Ann. Inst. Statist. Math.
29 (1977), Part A, 379-387

AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTIONS OF THE LATENT ROOTS OF THE WISHART MATRIX WITH MULTIPLE POPULATION ROOTS

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(Received Jan. 7, 1977; revised Mar. 18, 1977)

1. Introduction

Let S be a Wishart matrix having the distribution $W_p(n, \Sigma)$. The exact pdf of the latent roots of S depends on a definite integral over the group $O(p)$ of orthogonal matrices of order p, which was expressed by James [8] as a zonal polynomial series. From a practical point of view asymptotic expansions for the distribution of the roots are needed. The expressions obtained by expanding the definite integral were obtained by Anderson [1], James [8], Chattopadhyay and Pillai [4] and Muirhead and Chikuse [11], Chikuse [6], based on the representation of an orthogonal matrix by a skew symmetric matrix and a partial differential equation method respectively. Sugiura [12], [13] obtained another type of asymptotic expansion for the joint and marginal pdf's of the roots in the case of all population roots being simple, based on a perturbation method (for the method, see Girshick [7], Lawley [10], Anderson [3], etc.). This type of expansion was also obtained in the papers of Muirhead and Chikuse [11], Chikuse [5] and Sugiura [13] by rearranging the first type of expansion.

The purpose of this paper is to extend the perturbation method to the case of multiple population roots. We give a method for the derivation of the latter type of the asymptotic expansion of the joint pdf of the roots of S in the case of the population roots having arbitrary multiplicity. Asymptotic expansions of the marginal pdf's of the roots corresponding to the equal population roots are also obtained.

2. Expansions for the distribution of the roots

Let $d_1 \geq \cdots \geq d_p$ be the latent roots of S. In dealing with the distribution of $D = \text{diag}(d_1, \cdots, d_p)$, we may assume that $\Sigma = \Delta = \text{diag}(\delta_1, \cdots, \delta_p)$, where $\delta_1 \geq \cdots \geq \delta_p > 0$ are the latent roots of Σ. Let Δ have multiple roots as in (2.1),

$$\begin{aligned}\delta_1 &= \cdots = \delta_{q_1} = \lambda_1 ,\\ \delta_{q_1+1} &= \cdots = \delta_{q_1+q_2} = \lambda_2, \cdots, \delta_{p-q_r+1} = \cdots = \delta_p = \lambda_r ,\end{aligned} \tag{2.1}$$

where $\lambda_1 > \cdots > \lambda_r$. Let

$$\frac{1}{m} S = \Delta + \frac{1}{\sqrt{m}} V , \tag{2.2}$$

where $m = n - 2\eta$ and η is a fixed constant. It is probable that the value of η could be chosen to optimise the approximations derived below. An optimal value for η is not yet known, but one could simply put $\eta = 0$ in what follows. Based on the limiting distribution of V, Anderson [2] obtained the limiting distribution of $L = \mathrm{diag}\,(l_1, \cdots, l_p)$ defined by

$$l_j = \sqrt{\frac{m}{2\lambda_\alpha^2}} \left\{ \frac{1}{m} d_j - \lambda_\alpha \right\} , \qquad j \in J_\alpha,\ \alpha = 1, \cdots, r , \tag{2.3}$$

where J_α is the set of integers $q_1 + \cdots + q_{\alpha-1} + 1, \cdots, q_1 + \cdots + q_\alpha$ and $q_0 = 0$. The following Lemma 1 is fundamental in our asymptotic expansion method.

LEMMA 1. *Under $\Sigma = \Delta$ and (2.1), the $(q + \cdots + q_{\alpha-1} + j)$th largest root of S/m can be expressed as the jth largest latent root of*

$$\begin{aligned} Z_\alpha = \lambda_\alpha I &+ \frac{1}{\sqrt{m}} V_{\alpha\alpha} + \frac{1}{m} \sum_{\beta \neq \alpha} \lambda_{\alpha\beta} V_{\alpha\beta} V_{\beta\alpha} \\ &+ \frac{1}{m\sqrt{m}} \left\{ \sum_{\beta \neq \alpha} \sum_{\gamma \neq \alpha} \lambda_{\alpha\beta} \lambda_{\alpha\gamma} V_{\alpha\beta} V_{\beta\gamma} V_{\gamma\alpha} - \frac{1}{2} V_{\alpha\alpha} \sum_{\beta \neq \alpha} \lambda_{\alpha\beta}^2 V_{\alpha\beta} V_{\beta\alpha} \right. \\ &\left. - \frac{1}{2} \sum_{\beta \neq \alpha} \lambda_{\alpha\beta}^2 V_{\alpha\beta} V_{\beta\alpha} V_{\alpha\alpha} \right\} + O_p(m^{-2}) \end{aligned} \tag{2.4}$$

where j is any one of the integers $1, \cdots, q_\alpha$, $\lambda_{\alpha\beta} = (\lambda_\alpha - \lambda_\beta)^{-1}$, the matrices $V_{\alpha\beta}$: $q_\alpha \times q_\beta$ are the submatrices of V partitioned into $q_1, \cdots, q_r$ rows and columns, and O_p means the order in probability.

PROOF. This result shall be proved by a slight modification of a method due to Lawley [10]. We define a matrix M having the same latent roots of S/m by

$$M = H \frac{1}{m} S H' = H \left(\Delta + \frac{1}{\sqrt{m}} V \right) H' ,$$

where H is an orthogonal matrix defined by $H = \left(I - \frac{1}{2\sqrt{m}} E \right)^{-1} \left(I + \frac{1}{2\sqrt{m}} E \right)$ and E is a skew-symmetric matrix defined by

$$E=\begin{bmatrix} 0 & \lambda_{12}V_{12} & \cdots & \lambda_{1r}V_{1r} \\ \lambda_{21}V_{21} & 0 & \cdots & \lambda_{2r}V_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{r1}V_{r1} & \lambda_{r2}V_{r2} & \cdots & 0 \end{bmatrix}. \tag{2.5}$$

We expand M in a series in terms of Δ, V and E, getting

$$M=\Delta+\frac{1}{\sqrt{m}}M^{(1)}+\frac{1}{m}M^{(2)}+\frac{1}{m\sqrt{m}}M^{(3)}+O_p(m^{-2}),$$

where

$$M^{(1)}=E\Delta-\Delta E+V, \qquad M^{(2)}=\frac{1}{2}E(V+M^{(1)})-\frac{1}{2}(V+M^{(1)})E,$$

$$M^{(3)}=\frac{1}{2}E\left(M^{(2)}+\frac{1}{2}E\Delta E\right)-\frac{1}{2}\left(M^{(2)}+\frac{1}{2}E\Delta E\right)E.$$

Let $M^{(j)}$ be partitioned as in (2.5). Noting that $M^{(1)}_{\alpha\beta}=0$ $(\alpha\neq\beta)$, it can be demonstrated that neglecting the terms of order m^{-2} the $(q_1+\cdots+q_{\alpha-1}+j)$th largest latent root of M is equal to the jth largest latent root of

$$\lambda_\alpha I+\frac{1}{\sqrt{m}}M^{(1)}_{\alpha\alpha}+\frac{1}{m}M^{(2)}_{\alpha\alpha}+\frac{1}{m\sqrt{m}}M^{(3)}_{\alpha\alpha}+O_p(m^{-2})$$

which can be reduced to the matrix Z_α given by (2.4).

From Lemma 1 it follows that the latent roots $\{l_j;\ j\in J_\alpha\}$ are equal to the latent roots of the symmetric matrix W_α given by

$$\begin{aligned} W_\alpha=\sqrt{\frac{1}{2\lambda_\alpha^2}}\Bigg[&V_{\alpha\alpha}+\frac{1}{\sqrt{m}}\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}V_{\alpha\beta}V_{\beta\alpha} \\ &+\frac{1}{m}\left\{\sum_{\beta\neq\alpha}\sum_{\gamma\neq\alpha}\lambda_{\alpha\beta}\lambda_{\alpha\gamma}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma\alpha}-\frac{1}{2}V_{\alpha\alpha}\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}^2V_{\alpha\beta}V_{\beta\alpha}\right. \\ &\left.-\frac{1}{2}\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}^2V_{\alpha\beta}V_{\beta\alpha}V_{\alpha\alpha}\right\}+O_p(m^{-3/2})\Bigg]. \end{aligned} \tag{2.6}$$

First we consider the joint distribution of $\{W_1,\cdots,W_r\}$. The joint characteristic function (=ch.f.) of $\{W_1,\cdots,W_r\}$ can be expressed as follows:

$$\begin{aligned} C(T_1,\cdots,T_r)&=\mathrm{E}\left[\exp\left(i\sum_{\alpha=1}^{r}\operatorname{tr}T_\alpha W_\alpha\right)\right] \\ &=\mathrm{E}\left[\exp(i\operatorname{tr}AV)\left\{1+\frac{i}{\sqrt{m}}\sum_{\alpha=1}^{r}\operatorname{tr}Q_\alpha VR_\alpha V\right.\right. \\ &\qquad+\frac{i}{m}\left(\sum_{\alpha=1}^{r}\operatorname{tr}Q_\alpha VR_\alpha VR_\alpha V-\sum_{\alpha=1}^{r}\operatorname{tr}Q_\alpha VR_\alpha^2VB_\alpha V\right. \end{aligned} \tag{2.7}$$

$$+\frac{i}{2}\Big(\sum_{\alpha=1}^{r}\operatorname{tr} Q_\alpha AVR_\alpha V\Big)^2\Big)\Big\}+O_p(m^{-3/2})\Big] ,$$

where the expectation is taken with respect to the distribution of V, T_α is a symmetric matrix of order q_α having $(1+\delta_{ij})t_{ij}/2$ as its (i, j)th element with $\delta_{ij}=0$ $(i\neq j)$ and $\delta_{ii}=1$ and Q_α, R_α, B_α and A are block-diagonal matrices given by

$$Q_\alpha=\operatorname{diag}(0,\cdots,0,\underset{\alpha\text{th block}}{(2\lambda_\alpha^2)^{-1/2}T_\alpha},0,\cdots,0) ,$$

$$A=\sum_{\alpha=1}^{r} Q_\alpha ,$$

$$R_\alpha=\operatorname{diag}(\lambda_{\alpha1}I_{q_1},\cdots,\lambda_{\alpha\alpha-1}I_{q_{\alpha-1}},0\times I_{q_\alpha},\lambda_{\alpha\alpha+1}I_{q_{\alpha+1}},\cdots,\lambda_{\alpha r}I_{q_r}) ,$$

$$B_\alpha=\operatorname{diag}(\underbrace{0,\cdots,0}_{q_1+\cdots+q_{\alpha-1}},\underbrace{1,\cdots,1}_{q_\alpha},\underbrace{0,\cdots,0}_{p-q_1-\cdots-q_\alpha}) .$$

After some calculations as in Sugiura [12], we find that

$$\begin{aligned}(2.8)\quad C(T_1,\cdots,T_r)=\exp\Big(-\frac{1}{2}\sum_{\alpha=1}^{r}\operatorname{tr} T_\alpha^2\Big)\Big[1+\sqrt{\frac{2}{m}}\,i\Big\{\sum_{\alpha=1}^{r}\Big(\eta+\frac{1}{2}a_\alpha\Big)\operatorname{tr} T_\alpha\\
+\frac{i^2}{3}\sum_{\alpha=1}^{r}\operatorname{tr} T_\alpha^3\Big\}+\frac{i^2}{m}\Big\{\Big(\sum_{\alpha=1}^{r}\Big(\eta+\frac{1}{2}a_\alpha\Big)\operatorname{tr} T_\alpha\Big)^2\\
+\sum_{\alpha=1}^{r}\Big(\eta-\frac{1}{2}b_\alpha\Big)\operatorname{tr} T_\alpha^2+\frac{1}{2}\sum_{\alpha\neq\beta}\sum\lambda_{\alpha\beta}^2\lambda_\alpha\lambda_\beta(\operatorname{tr} T_\alpha)\operatorname{tr} T_\beta\\
+\frac{i^2}{2}\sum_{\alpha=1}^{r}\operatorname{tr} T_\alpha^4+\frac{2}{3}i^2\Big(\sum_{\alpha=1}^{r}\Big(\eta+\frac{1}{2}a_\alpha\Big)\operatorname{tr} T_\alpha\Big)\sum_{\alpha=1}^{r}\operatorname{tr} T_\alpha^3\\
+\frac{i^4}{9}\Big(\sum_{\alpha=1}^{r}\operatorname{tr} T_\alpha^3\Big)^2\Big\}+O(m^{-3/2})\Big] ,\end{aligned}$$

where

$$(2.9)\qquad a_\alpha=\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}\lambda_\beta q_\beta ,\qquad b_\alpha=\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}^2\lambda_\beta^2 q_\beta .$$

The inverse Fourier transform of (2.8) is obtained by term-by-term application of Lemmas 2 and 3 given in Section 3. After some simplification, we obtain the joint pdf of $\{W_1,\cdots,W_r\}$ in an expanded form given in (2.10).

$$\begin{aligned}(2.10)\quad g(W_1,\cdots,W_r)=\prod_{\alpha=1}^{r}\frac{\exp(-(1/2)\operatorname{tr} W_\alpha^2)}{\pi^{q_\alpha(q_\alpha+1)/4}2^{q_\alpha/2}}\\
\times\Big[1+\frac{1}{\sqrt{m}}K_1(W_1,\cdots,W_r)+\frac{1}{m}K_2(W_1,\cdots,W_r)\\
+O(m^{-3/2})\Big] ,\end{aligned}$$

where

(2.11)

$$K_1(W_1,\cdots,W_r)=\sqrt{2}\left[\left(\eta-\frac{1}{2}(p+1)\right)\sum_{\alpha=1}^{r}\operatorname{tr}W_\alpha+\frac{1}{3}\sum_{\alpha=1}^{r}\operatorname{tr}W_\alpha^3 +\frac{1}{2}\sum\sum_{\alpha<\beta}\lambda_{\alpha\beta}\{\lambda_\alpha q_\beta\operatorname{tr}W_\alpha-\lambda_\beta q_\alpha\operatorname{tr}W_\beta\}\right]$$

$$K_2(W_1,\cdots,W_r)=\frac{1}{2}K_1(W_1,\cdots,W_r)^2-p\eta^2+\frac{1}{2}p(p+1)\eta -\frac{1}{24}p(2p^2+3p-1)-\left(\eta-\frac{1}{2}(p+1)\right)\sum_{\alpha=1}^{r}\operatorname{tr}W_\alpha^2 -\frac{1}{2}\sum_{\alpha=1}^{r}\operatorname{tr}W_\alpha^4-\frac{1}{2}\sum\sum_{\alpha<\beta}\lambda_{\alpha\beta}^2\{\lambda_\alpha^2 q_\beta\operatorname{tr}W_\alpha^2 +\lambda_\beta^2 q_\alpha\operatorname{tr}W_\beta^2-2\lambda_\alpha\lambda_\beta(\operatorname{tr}W_\alpha)\operatorname{tr}W_\beta-\lambda_\alpha\lambda_\beta q_\alpha q_\beta\}\ .$$

Let $L_\alpha=\operatorname{diag}(l_{q_1+\cdots+q_{\alpha-1}+1},\cdots,l_{q_1+\cdots+q_\alpha})$. Making the transformation

$$W_\alpha=H_\alpha L_\alpha H_\alpha'\ ,\quad H_\alpha\in O(q_\alpha)\ ,\qquad \alpha=1,\cdots,r$$

and noting that the density in (2.10) is a function of the L_α's alone, we obtain an asymptotic expansion for the pdf of L given by

$$f(L)=\prod_{\alpha=1}^{r}\frac{\pi^{q_\alpha(q_\alpha-1)/4}}{2^{q_\alpha/2}\Gamma_{q_\alpha}(q_\alpha/2)}\exp\left(-\frac{1}{2}\sum_{j\in J_\alpha}l_j^2\right)\prod_{\substack{i<j\\ i,j\in J_\alpha}}(l_i-l_j) \times\left[1+\frac{1}{\sqrt{m}}K_1(L_1,\cdots,L_r)+\frac{1}{m}K_2(L_1,\cdots,L_r)+O(m^{-3/2})\right]\ , \tag{2.12}$$

where $\Gamma_p(a)=\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(a-(i-1)/2)$ and $K_j(\cdot)$ are defined by (2.11). The first term, which is the limiting distribution of L, agrees with the result obtained by Anderson [2]. Further, the expansion (2.12) agrees with the result of Sugiura [13] which was obtained by substituting (2.3) in the result of Chattopadhyay and Pillai [4].

Next we consider the distribution of L_α itself. Similarly, inverting the ch.f. of W_α obtained by letting $T_\beta=0$ ($\beta\neq\alpha$) in equation (2.8), we can write the pdf of W_α in an expanded form as

$$g_\alpha(W_\alpha)=\frac{\exp(-(1/2)\operatorname{tr}W_\alpha^2)}{\pi^{q_\alpha(q_\alpha+1)/4}2^{q_\alpha/2}}\left[1+\frac{1}{\sqrt{m}}K_{1\alpha}(W_\alpha)+\frac{1}{m}K_{2\alpha}(W_\alpha) +O(m^{-3/2})\right]\ , \tag{2.13}$$

where

$$K_{1\alpha}(W_\alpha)=\sqrt{2}\left[\left\{\eta-\frac{1}{2}(p+1)+\frac{1}{2}\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}\lambda_\alpha q_\beta\right\}\operatorname{tr}W_\alpha+\frac{1}{3}\operatorname{tr}W_\alpha^3\right]\ , \tag{2.14}$$

$$K_{2\alpha}(W_\alpha)=\frac{1}{2}K_{1\alpha}(W_\alpha)^2-q_\alpha\eta^2+\left\{\frac{1}{2}q_\alpha(2p+1-q_\alpha)-\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}\lambda_\alpha q_\alpha q_\beta\right\}\eta$$
$$-\frac{1}{24}q_\alpha(2q_\alpha^2+3q_\alpha-1)+\frac{1}{4}q_\alpha(q_\alpha-p)(p+1)$$
$$+\frac{1}{4}\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}^2 q_\alpha q_\beta\{(2p+1-q_\alpha-q_\beta)\lambda_\alpha^2+2(q_\alpha-p)\lambda_\alpha\lambda_\beta\}$$
$$-\frac{1}{4}\sum_{\beta\neq\gamma\neq\alpha}\sum\lambda_{\alpha\beta}\lambda_{\alpha\gamma}\lambda_\alpha^2 q_\alpha q_\beta q_\gamma-\left\{\eta-\frac{1}{2}(p+1)\right.$$
$$\left.+\frac{1}{2}\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}^2\lambda_\alpha q_\beta\right\}\operatorname{tr}W_\alpha^2-\frac{1}{2}\operatorname{tr}W_\alpha^4\,.$$

Making the transformation $W_\alpha=H_\alpha L_\alpha H_\alpha'$, $H_\alpha\in O(q_\alpha)$, we have an asymptotic expansion for the pdf of L_α given by

$$f_\alpha(L_\alpha)=\frac{\pi^{q_\alpha(q_\alpha-1)/4}}{2^{q_\alpha/2}\Gamma_{q_\alpha}(q_\alpha/2)}\exp\left(-\frac{1}{2}\sum_{j\in J_\alpha}l_j^2\right)\prod_{\substack{i<j\\ i,j\in J_\alpha}}(l_i-l_j)$$
$$\times\left[1+\frac{1}{\sqrt{m}}K_{1\alpha}(L_\alpha)+\frac{1}{m}K_{2\alpha}(L_\alpha)+O(m^{-3/2})\right],\tag{2.15}$$

where $K_{j\alpha}(L_\alpha)$ are given by (2.14). By letting $q_\alpha=1$ in equation (2.15) we obtain an asymptotic expansion of the marginal pdf of l_j when δ_j is simple. The special case was considered in Muirhead and Chikuse [11] and Sugiura [12].

We note that L_α's are independent in the following asymptotic sense, i.e., it follows that with an error of order n^{-1},

$$f(L)=\prod_{\alpha=1}^{r}f_\alpha(L_\alpha)\,.$$

However, the property does not work for the terms of $O(n^{-1})$.

3. Some integration results

In this section we consider the following integrals needed to find the inverse Fourier transformations of (2.8) with $T_\beta=0$ ($\beta\neq\alpha$) and (2.8) itself:

$$I_\alpha[h(T_\alpha)]=\left(\frac{1}{2\pi}\right)^{q_\alpha(q_\alpha+1)/2}\int_{T_\alpha}\exp(-i\operatorname{tr}T_\alpha W_\alpha)h(T_\alpha)$$
$$\times\exp(-(1/2)\operatorname{tr}T_\alpha^2)dT_\alpha\tag{3.1}$$

and

$$I[g(T_1,\cdots,T_r)]=\left(\frac{1}{2\pi}\right)^{\sum_{\alpha=1}^{r}q_\alpha(q_\alpha+1)/2}\tag{3.2}$$

$$\times \int_{T_1} \cdots \int_{T_r} \exp\left(-i \sum_{\alpha=1}^{r} \operatorname{tr} T_\alpha W_\alpha\right) g(T_1, \cdots, T_r)$$
$$\times \exp\left(-\frac{1}{2} \sum_{\alpha=1}^{r} \operatorname{tr} T_\alpha^2\right) dT_1 \cdots dT_r ,$$

where $T_\alpha = [(1+\delta_{ij})t_{ij}^{(\alpha)}/2]$ with $\delta_{ij}=0$ $(i \neq j)$ and $\delta_{ii}=1$, $dT_\alpha = \prod_{i<j} dt_{ij}^{(\alpha)}$, and $W_\alpha = [w_{ij}^{(\alpha)}]$ are symmetric matrices of order q_α. Then

$$I_\alpha[1] = \phi_\alpha(W_\alpha) , \qquad I[1] = \prod_{\alpha=1}^{r} \phi_\alpha(W_\alpha) . \tag{3.3}$$

where $\phi_\alpha(W_\alpha) = [\pi^{q_\alpha(q_\alpha+1)/4} 2^{q_\alpha/2}]^{-1} \exp(-(1/2) \operatorname{tr} W_\alpha^2)$. Differentiating (3.3) with respect to the elements of W_α's, we have that for polynomials h and g

$$I_\alpha[h(-iT_\alpha)] = h(\partial_\alpha)\phi_\alpha(W_\alpha) \tag{3.4}$$

and

$$I[g(-iT_1, \cdots, -iT_r)] = g(\partial_1, \cdots, \partial_r) \prod_{\alpha=1}^{r} \phi_\alpha(W_\alpha) \tag{3.5}$$

where ∂_α denotes the matrix of differential operators having $(1/2)(1+\delta_{ij})(\partial/\partial w_{ij}^{(\alpha)})$ as its (i, j)th element. Carrying out the operators ∂_α's in (3.4) and (3.5) for some special polynomials h and g, we obtain the following Lemmas 2 and 3.

LEMMA 2. *Letting* $M_\alpha[h(T_\alpha)] = I_\alpha[h(T_\alpha)]\{\phi_\alpha(W_\alpha)\}^{-1}$, *then the following equalities hold.*

$$M_\alpha[i \operatorname{tr} T_\alpha] = \operatorname{tr} W_\alpha ,$$

$$M_\alpha[(i \operatorname{tr} T_\alpha)^2] = (\operatorname{tr} W_\alpha)^2 - q_\alpha ,$$

$$M_\alpha[i^2 \operatorname{tr} T_\alpha^2] = \operatorname{tr} W_\alpha^2 - \frac{1}{2} q_\alpha(q_\alpha+1) ,$$

$$M_\alpha[i^3(\operatorname{tr} T_\alpha^3)] = \operatorname{tr} W_\alpha^3 - \frac{3}{2}(q_\alpha+1) \operatorname{tr} W_\alpha ,$$

$$M_\alpha[i^4(\operatorname{tr} T_\alpha) \operatorname{tr} T_\alpha^3] = (\operatorname{tr} W_\alpha) \operatorname{tr} W_\alpha^3 - \frac{3}{2}(q_\alpha+1)(\operatorname{tr} W_\alpha)^2 - 3 \operatorname{tr} W_\alpha^2 + \frac{3}{2} q_\alpha(q_\alpha+1) ,$$

$$M_\alpha[i^4 \operatorname{tr} T_\alpha^4] = \operatorname{tr} W_\alpha^4 - (\operatorname{tr} W_\alpha)^2 - (2q_\alpha+3) \operatorname{tr} W_\alpha^2 + \frac{1}{4} q_\alpha(2q_\alpha^2+5q_\alpha+5) ,$$

$$M_\alpha[i^6(\operatorname{tr} T_\alpha^3)^2]=(\operatorname{tr} W_\alpha^3)^2-3(q_\alpha+1)(\operatorname{tr} W_\alpha)\operatorname{tr} W_\alpha^3-9\operatorname{tr} W_\alpha^4$$
$$+\frac{9}{4}(q_\alpha^2+2q_\alpha+3)(\operatorname{tr} W_\alpha)^2+\frac{9}{2}(3q_\alpha+4)\operatorname{tr} W_\alpha^2$$
$$-\frac{3}{4}q_\alpha(4q_\alpha^2+9q_\alpha+7)\,.$$

LEMMA 3. *Let* $M[g(T_1,\cdots,T_r)]=I[g(T_1,\cdots,T_r)]\left[\prod_{\alpha=1}^{r}\phi_\alpha(W_\alpha)\right]^{-1}$ *and* $\{a_1,\cdots,a_r\}$ *be arbitrary scalars. Then the following equalities hold.*

$$M\left[\sum_{\alpha=1}^{r}a_\alpha i^k\operatorname{tr} T_\alpha^k\right]=\sum_{\alpha=1}^{r}a_\alpha M_\alpha[i^k\operatorname{tr} T_\alpha^k]\,,\qquad k=1,2,\cdots,$$

$$M\left[i^2\left(\sum_{\alpha=1}^{r}a_r\operatorname{tr} T_\alpha\right)^2\right]=\left(\sum_{\alpha=1}^{r}a_\alpha\operatorname{tr} W_\alpha\right)^2-\sum_{\alpha=1}^{r}a_\alpha^2 q_\alpha\,,$$

$$M\left[i^4\left(\sum_{\alpha=1}^{r}a_\alpha\operatorname{tr} T_\alpha\right)\sum_{\alpha=1}^{r}\operatorname{tr} T_\alpha^3\right]$$
$$=\left(\sum_{\alpha=1}^{r}a_\alpha\operatorname{tr} W_\alpha\right)\left\{\sum_{\alpha=1}^{r}\operatorname{tr} W_\alpha^3-\frac{3}{2}\sum_{\alpha=1}^{r}(q_\alpha+1)\operatorname{tr} W_\alpha\right\}$$
$$-3\sum_{\alpha=1}^{r}a_\alpha\left\{\operatorname{tr} W_\alpha^2-\frac{1}{2}q_\alpha(q_\alpha+1)\right\}\,,$$

$$M\left[i^6\left(\sum_{\alpha=1}^{r}\operatorname{tr} T_\alpha^3\right)^2\right]=\left[\sum_{\alpha=1}^{r}\left\{\operatorname{tr} W_\alpha^3-\frac{3}{2}(q_\alpha+1)\operatorname{tr} W_\alpha\right\}\right]^2$$
$$-9\sum_{\alpha=1}^{r}\left\{\operatorname{tr} W_\alpha^4-\frac{1}{2}(\operatorname{tr} W_\alpha)^2-\frac{1}{2}(3q_\alpha+1)\operatorname{tr} W_\alpha^2\right.$$
$$\left.+\frac{1}{12}q_\alpha(4q_\alpha^2+9q_\alpha+7)\right\}\,.$$

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