denote the partial derivatives defined in Section 2 and $l_\alpha^{(j)}$ are given by (3.3).

Now let us consider this result in terms of a Wishart matrix S having $W_p(n, \Sigma)$. In dealing with the distribution of the latent roots, we can assume without loss of generality that $\Sigma=\Lambda=\mathrm{diag}\,[\lambda_1, \lambda_2, \cdots, \lambda_p]$ $(\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_p>0)$. Let

$$\frac{1}{m}S=\Lambda+\frac{1}{\sqrt{m}}V , \tag{3.6}$$

where $m=n-2\Delta$ for a correction factor Δ. Put $\varepsilon=1/\sqrt{m}$ and $V^{(j)}=0$ $(j=2, 3, \cdots)$ in (3.1). Then, it follows from (3.4) that the function of the latent roots l_α of S/m can be expanded in a power series of order $1/\sqrt{m}$ as

$$f(l_1,\cdots,l_p)=f(\lambda_1,\cdots,\lambda_p)+\frac{1}{\sqrt{m}}F_1+\frac{1}{m}F_2+\frac{1}{m\sqrt{m}}F_3+\cdots, \tag{3.7}$$

where

$$F_1=\sum_\alpha v_{\alpha\alpha}f_\alpha ,$$

$$F_2=\sum_\alpha\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}v_{\alpha\beta}^2 f_\alpha+\frac{1}{2}\sum_\alpha\sum_\beta v_{\alpha\alpha}v_{\beta\beta}f_{\alpha\beta} , \tag{3.8}$$

$$F_3=\sum_\alpha\sum_{\beta\neq\alpha}\sum_{\gamma\neq\alpha}\lambda_{\alpha\beta}\lambda_{\alpha\gamma}v_{\alpha\beta}v_{\beta\gamma}v_{\gamma\alpha}f_\alpha-\sum_\alpha\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}^2 v_{\alpha\alpha}v_{\alpha\beta}^2 f_\alpha$$

$$+\frac{1}{2}\sum_{\alpha}\sum_{\beta}(v_{\alpha\alpha}\sum_{\gamma\neq\beta}\lambda_{\beta\gamma}v_{\beta\gamma}^2+v_{\beta\beta}\sum_{\gamma\neq\alpha}\lambda_{\alpha\gamma}v_{\alpha\gamma}^2)f_{\alpha\beta}$$
$$+\frac{1}{6}\sum_{\alpha}\sum_{\beta}\sum_{\gamma}v_{\alpha\alpha}v_{\beta\beta}v_{\gamma\gamma}f_{\alpha\beta\gamma} \qquad \text{for } V=(v_{\alpha\beta}) .$$

Then the characteristic function of $\sqrt{m}\{f(l_1,\cdots,l_p)-f(\lambda_1,\cdots,\lambda_p)\}$ can be expressed as

$$\text{(3.9)} \qquad \mathrm{E}\left[\operatorname{etr}(itAV)\left\{1+\frac{(it)}{\sqrt{m}}F_2+\frac{1}{m}\left((it)F_3+\frac{(it)^2}{2}F_2^2\right)+O(m^{-3/2})\right\}\right],$$

where $A=\operatorname{diag}[f_1,f_2,\cdots,f_p]$ and F_2, F_3 are given by (3.8). Computing each expectation in (3.9), using Lemma 5.1 in Sugiura [13], we have the following form for the characteristic function of $\sqrt{m}\{f(l_1,\cdots,l_p)-f(\lambda_1,\cdots,\lambda_p)\}$:

$$\text{(3.10)} \qquad \exp\left(-t^2\sum_{\alpha}\lambda_\alpha^2 f_\alpha^2\right)\left[1+\frac{1}{\sqrt{m}}\{(it)g_1+(it)^3g_3\}\right.$$
$$\left.+\frac{1}{m}\sum_{j=1}^{3}(it)^{2j}h_{2j}+O(m^{-3/2})\right],$$

where the coefficients g_j, h_j are given by (2.2) in Section 2.

Inverting the characteristic function (3.10) gives the asymptotic expansion (2.1) in Theorem 2.1.

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