THE LIKELIHOOD RATIO CRITERION AND THE ASYMPTOTIC EXPANSION OF ITS DISTRIBUTION

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Summary

Asymptotic expansion of the distribution of the likelihood ratio criterion (LRC) for testing a composite hypothesis is derived under null hypothesis and a correction factor $\rho$ which makes the term of order $1/n$ in the asymptotic expansion of the distribution of it vanish is obtained. The problem is extended to the case of a general composite hypothesis and of Pitman’s local alternatives. The asymptotic distribution of LRC for a simple hypothesis is studied under a fixed alternative.

1. Introduction

Let $X = [x_1, \ldots, x_n]$ be an $m \times n$ observation matrix, where $x_n$’s are independently and identically distributed with probability density function $f(x|\theta)$ depending on an unknown parameter $\theta = (\theta_1, \ldots, \theta_p)$ with $p$ components.

The problem considered is that of testing a composite hypothesis $H_0: \theta_1 = \theta_2$, against $H: \theta_1 \neq \theta_2$, where $\theta' = (\theta_1, \theta_2)$, $\theta_1 = (\theta_{q+1}, \ldots, \theta_p)$ and $\theta_{p,q} = (\theta_{q+1,0}, \ldots, \theta_{p,0})$ is a specified $(p-q)$-dimensional vector. The likelihood ratio criterion (LRC) was proposed by Neyman and Pearson [8] as a method for testing a composite hypothesis. Wilks [12] showed that the limiting distribution of the log-likelihood ratio criterion $-2 \log \lambda$, based on $n$ observations, is a central chi-square distribution with $(p-q)$ degrees of freedom when the null hypothesis $H_0$ is true. The limiting distribution of the LRC under a particular sequence of Pitman’s alternative $H_a: \theta_1 = \theta_2 + e/\sqrt{n}$, $e = (e_{q+1}, \ldots, e_p)$, has been studied by Davidson and Lever [2] and the asymptotic expansion of its distribution up to order $1/\sqrt{n}$ under $H_a$ has been studied by Hayakawa [4]. (Peers [9] studied the case of simple hypothesis.)

The purpose of this paper is to present the asymptotic expansion of the distribution of LRC up to the order $1/n$ under the null hypothesis.
and a correction factor $\rho$ which makes the term of order $1/n$ in the asymptotic distribution of $-2 \log \lambda$ vanish in Section 2, and the asymptotic expansion of the distribution of LRC under a general composite hypothesis and Pitman’s alternatives, respectively, in Section 3. We also study the asymptotic behavior of the distribution of $-2 \log \lambda$ for a simple hypothesis under a fixed alternative $H_0 : \theta = \theta_1 + \epsilon$ in Section 4.

2. Expansion of the LRC under the null hypothesis

In this section we discuss some notations needed for our consideration and the asymptotic expansion of the distribution of a LRC under the composite hypothesis.

Let the likelihood ratio criterion $\lambda$ for $H_0$ versus $H$ be

\[ \lambda = \prod_{s=1}^{n} \frac{f(x_s | \hat{\theta}_1, \theta_m)}{f(x_s | \theta_1, \hat{\theta}_1)} \]

where $\hat{\theta}' = (\hat{\theta}_1', \hat{\theta}_2')$ is the maximum likelihood estimator for $\theta$ under $H$ and $\hat{\theta}_1$ is that for $\theta_1$ under $H_0$.

Defining the log-likelihood function by $L(\theta) = \sum_{s=1}^{n} \log f(x_s | \theta)$ we have

\[ \log \lambda = L(\hat{\theta}_1, \theta_m) - L(\hat{\theta}_1, \hat{\theta}_2). \]

In the asymptotic expansion of the log-likelihood function the following notations and assumptions will be adopted.

(i) The function $L(\theta)$ is regular with respect to $\theta$ derivatives up to and including those of fourth order.

(ii) Any function evaluated at the point $\theta = \hat{\theta}$ will be distinguished by the addition of a circumflex.

(iii) Any function evaluated at the point $\theta_1 = \theta_1$, $\theta_2 = \theta_2$ will be distinguished by the addition of a tilde.

(iv) Let

\[ v_i = \sqrt{n}(\hat{\theta}_i - \theta_i), \quad v' = (v_1, \ldots, v_p), \]

\[ w_i = \sqrt{n}(\tilde{\theta}_i - \theta_i), \quad w' = (w_1, \ldots, w_q), \]

i.e., $v = \sqrt{n}(\hat{\theta} - \theta)$, $w = \sqrt{n}(\tilde{\theta} - \theta)$.

(v) Let

\[ Y_{i_1 \ldots i_p} = n^{-1/2} \sum_{a=1}^{n} y_{a i_1 \ldots i_p}, \quad Y_{i_1 \ldots i_p} = \frac{\partial^{2} \log f(x_a | \theta)}{\partial \theta_{i_1} \ldots \partial \theta_{i_p}} \]

\[ j_1, \ldots, i_p = 1, \ldots, p; \quad l = 1, 2, 3, 4. \]

\[ y = (y_1, \ldots, y_p)' , \quad Y = (y_{ij}) , \quad Y_{ij} = (y_{ij}) , \quad Y_{i j k l} = (y_{ijkl}) \]
\[ \kappa_{i,j} = E(y_{ij}) = E(y_{atij}) \]
\[ \kappa_{i,j,k} = \sqrt{n} E(y_{ij}y_{ik}) = E(y_{atij}y_{atk}) \]
\[ \kappa_{i,j,k} = \sqrt{n} E(y_{ij}y_{jk}) = E(y_{atij}y_{atjk}) \]
\[ \kappa_{i,j,k} = \sqrt{n} E(y_{ijk}) = E(y_{atijk}) \]
\[ \kappa_{i,j,k} = E(y_{atij}y_{atk}) \]
\[ \kappa_{i,j,k} = E(y_{atij}y_{atk}) \]
\[ \kappa_{i,j,k} = E(y_{atij}y_{atjk}) \]
\[ \kappa_{i,j,k} = E(y_{atij}y_{atjk}) \]

By the regularity conditions we have

\[ K = (\kappa_{i,j}) = E(yy') = -E(Y) = -K_{..} \]

We assume the non-singularity of \( Y \) with probability one and the positive definiteness of \( K \) which is a Fisher's information matrix. The partitioned matrices are denoted as

\[
Y = \begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{12} & Y_{22}
\end{bmatrix}
K = \begin{bmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{bmatrix}
\]

(vi) For three and four suffix quantities, the following summation notations will be used: Let \( A \) be a \( p \times p \) matrix, and \( a, b \) and \( c \) be \( p \times 1 \) column vectors,

\[ K_{..} \circ a \circ b \circ c = \sum_{i,j,k} \kappa_{ijk} a_i b_j c_k \text{ scalar} \]

\[ K_{..} \circ a \circ b = (\sum_{j,k} \kappa_{ijk} a_j b_k) \text{ column vector with } p \text{ components designated by index } i \]

\[ K_{..} \circ a = (\sum_k \kappa_{ijk} a_k) \text{ } p \times p \text{ matrix with } (i, j)\text{th element } \sum_k \kappa_{ijk} a_k \]

\[ K_{..} \circ A \circ b = \sum_{i,j,k} \kappa_{ijk} a_i b_j c_k \text{ scalar} \]

\[ K_{..} \circ A \circ B \circ C = \sum \kappa_{ijk} a_i b_j c_k \]

We also use the following abbreviate notations:

\[ K_{..} (\circ a)^3 = K_{..} \circ a \circ a \circ a \]

Let the set of indices \( \{1, 2, \ldots, p\} \) be partitioned into \( \{1, 2, \ldots, q\} \) and \( \{q+1, \ldots, p\} \) and let vectors and matrices be correspondingly partitioned.
\[ \alpha' = (\alpha_1', \alpha_2'), \text{ etc.} \] Summation with respect to the indices in the divided set will be denoted as follows:

\[ K_{111} \circ \alpha_1 \circ \beta_1 \circ \gamma_1 = \sum_{i=1}^{q} \kappa_{ijk} a_i b_j c_k \]

\[ K_{112} \circ \alpha_1 \circ \beta_1 \circ \gamma_2 = \sum_{i,j=1}^{q} \kappa_{ijk} a_i b_j c_a \]

\[ K_{112} \circ \alpha_2 \circ \beta_2 = \sum_{i=1}^{q} \sum_{a,b=1}^{q+1} \kappa_{ab} a_i b_a \]

(vii) Symbols \( o_p \) and \( O_p \) denote the orders of magnitude in probability sense.

Expanding \( L(\hat{\theta}_1, \theta_2) \) in a Taylor series about the point \( \theta_1 = \hat{\theta}_1, \theta_2 = \hat{\theta}_2 \), we have

\[
2 \log \lambda = 2 \{ L(\hat{\theta}_1, \theta_2) - L(\hat{\theta}_1, \hat{\theta}_2) \}
= (u - v)' \hat{Y}(u - v) + \frac{1}{3} \hat{Y} \cdot (u - v)^3
+ \frac{1}{12} \hat{Y} \cdot (u - v)^4 + o_p(1/n)
\]

where

\[ u' = (u', 0'), \quad v' = (v', v_2'). \]

Noting that

\[
\hat{Y} = Y + Y_{...} \circ v + \frac{1}{2} Y_{...} (u - v)^2 + o_p(1/n),
\]

\[
\hat{Y}_{...} = Y_{...} + Y_{...} \circ v + o_p(1/n),
\]

\[
\hat{Y}_{...} = Y_{...} + o_p(1/n),
\]

we have

\[
(2) \quad 2 \log \lambda = (u - v)' Y(u - v)
+ \frac{1}{3} Y_{...} (u - v)^3 + Y_{...} (u - v) \circ (u - v) \circ v
+ \frac{1}{12} Y_{...} (u - v)^4 + \frac{1}{3} Y_{...} (u - v)^3 \circ v
+ \frac{1}{2} Y_{...} (u - v)^2 \circ v^2 + o_p(1/n).
\]

The equation satisfied by \( v \) can be written as
0 = \hat{y} = y + Yv + \frac{1}{2} Y\cdots v \cdots v + \frac{1}{6} Y\cdots (\cdots v)^3 + o_p(1/n).

Inverting this equation with respect to $v$, recursively, we have

$$v = -Y^{-1}y - \frac{1}{2} Y^{-1}[Y\cdots (\cdots Y^{-1}y)^3]$$
$$- \frac{1}{2} Y^{-1}[Y\cdots Y^{-1}y \cdots Y^{-1}(Y\cdots Y^{-1}y)]$$
$$+ \frac{1}{6} Y^{-1}Y\cdots (\cdots Y^{-1}y)^3 + o_p(1/n).$$

Putting $\theta_1 = \theta_3$ in (3) as we are handling the asymptotic theory of $\lambda$ under $H_0: \theta_1 = \theta_3$, we have the asymptotic expansion of $v$ at $(\theta_1', \theta_3') = (\theta_1', \theta_3')$. Similarly, expanding $\hat{y}_1$ at $\theta_1 = \theta_1$, $\theta_3 = \theta_3$,

$$0 = \hat{y}_1 = y_1 + w + \frac{1}{2} w + \frac{1}{6} w + o_p(1/n)$$

and therefore

$$u = -Z_0\hat{y} - \frac{1}{2} Z_0(Y\cdots (\cdots Z_0y)^3)$$
$$- \frac{1}{2} Z_0[Y\cdots Z_0y \cdots Z_0(Y\cdots Z_0y)]$$
$$+ \frac{1}{6} Z_0(Y\cdots (\cdots Z_0y)^3) + o_p(1/n),$$

where

$$Z_0 = \begin{bmatrix} Y_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Inserting these values of $u$ and $v$ into (2), we have the asymptotic expansion of the likelihood ratio criterion for the composite hypothesis up to order $O_p(1/n)$ as follows:

$$2 \log \lambda = y'Zy + q_1 + q_2 + o_p(1/n),$$

$$q_1 = \frac{1}{3} Y\cdots Y + Y\cdots Zy + Z_0y \cdots Y^{-1}y,$$

$$q_2 = \frac{1}{4} Y\cdots Y^{-1}y^3 + Y^{-1}(Y\cdots Y^{-1}y)^2$$
$$- \frac{1}{4} Y\cdots (Z_0y)^3 + Z_0(Y\cdots (\cdots Z_0y)^3) + \frac{1}{12} Y\cdots (\cdots Z_0y)^4.$$
\[ \frac{1}{6} Y_{\cdots} \circ (Zy)^2 \circ Y^{-1}y \circ (Z + 3Z_0)y \]

\[ -\frac{1}{3} Y_{\cdots} \circ Y^{-1}y^3 \circ Zy , \]

where

\[ Z = Y^{-1} - Z_0 \quad \text{and} \quad ZYZ = Z , \quad ZYZ_0 = 0 . \]

It should be noted that rank \( Z = p - q . \)

To obtain the moment generating function (MGF) of \( S = -2 \log \lambda \) we use the multivariate Edgeworth expansion for the joint density function of \( y, Y, Y_{\cdots} \) and \( Y_{\cdots} \) up to the order \( 1/n \), which is stated as follows:

\[ (6) \quad f_i = f_0 \{ 1 + A/\sqrt{n} + B/n \} + o(1/n) \]

where

\[ f_0 = (2\pi)^{-n/2} |K|^{-1/2} \exp \left( -\frac{1}{2} y'K^{-1}y \right) \prod \delta(y_{ij} - \kappa_{ij}) \cdot \prod \delta(y_{ijk} - \kappa_{ijk}/\sqrt{n}) \prod \delta(y_{ijkl}/n) \]

\[ A = \frac{1}{6} \{ K_{\cdots} \circ K^{-1}y^4 - 3K_{\cdots} \circ K^{-1} K^{-1}y \} - K_{\cdots} \circ K^{-1}y \circ D.. \]

\[ B = \frac{1}{2} \{ K_{\cdots} \circ (D..)^2 - (\text{tr } KD..)^2 \} \]

\[ + \frac{1}{2} (K_{\cdots} \circ K^{-1} \circ D.. - K_{\cdots} \circ (K^{-1}y)^2 \circ D..) \]

\[ + \frac{1}{2} (p - y'K^{-1}y) \text{tr } KD.. - K_{\cdots} \circ K^{-1}y \circ D.. \]

\[ - \frac{1}{2} K_{\cdots} \circ K^{-1}(K_{\cdots} \circ D..) \circ D.. + \frac{1}{2} (K_{\cdots} \circ K^{-1}y \circ D..)^2 \]

\[ + \frac{1}{24} \{ K_{\cdots} \circ (K^{-1}y)^4 - 6K_{\cdots} \circ K^{-1}(K^{-1}y)^3 \]

\[ + 3K_{\cdots} \circ (K^{-1})^2 \} \]

\[ - \frac{1}{8} \{ p(p+2) - 2(p+2)y'K^{-1}y + (y'K^{-1}y)^2 \} \]

\[ - \frac{1}{2} K_{\cdots} \circ K^{-1} \circ K^{-1}(K_{\cdots} \circ D..) \]

\[ + \frac{1}{2} K_{\cdots} \circ (K^{-1}y)^2 \circ K^{-1}(K_{\cdots} \circ D..) \]
\[
\frac{1}{2} K_{..} \circ K^{-1} \circ K^{-1} y \cdot K_{..} \circ K^{-1} y \circ D..
\]

\[
-\frac{1}{2} K_{..} \circ (\circ K^{-1} y)^2 K_{..} \circ K^{-1} y \circ D..
\]

\[
+ \frac{1}{72} (-9 K_{..} \circ K^{-1} \circ K^{-1}(K_{..} \circ K^{-1}) - 6 K_{..} \circ (\circ K^{-1} y)^2 K_{..}
\]

\[
+ 18 K_{..} \circ K^{-1} \circ K^{-1}(K_{..} \circ (\circ K^{-1} y)^2)
\]

\[
+ 9(K_{..} \circ K^{-1} \circ K^{-1} y)^2
\]

\[
+ 18 K_{..} \circ (\circ K^{-1} y)^2 \circ K^{-1} y \circ K_{..}
\]

\[
+(K_{..} \circ (\circ K^{-1} y)^2 - 9 K_{..} \circ (\circ K^{-1} y)^2 \circ K^{-1}(K_{..} \circ (\circ K^{-1} y))
\]

\[
- 6 K_{..} \circ K^{-1} \circ K^{-1} y \cdot K_{..} \circ (\circ K^{-1} y)^2
\]

\[
D_{..} = (d_{ab}) , \quad d_{ab} = \delta'(y_{ab} - \kappa_{ab})/\delta y_{ab}
\]

\[
D_{...} = (d_{abc}) , \quad d_{abc} = \delta'(y_{abc} - \kappa_{abc})/\delta y_{abc}/\sqrt{n}
\]

where \(\delta\)'s are Dirac delta functions satisfying

\[
\delta(x-a) = 0 , \quad x \neq a ,
\]

\[
\int \delta(x-a) dx = 1,
\]

\[
\int h(\cdots, x, \cdots) \delta(x-a) dx = h(\cdots, a, \cdots),
\]

\[
\int h(\cdots, x, \cdots) \delta^{(i)}(x-a) dx = (-1)^i \frac{\partial h(\cdots, x, \cdots)}{\partial x^i} \bigg|_{x=a}.
\]

As this expression is represented at the any point \(\theta\), the MGF \(M_{(t)}\) should be evaluated at the value \(H_0 : \theta_2 = \theta_{20}\), that is,

\[
M_{(t)} = E \left[ \exp (tS) | H_0 \right] = \int \exp (tS)f(y, Y, Y..., Y... | \theta_1, \theta_{20}) dy dY dY...dY... + o(1/n).
\]

By the use of some relations with respect to \(K, K_{..}, K_{...}, \) etc., which are derived from the regularity conditions for the density function,

\[
K_{...} \circ K^{-1} \circ K^{-1} + 4 K_{...} \circ K^{-1} \circ K^{-1} + K_{...} \circ K^{-1} = 0
\]

\[
K_{...} \circ K^{-1} \circ (K_{...} \circ K^{-1}) + K_{...} \circ K^{-1} \circ K^{-1}(K_{...} \circ K^{-1})
\]

\[
+ K_{...} \circ K^{-1} \circ K^{-1}(K_{...} \circ K^{-1}) = 0
\]

\[
+ 2 K_{...} \circ K^{-1} \circ K^{-1}(K_{...} \circ K^{-1}) = 0
\]
we find the MGF of \(-2 \log \lambda\) as given by (5), to

\[
M(t) = (1 - 2t)^{-(p-q)/2} \left[ 1 + \frac{1}{24n} \{ A_2d^2 + A_1d \} + o(1/n) \right],
\]

where

\[
d = 2t/(1 - 2t),
\]

\[
A_2 = 12 \{ K_{..} \ast M \ast M \ast A \ast K_{..} \ast K_{..} \ast A \ast M \ast M \ast K_{..} + K_{..} \ast M \ast M \ast A \ast K_{..} + 2K_{..} \ast A \ast M \ast M \ast K_{..} \}
\]

\[
A_1 = u(K^{-1}) - u(A),
\]

\[
l(K^{-1}) = 3K_{..} \ast (\ast K^{-1})^2 + 12K_{..} \ast \ast K^{-1} \ast (\ast K^{-1})^2 + 12K_{..} \ast \ast K^{-1} \ast (\ast K^{-1})^2
\]

\[
+ 12K_{..} \ast (\ast K^{-1})^2 + 3K_{..} \ast K^{-1} \ast K^{-1}(K_{..} \ast K^{-1})
\]

\[
+ 12K_{..} \ast K^{-1} \ast K^{-1}(K_{..} \ast K^{-1}) + 6K_{..} \ast (\ast K^{-1})^2 \ast K_{..}
\]

\[
+ 4K_{..} \ast (\ast K^{-1})^2 \ast K_{..} + 24K_{..} \ast (\ast K^{-1})^2 \ast K_{..}
\]

\[
M = K^{-1} - A, \quad A = \begin{bmatrix} K_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{p-q}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_{p-q}.
\]

The inversion of (7) gives the following theorem.

**Theorem 1.** The asymptotic expansion of the distribution of the likelihood ratio criterion for a composite hypothesis under the null hypothesis is as follows:

\[
P \{ -2 \log \lambda \leq x | H_0 \}
= P_r + \frac{1}{24n} \{ A_2P_{r+t} - (2A_2 - A_1)P_{r+t} + (A_2 - A_1)P_r \} + o(1/n)
\]

where \(P_r = P \{ \chi^2 \leq x \}, \chi^2\) is a central chi-square random variable with \(f\) degrees of freedom and \(f = p - q\), and \(A_1\) and \(A_2\) are given in (7), respectively.
Remark 1. In case of a simple hypothesis $H_0: \theta = \theta_0$, the vanishing of $A_z = 0$ in (7), and (8) becomes

$$P \{ -2 \log \lambda \leq x | H_0 \} = P_f + \frac{u(K^{-1})}{24n} (P_{r+1} - P_r) + o(1/n)$$

(9)

and $f = p$.

Differentiating $M(t)$ under $H_0$ and setting $t = 0$, we have the asymptotic expectation of $-2 \log \lambda$ as follows:

$$E \{ -2 \log \lambda | H_0 \} = p + \frac{u(K^{-1})}{12n} + o(1/n).$$

(10)

Combining (9) and (10), we are able to give a correction factor $\rho$ which makes the term of order $1/n$ of these expressions vanish simultaneously:

$$\rho = 1 - \frac{u(K^{-1})}{12np} + o(1/n).$$

(11)

Wilks [12] expected that the magnitude of the second term of the asymptotic expansion of the distribution would be of order $1/\sqrt{n}$. (9) shows that the convergence of $-2 \log \lambda$ to a central chi-square is more rapid than it is expected. It is also of interest to note that we can find a correction factor $\rho$ for a simple hypothesis.

Remark 2. It is of interest to note that the term $A_z$ in the expression (8) vanishes for a particular distribution. Let $x$ be a random variable with Darmois-Koopman type probability density function expressed as

$$f(x|\theta) = h(x) \exp \left\{ \sum_{i=1}^{p} \theta_i u_i(x) + V(\theta) \right\}$$

where $V(\theta)$ is differentiable with respect to $\theta$ up to the fourth order. We assume that $f(x|\theta)$ also satisfies the regularity conditions. As the second derivatives of $\log f(x|\theta)$ with respect to $\theta$ become a function of $\theta$, not depending on a random variable $x$, that is,

$$\frac{\partial^2 \log f}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 V}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, 2, \ldots, p,$$

we have by the regularity condition

$$\kappa_{i, j} = E \left[ \frac{\partial^2 \log f}{\partial \theta_j \partial \theta_k} \frac{\partial \log f}{\partial \theta_i} \right] = \frac{\partial^2 V}{\partial \theta_j \partial \theta_k} E \left[ \frac{\partial \log f}{\partial \theta_i} \right] = 0.$$

This implies that $K_{i,j} = 0$, that is, $A_z = 0$ in (8). Noting the relations
\[ \kappa_{i.f.k} = -\kappa_{f.k} \]
\[ \kappa_{i.f.k} = -\kappa_{i.f.k}, \quad \kappa_{i.f.k} = 0 \]
\[ \kappa_{i.f.k} + \kappa_{i.f.k} = \kappa_{i.f.k} + \kappa_{i.f.k} + \kappa_{i.f.k} , \]

the coefficient \( A_i \) in (7) can be simply represented as follows:

\[ A_i = l(K^{-1}) - l(A) , \]
\[ l(K^{-1}) = 3K_{n..}(K^{-1})^3 + 3K_{n..}(K^{-1})^3 + K_{n..}(K^{-1})^3 + K_{n..}(K^{-1})^2 \]
\[ + 2K_{n..}(K^{-1})^2 \times K_{n..} . \]

Thus we have again a correction factor \( \rho \) as

\[ \rho = 1 - \frac{A_i}{12np} + o(1/n) . \]

Recently the asymptotic distribution theory of the LRC's for various hypotheses about the parameters of the multivariate normal population has been studied by the method developed for the normal distribution theory (Box [1], Sugiura and Fujikoshi [10], Sugiura [11], Fujikoshi [3], etc.). Lee, Krishnaiah and Chang [6] have given the asymptotic expansion of the distribution of a statistic, which has the moments expressed by Gamma functions, up to the order \( 1/n^4 \), where \( n \) is a sample size. For these expressions it is of some interest that we are able always to obtain a correction factor \( \rho \) which makes a rapid convergence of LRC to a limiting distribution.

We will show in Example 1 how to get the asymptotic expansion of the distribution of LRC for a particular hypothesis of a multivariate normal population by the use of Theorem 1. However, the derivation is laborious.

**Example 1** (The calculation of this example is partly due to Miss Y. Kikuchi). Let \( x \) be an \( m \) dimensional normal random vector with mean \( \mu \) and covariance matrix \( \Sigma \), and \( x_1, \ldots, x_N \) a random sample from this population. We test the simple hypothesis

\[ H_0 : \Sigma^{-1} = \Theta = \Theta_0 \text{ (specified), given } \mu = 0 \]

against

\[ H : \Sigma^{-1} = \Theta \neq \Theta_0 \text{, given } \mu = 0 . \]

The probability density function of this normal population belongs to the family of Darmois and Koopman and the likelihood ratio criterion for this \( H_0 \) is given by
$$\lambda = \left( \frac{e}{N} \right)^{n/2} |\Theta_0 A|^{n/2} \text{etr} \left( -\frac{1}{2} \Theta_0 A \right)$$

where \( S = \sum_{a=1}^{N} x_a x_a' \). Without loss of generality we are able to assume \( \Theta_0 = I_m \). Rearranging parameters as \( \theta = (\theta_{11}, \theta_{22}, \cdots, \theta_{mm}, \theta_{12}, \cdots, \theta_{m-1,m}) \), we have the expectations of first and second derivatives of log-likelihood function,

$$E \left[ \frac{\partial^2 L}{\partial \theta_{ii} \partial \theta_{jj}} \right] = \frac{1}{2} (\theta_{ij})^2 , \quad E \left[ \frac{\partial L}{\partial \theta_{ii}} \frac{\partial L}{\partial \theta_{pq}} \right] = \theta_{ip} \theta_{jq} .$$

$$E \left[ \frac{\partial^2 L}{\partial \theta_{pq} \partial \theta_{rs}} \right] = \theta_{pr} \theta_{qs} + \theta_{ps} \theta_{qr} ,$$

respectively.

Making third and fourth derivatives of \( L \) with respect to \( \theta_{ij} \)'s, we have the following identities with a little algebra

\[
\begin{align*}
K_{..} & \circ K^{-1} \circ K^{-1} = -(2m^3 + 5m^2 + 5m) \\
K_{..} & \circ K^{-1} \circ K^{-1}(K_{..} \circ K^{-1}) = 2m^3 + 4m^2 + 2m \\
K_{..} \ast K^{-1} \ast K^{-1} \ast K^{-1} \ast K_{..} = m^3 + 3m^2 + 4m ,
\end{align*}
\]

which gives

\[
A_1 = 2m^3 + 3m^2 - m .
\]

Thus we have the following asymptotic expression of the distribution of \( \lambda \) under \( H_0 \):

\[
P \{-2 \log \lambda \leq x\} = P_f + \frac{1}{24N} m(2m^2 + 3m - 1)(P_{f+1} - P_f) + o(1/n) ,
\]

where \( P_f = P \{ \lambda \leq x \} \), and \( f = m(m+1)/2 \). This result agrees with (2.10) in Sugiura [11] up to order \( 1/N \). Sugiura considered testing a hypothesis \( H_0 : \Sigma = \Sigma_0 \), against \( \Sigma \neq \Sigma_0 \) by the use of a modified likelihood ratio criterion having an unbiasedness,

\[
\lambda^* = \left( \frac{e}{n} \right)^{mn/2} |\Theta_0 A|^{n/2} \text{etr} \left( -\frac{1}{2} \Theta_0 A \right) ,
\]

where \( n = N - 1 \), \( \Theta = \Sigma^{-1} \), \( A = \sum_{a=1}^{N} (x_a - \bar{x})(x_a - \bar{x})' \).

3. Expansion of \( \lambda \) under a general composite hypothesis

In the discussion of Section 2 the attention has been restricted to the test of null hypothesis in which certain components of parameters
\( \theta \) have been specified. We generalize this result to a test of hypothesis \( H_0 : \theta \in \omega \), where \( \omega \) is the subspace of all vectors \( \theta \in \Theta \) for which there is a transformation \( \xi(\theta) \) such that

\[
\xi = (\xi_{q+1}(\theta), \ldots, \xi_p(\theta)) = (\xi_{q+1,0}, \ldots, \xi_{p,0}) = \xi_{20},
\]

where \( \xi_{q+1,0}, \ldots, \xi_{p,0} \) are constants. The transformation \( \xi(\theta) \) is required to satisfy the following properties:

(a) There exists a vector \( \xi(\theta) = (\xi_1(\theta), \ldots, \xi_q(\theta))' \) such that the inverse transformation \( \theta(\xi) = (\theta_1(\xi), \ldots, \theta_p(\xi))' \) exists, where \( \xi = (\xi', \xi_i) \).

(b) The partial derivatives of \( \theta(\xi) \) with respect to \( \xi \) exist up to order four, and are bounded and continuous function of \( \xi \).

(c) The transformation should be one to one, that is, the Jacobian \( |\partial(\xi)/\partial(\theta)| \neq 0 \).

Let \( \hat{\theta} \) be a maximum likelihood estimator of \( \theta \in \Theta \), then by the one to one correspondence between \( \Theta \) and \( \xi \), \( \xi(\hat{\theta}) = \hat{\xi} \) becomes also a maximum likelihood estimator of \( \xi \). By this reason the likelihood ratio criterion of testing the general composite hypothesis

\[
H_0 : \xi = \xi_{20} \quad \text{against} \quad H_* : \xi \neq \xi_{20}
\]

is expressed as

\[
\lambda = \prod_{i=1}^n \frac{f(x|\hat{\xi}_i, \xi_{20})}{f(x|\hat{\xi}_i, \hat{\xi}_i)},
\]

where \( \hat{\xi}_i = (\hat{\xi}_1, \ldots, \hat{\xi}_q) \) is a m.l.e. of \( \xi_i \) under \( H_0 \). The expression (12) is completely similar to (1) except the notation \( \theta \) and \( \xi \). This implies that the distribution theory with respect to \( \lambda \) under the general composite hypothesis can be handled by similar way as the case of a composite hypothesis.

**Theorem 2.** Under the required conditions for the density function and \( \xi(\Theta) \), the asymptotic expansion of the distribution of the likelihood ratio criterion \( \lambda \) under \( H_0 \) can be expressed as follows:

\[
P \{ -2 \log \lambda \leq x | H_0 \}
\]

\[
= P_f + \frac{1}{24n} [A_{2}\mu P_{q+1} - (2A_{2} - A_{1})\mu P_{q+1} + (A_{2} - A_{1})\mu P_f] + o(1/n)
\]

where \( A_{2} \) and \( A_{1} \) are the values of \( A_3 \) and \( A_1 \) in (7) at \( \xi = (\xi_1, \xi_{20}) \) and \( \mu = p - q \).

To express (13) in terms of \( \theta \), we have to know the values of \( A_{2} \) and \( A_{1} \) at \( \omega \). By the regularity conditions required for transformation,

\[
\kappa_{i,j}(\xi) = \kappa_{ip} \tau_{ip} \tau_{jq}, \quad \kappa_{i,j}(\xi) = \kappa_{i,j} \tau_{ip} \tau_{jq}
\]
(14)  

\[ \kappa_{i,j,k} = \kappa_{pq} \tau_{ip} \tau_{j,k} + \kappa_{pq} \tau_{ip} \tau_{jq} + \kappa_{pq} \tau_{j,k} \tau_{ip} + \kappa_{pq} \tau_{jq} \tau_{ip} \]

\[ \kappa_{i,j,k} = \kappa_{p,q} \tau_{ip} \tau_{jq} \tau_{kr} + \kappa_{p,q} \tau_{ip} \tau_{jq} \tau_{kr} \]

\[ \kappa_{i,j,k,l} = \kappa_{p,q,r} \tau_{ip} \tau_{jq} \tau_{kr} \] etc.,

where

\[ \tau_{ip} = \frac{\partial \theta_p}{\partial \xi_i}, \quad \tau_{ip,pp} = \frac{\partial^2 \theta_p}{\partial \xi_i \partial \xi_j}. \]

The functions of the fourth order derivatives can be obtained in a similar way.

If the transformation for \( \theta \) to \( \xi \) is an Affine transformation, i.e., \( \theta = G \xi + \eta \), where \( G = (g_{ij}) \), \( i, j = 1, 2, \ldots, p \), is a non-singular constant matrix, the expressions given in (14) become more simpler ones:

\[ \kappa_{i,j} = \kappa_{pq} \tau_{ip} \tau_{jq}, \quad \kappa_{i,j,k} = \kappa_{p,q} \tau_{ip} \tau_{jq} \tau_{kr} \]

\[ \kappa_{i,j,k,l} = \kappa_{p,q,r} \tau_{ip} \tau_{jq} \tau_{kr}, \quad \kappa_{i,j,k,l,m} = \kappa_{p,q,r,s} \tau_{ip} \tau_{jq} \tau_{kr} \tau_{lm} \] etc.

It is easy to see that \( l(K^{-1}) \) of (7) is invariant under this Affine transformation, but \( l(A) \) is not invariant.

We discuss two examples which cover the testing hypothesis of a multivariate normal population as the particular cases.

**Example 2.1.** We consider the following composite hypothesis

\( H_0: \theta_1 = \theta e, \quad \theta_2 = \theta_0, \) against \( H: H_0 \) is not true,

where \( \theta = (\theta_1, \ldots, \theta_q) \), \( \theta_0 = (\theta_{q+1}, \ldots, \theta_p), \quad e = (1, \ldots, 1)_{q \times 1} \quad \theta_0 = (\theta_{q+1,0}, \ldots, \theta_{p,0}) \) specified vector. By considering a following Affine transformation such that

\[ \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = G^{-1} \begin{bmatrix} \theta_1 \\ \theta_e \\ \theta_0 \end{bmatrix} - \begin{bmatrix} 0 \\ \theta_0 \end{bmatrix} \]

where

\[ G^{-1} = \begin{bmatrix} 1 \\ -1 & 1 \\ -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \]

\[ q \\ p-q \]
the hypothesis $H_0$ is reduced to a simple form,

$H_{\alpha} : \xi_1 = \xi_2 = \cdots = \xi_p = 0$, against $H_0 : H_{\alpha}$ is not true

and the likelihood ratio criterion becomes

$$(15) \quad \lambda = \prod_{i=1}^{n} \frac{f(x_i | \hat{\xi}_i, 0, \ldots, 0)}{f(x_i | \xi_i, \hat{\xi}_i)} ,$$

where $\hat{\xi}_i$ is a maximum likelihood estimator of $\xi_i$ under $H_{\alpha}$. It is easy to check that the first term of $-2 \log \lambda$ becomes

$$2 \log \lambda = u'Vu + O_p(1/n) ,$$

where

$$u = \frac{1}{\sqrt{n}} \frac{\partial L}{\partial \xi} = G \frac{1}{\sqrt{n}} \frac{\partial L}{\partial \theta} = Gy ,$$

$$V = U^{-1} - V = U^{-1} - \begin{bmatrix} a & 0' \\ 0 & O \end{bmatrix} , \quad a = u_{ii}^{-1} = \left( \frac{1}{n} \frac{\partial^2 L}{\partial \xi_i^2} \right)^{-1}$$

$$U = \frac{1}{n} \frac{\partial^2 L}{\partial \xi \partial \xi'} = GYG' .$$

Noting $u_{ii} = (e', 0')Y(e', 0') = e'Y_{ii}e$, we have a quadratic form with respect to $y$ and $Y$ as

$$u'Vu = y' \left( Y^{-1} - \frac{1}{e'Y_{ii}e} \begin{bmatrix} ee' & O \\ O & O \end{bmatrix} \right) y = y'Zy .$$

It is seen that

$$YZ = Z \quad \text{and} \quad \text{rank } Z = \text{rank } Y = \text{tr } ZY = p - 1 ,$$

which implies that $u'Vu$ approaches to a central chi-square random variable with $p - 1$ degrees of freedom in law as $n$ tends to infinity.

This type of hypothesis can be found in testing hypothesis concerning with a multivariate normal population.

(a) Let $x$ be an $m$ dimensional multivariate normal random vector with mean zero and covariance matrix $\Sigma$. Testing the sphericity $H_0 : \Sigma = \sigma^2 I_m$, against $H : \Sigma \neq \sigma^2 I_m$, $\sigma^2$ unspecified, is included in this case.

(b) Let $x$ be distributed as above (a). Testing the hypothesis of an intraclass correlation model $H_0 : \sigma_{ij} = \rho, i \neq j$ and $\sigma_{ii} = 1, i, j = 1, \ldots, m$, against $H : H_0$ is not true, is included in this case.

Example 2.2. For testing a hypothesis

$H_0 : \theta = 0e$, against $H : \theta \neq 0e, e = (1, 1, \ldots, 1)'(p-q) \times 1$
the first term of 2 log \( \lambda \) can be expressed by doing a similar way as

\[
u'Vu = y' \begin{pmatrix} Y^{-1} & I_t' \\ O & e \end{pmatrix} Y_{11}^{-1}(\xi) \begin{pmatrix} I_t & O \\ 0' & e' \end{pmatrix} y = y'Zy,
\]

where

\[
Y_{11}(\xi) = \begin{pmatrix} I_t & O \\ 0' & e' \end{pmatrix} Y \begin{pmatrix} I_t & O \\ O & e \end{pmatrix},
\]

and

\[ZYZ = Z, \quad \text{rank Z = tr ZY} = p - q - 1.\]

This implies that \(-2 \log \lambda\) approaches to a central chi-square with \(p - q - 1\) degrees of freedom. We are able to construct many hypothesis for parameters of multivariate normal population.

To close this section we show the asymptotic behavior of \(-2 \log \lambda\) under a sequence of Pitman's alternative \(H_{\omega_n}\) such that

\[H_{\omega_n} : \xi_z = \xi_{20} + s/\sqrt{n}\]

where \(n\) is a sample size. The asymptotic expansion of the distribution of LRC under \(H_n : \theta = \theta_{20} + s/\sqrt{n}\) has been obtained by Theorem 1 in Hayakawa [4]. By using a representation (12) of LRC and Theorem 1 of Hayakawa, the asymptotic expansion of the distribution of \(-2 \log \lambda\) under \(H_{\omega_n}\) is given as follows:

(16) \[P \{ -2 \log \lambda \leq x \} = P(f(\delta^2)) + \frac{1}{\sqrt{n}} \sum_{k=0}^{3} a_k P(f+k(\delta^2)) + O(1/n)\]

where

\[f = p - q, \quad \delta^2 = \frac{1}{2} s'K_{22}^{-1}s,\]

\[a_4 = -\frac{1}{6} K_{\cdot\cdot\cdot}(\circ s^4),\]

\[a_4 = -\frac{1}{6} \{ K_{\cdot\cdot\cdot}(\circ s^4) - 2K_{\cdot\cdot\cdot}(\circ s^3) + 3K_{\cdot\cdot\cdot} \circ A \circ s^3 + 6K_{\cdot\cdot\cdot} \circ A \circ s + 3K_{\cdot\cdot\cdot} \circ s \circ s^2 \circ s + 3K_{\cdot\cdot\cdot} \circ s \circ s \circ s \circ s \}
\]

\[a_0 = -\frac{1}{6} \{ K_{\cdot\cdot\cdot}(\circ s^3) - K_{\cdot\cdot\cdot}(\circ s^2) - 3K_{\cdot\cdot\cdot} \circ A \circ s^2 + 6K_{\cdot\cdot\cdot} \circ A \circ s + 3K_{\cdot\cdot\cdot} \circ s \circ s \circ s \circ s \}
\]

\[s^* = \begin{pmatrix} K_{12}^{-1}K_{12} \\ -I_{p-q} \end{pmatrix}s.
\]
\( K_{1\ldots N}, K_{1\ldots N}, A \) and \( \epsilon^* \) are the values of these at \((\xi_1, \xi_2)\).

4. Expansion of \( \lambda \) under a fixed alternative

In this section we shall only consider the asymptotic distribution of the normalized likelihood ratio criterion under a simple fixed alternative \( H_0: \theta=\theta_0+\epsilon, \epsilon \) is a fixed vector.

Expanding \( L(\theta_0) \) at \( \theta=\hat{\theta} \) and noting \( \frac{1}{\sqrt{n}}(\hat{\theta}-\theta_0)=v+\frac{1}{\sqrt{n}}\epsilon, \)

\[
2(L(\theta_0)-L(\hat{\theta})) = (v+\frac{1}{\sqrt{n}}\epsilon)'\hat{Y}(v+\frac{1}{\sqrt{n}}\epsilon)-\frac{1}{3}\hat{Y}...\circ(v+\frac{1}{\sqrt{n}}\epsilon)'^3 \\
+\frac{1}{12}\hat{Y}...\circ(v+\frac{1}{\sqrt{n}}\epsilon)'^4-\cdots .
\]

Again expanding \( \hat{y}_{ij}, \hat{y}_{ijk}, \cdots \) at \( \theta \) and rearranging (17), we have the following asymptotic expansion;

\[
2n[K(\theta_0)-K(\theta)]+2\sqrt{n}[u-u+e'y-e'K.v+e'W.v] \\
+v'K.v+K...\circ v\circ v\circ v'+v'W.v+W...\circ v\circ v\circ v+o_p(1),
\]

where

\[
K(\theta)=E[\log f(x|\theta)]=\int[\log f(x|\theta)]f(x|\theta)dx, \\
K(\theta_0)=E[\log f(x|\theta_0)]
\]

\[
u_o=\frac{1}{\sqrt{n}}[L(\theta_0)-nK(\theta_0)], \quad u=\frac{1}{\sqrt{n}}[L(\theta)-nK(\theta)],
\]

\[
W..=(w_{ij})=Y-K.., \quad w_{ij}=\frac{1}{n}[L_{ij}(\theta)-n\kappa_{ij}(\theta)],
\]

\[
W...=(w_{ijk})=\sqrt{n}(Y...-K...), \quad w_{ijk}=\frac{1}{n}[L_{ijk}(\theta)-n\kappa_{ijk}(\theta)].
\]

where

\[
L_{ij}(\theta)=\frac{\partial^2 L}{\partial \theta_i \partial \theta_j} \quad \text{and} \quad L_{ijk}(\theta)=\frac{\partial^3 L}{\partial \theta_i \partial \theta_j \partial \theta_k}.
\]

Replacing \( v \) by (3), we have the normalized asymptotic expansion of 

\[
S_\lambda=[-2\log \lambda+2n[K(\theta_0)-K(\theta)]]/\sqrt{n} \\
=2(u-u_o)-y'Y^{-1}y/\sqrt{n}+o_p(1/\sqrt{n}).
\]
To find the MGF of $S_\theta$ we need the Edgeworth expansion of $(u_0, u, y, Y)$, which is given as follows.

$$
 f_z = f_0 \left[ 1 + \frac{1}{\sqrt{n}} \left\{ \frac{1}{6} \left\{ H_{00} \circ M^{-1} z + 3H_{i,j} \circ M^{-1} z \right\} - H_{i,j} \circ M^{-1} z \circ D_{ij} \right\} + o(1/n) \right],
$$

where

$$
 f_0 = (2\pi)^{-(p+1)/2} |M|^{-1/2} \exp \left\{ -\frac{1}{2} z' M^{-1} z \right\} \prod_{i,j} \delta(y_{ij} - \kappa_{ij})
$$

$$
 z' = (u_0, u, y')
$$

$$
 M = E(zz') = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & K_{pp} \end{bmatrix}_p
$$

$$
 H_{i,j} = (h_{i,j,k}), \quad h_{i,j,k} = E(z_i z_j z_k)
$$

$$
 H_{i,j} = (h_{i,j,k}), \quad h_{i,j,k} = E(z_i y_{jk})
$$

$$
 D_{ij} = (d_{ij}), \quad d_{ij} = \delta'(y_{ij} - \kappa_{ij}) / \delta(y_{ij} - \kappa_{ij}).
$$

We find after some lengthy algebra with respect to the normal distribution that the MGF of $S_\theta$ is given by

$$
 M_0(t) = \exp \left\{ 2t' e' M_{11} e \right\} \left[ 1 + \frac{q_1}{\sqrt{n}} + o(1/n) \right],
$$

where

$$
 q_1 = pt + t^4 \left\{ 4e' M_{12} M_{22}^{-1} M_{11} e + \frac{4}{3} H_{1,1,1}(\circ e) \right\},
$$

$$
 e' = (-1, 1),
$$

$$
 H_{1,1,1}(\circ e) = h_{0,0,0} + h_{1,0,0} + h_{1,1,1} - \{ h_{0,0,0} h_{0,1,1} + h_{1,0,1} + h_{1,1,0} \}. 
$$

Putting $\tau^2 = 4e' M_{11} e$, we have the following theorem.

**Theorem 3.** Under the fixed alternative, the normalized likelihood ratio criterion $S_\theta = [-2 \log \lambda + 2n \{ K(\theta_0) - K(\theta) \}] / \sqrt{n}$ has the following asymptotic expansion of the distribution.

$$
 P \left\{ S_\theta / \tau \leq x \right\} = \Phi(x) - \frac{1}{\sqrt{n}} \left\{ p \Phi^{(1)}(x) / \tau + \left\{ 4e' M_{12} M_{22}^{-1} M_{11} e \right. \right. \\
 + \left. \left. \frac{4}{3} H_{1,1,1}(\circ e)^4 \Phi^{(3)}(x) / \tau^3 \right\} + o(1/n) \right\},
$$
where \( r^2 = 4e'M_{11}e \) and \( \Phi^{(r)}(x) \) means the \( r \)-th derivative of the standard normal distribution function \( \Phi(x) \).

Under the null hypothesis \( H_0: \theta = \theta_0 \), we have \( u = u_0 \). This means \( r^2 = 0 \), by which the asymptotic non-null distribution \( -2 \log \lambda \) has a singularity at the null hypothesis, so that the formula given by Theorem 3 does not give the good approximation near the null hypothesis.

**Example 3.** We shall examine the same hypothesis testing as Example 1. The parameter \( \theta \) is arranged as

\[
\theta' = (\sigma_{11}, \ldots, \sigma_{mm}, \sigma_{12}, \ldots, \sigma_{m-1,m})
\]

To find the covariance matrix \( M \) of \( z \), without loss of generality, we take the expectations of \( z \) at \( \Sigma = I_m \). By doing this, \( z \) should be read as

\[
u_0 = -\frac{1}{2} \log |2A| - \frac{1}{2} \operatorname{tr} Axx',
\]

\[
u = -\frac{1}{2} \log |2I_m| - \frac{1}{2} \operatorname{tr} xx',
\]

\[

\begin{bmatrix}
\frac{1}{\sqrt{n}} \left( \frac{\partial L}{\partial \sigma_{11}} \bigg|_{\Sigma = I_m} , \ldots, \frac{\partial L}{\partial \sigma_{mm}} \bigg|_{\Sigma = I_m} , \frac{\partial L}{\partial \sigma_{12}} \bigg|_{\Sigma = I_m} , \ldots, \frac{\partial L}{\partial \sigma_{m-1,m}} \bigg|_{\Sigma = I_m} \right)
\end{bmatrix},
\]

where

\[
x \sim N(0, I_m) \quad \text{and} \quad A = \Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2}.
\]

Then,

\[
M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix},
\]

\[
M_{11} = \frac{1}{2} \begin{bmatrix}
\operatorname{tr} A^2 & \operatorname{tr} A \\
\operatorname{tr} A & m
\end{bmatrix},
\]

\[
M_{22} = \frac{1}{2} I_m
\]

\[
M_{22} = \begin{bmatrix}
\frac{1}{2} I_m \\
I_{m(m-1)/2}
\end{bmatrix}
\]

\[
M_{12} = \begin{bmatrix}
\frac{1}{2} a_{11}, \ldots, \frac{1}{2} a_{mm} & a_{12}, \ldots, a_{m-1,m} \\
\frac{1}{2}, \ldots, \frac{1}{2} & 0, \ldots, 0
\end{bmatrix},
\]

We have
\[ \tau^2 = 4e'M_1e = 2 \text{ tr } (A - I)^2, \]
\[ 4e'M_2M_2^{-1}M_1e = 2 \text{ tr } (A - I)^3, \]
\[ H_{1,1,1}(\circ e)^3 = h_{1,1,1} + 3h_{1,0,0} - \{h_{0,0,0} + 3h_{1,1,1}\}, \]
\[ h_{0,0,0} = E(u_0^3) = - \text{ tr } A^3, \]
\[ h_{1,0,0} = h_{0,1,0} = h_{0,0,1} = E(u_0^3u) = - \text{ tr } A^4, \]
\[ h_{1,1,0} = h_{1,0,1} = h_{0,1,1} = E(u_0u^2) = - \text{ tr } A, \]
\[ h_{1,1,1} = E(u^3) = - m. \]

This gives
\[ H_{1,1,1}(\circ e)^3 = \text{ tr } (A - I)^3. \]

Combining these results, we have the following expression.
\[ P \left\{ -2 \log \lambda - n \{ \text{ tr } (A - I) - \log |A| \} / \sqrt{n} \tau \leq x \right\} \]
\[ = \Phi(x) - \frac{1}{6\sqrt{n}} \left\{ 3m(m+1)\Phi'(x)/\tau + 4(m+2 \text{ tr } A^3 \right. \]
\[ \left. - 3 \text{ tr } A^3)\Phi''(x)/\tau^3 \right\} + o(1/\sqrt{n}). \]

This agrees with (2.16) in Sugiura [11].

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REFERENCES

[8] Neyman, J. and Pearson, E. S. (1928). On the use and interpretation of certain test

587.

butions of the likelihood ratio criteria for multivariate linear hypothesis and inde-


CORRECTION TO

"THE LIKELIHOOD RATIO CRITERION AND THE ASYMPTOTIC EXPANSION OF ITS DISTRIBUTION"

TAKESI HAYAKAWA

(This Annals Vol. 29, No. 3 (1977), pp. 359-378)

The expressions for the constants $A_s$ in (7) and $A_{3s}$ in (13) are incorrect. The correct expressions are $A_s = A_{3s} = 0$.

This fact is also noted in Cordeiro [1] and Harris [2].

REFERENCES