Ann. Inst. Statist. Math.
29 (1977), Part A, 333-358

APPROXIMATIONS TO THE PROBABILITIES OF BINOMIAL AND MULTINOMIAL RANDOM VARIABLES AND CHI-SQUARE TYPE STATISTICS

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(Received Sept. 7, 1976; revised Apr. 22, 1977)

Summary

New lower and upper bounds are given to the probabilities of binomial and multinomial random variables. Exact bounds are also presented for the sampling distributions of chi-square type statistics and the K-L information number from a multinomial distribution.

1. Introduction

Many parts of the theory of statistics are based on approximate results. Suppose that the exact sampling distribution of a certain statistic is unknown or is very complicated to use for statistical purposes. Then it is quite natural to use approximate distributions (or functions) by the aid of approximation theories. In fact, large numbers of approaches to such problems have been done in various situations, in which the limiting or asymptotic distributions are frequently of main concern as the underlying sample size n tends to infinity. However, it should be noted that they are valid theoretically only for the sample size n is *sufficiently large* and that the magnitude of the approximation error is hidden behind the convenient but vague symbol $O(n^{-\alpha})$ $(\alpha>0)$.

In reality, the size n of the sample at our hand is always finite, and frequently it would not be assumed sufficiently large; to use asymptotic approximations in such cases seems to be dangerous. So, from both theoretical and practical point of views, it is desired to give exact error evaluations being applicable to a general case of any given n. From the above standpoint, bounds on some basic approximations will be investigated in subsequent sections.

The main purpose of this paper is two-fold. The first one is to approximate the binomial probabilities and related quantities more precisely than the usual normal approximation. Another one is to evaluate the χ^2-approximations for the distribution of chi-square type statistics by using double inequalities. To the former problem a few exact bounds have been presented so far to evaluate the accuracy of the De'Moivre-Laplace theorem (e.g. Uspensky [8] and Feller [2]). However, as far as the present author is aware of, exact evaluations for the χ^2-approximations have hardly been examined except for Vora [9].

In the following section some basic inequalities are proved, of which two new forms of the Stirling formula are presented, assuming n not to be large. In Section 3 approximation theory on the binomial distribution is treated. A set of lower and upper bounds for individual binomial probabilities is given in Theorem 3.1. Further, bounds for the tail probabilities of the distribution are estimated in Theorem 3.2. In Section 4 bounds on multinomial probabilities are given in Theorem 4.1. In Section 5 exact evaluations are made for the sampling distributions of chi-square type statistics in Theorem 5.1, which are improvements of Vora's results. Moreover, bounds are given in Theorem 5.2 for the sampling distribution function of the K-L information number from the multinomial distribution. Some numerical results are given in Table 5.1 on the evaluations of the distribution function of the chi-square type statistics from a number of different multinomial distributions.

2. Preliminary lemmas

In this section some basic lemmas are given which play fundamental roles in subsequent sections. First, some inequalities are obtained based on inverse factorial series which are more accurate than those so far obtained (cf. Theorem 2.1 of Matsunawa [5]).

LEMMA 2.1. *For $u>0$ and for any positive integer $K\geqq 2$, it holds that*

$$\ln\left(1+\frac{1}{u}\right)=\frac{1}{u}-\sum_{i=1}^{K} b_i u^{-[i+1]}-S_K(u) \tag{2.1}$$

with

$$\underline{S}_K(u)<S_K(u)<\bar{S}_K(u)\ , \tag{2.2}$$

where $u^{-[r]}$ denotes the inverse of an advancing factorial $u^{[r]}=u(u+1)\cdots(u+r-1)$ (r: a positive integer),

$$S_K(u)=\sum_{i=K+1}^{\infty} b_i u^{-[i+1]} \tag{2.3}$$

$$\underline{S}_K(u)=(K-1)b_K(u+1)^{-1}u^{-[K+1]} \tag{2.4}$$

$$\bar{S}_K(u)=Kb_K u^{-1}u^{-[K+1]}\ , \tag{2.5}$$

and $b_1=1/2,\ b_2=1/6,\ b_3=1/4,\ b_4=19/30,\ b_5=9/4,\ b_6=863/84,\ b_7=1375/24,$ $b_8=33953/90,\ b_9=57281/20,\ b_{10}=3250433/132,\cdots$ *and in general*

$$b_i=\sum_{j=0}^{i-1}(-1)^{i+j-1}d_{j,i-1}(i-j+1)^{-1}\,,\qquad (i\geqq 1)\,, \tag{2.6}$$

where d's are absolute values of Stirling's numbers of the first kind, satisfying

$$\begin{aligned}&d_{0,l}=1\,,\quad d_{l,l}=l!\,,\quad (l\geqq 0)\,;\qquad d_{k,l}=0\,,\quad (k>l)\,,\\ &d_{k,l}=l\cdot d_{k-1,l-1}+d_{k,l-1}\quad (1\leqq k\leqq l-1)\,.\end{aligned} \tag{2.7}$$

PROOF. The right hand expression of (2.1) with (2.3) was shown in the previous paper [5] but with the coefficients b's of the different form

$$b_1=\frac{1}{2}\,,\qquad b_i=\int_0^1 t(1-t)(2-t)\cdots(i-1-t)dt\qquad (i\geqq 2)\,. \tag{2.8}$$

This representation, however, reduces to (2.6) by noticing the fact that

$$t(1-t)(2-t)\cdots(i-1-t)\equiv(-1)^{i-1}t^{(i)}=\sum_{j=0}^{i-1}(-1)^{i+j-1}d_{j,i-1}t^{i-j}\,. \tag{2.9}$$

Thus, it suffices to prove the inequality (2.2) with the bounds (2.4) and (2.5). Since it is easily verified by evaluating (2.8) that

$$\frac{\Gamma(i-1)}{\Gamma(K-1)}b_K<b_i<\frac{\Gamma(i)}{\Gamma(K)}b_K \tag{2.10}$$

for $i\geqq K+1$ and $K\geqq 2$, then by using Lemma 3.3 of [5] we can evaluate $S_K(u)$ as

$$\begin{aligned}S_K(u)&>\frac{b_K}{\Gamma(K-1)}\sum_{i=K+1}^{\infty}\Gamma(i-1)u^{-[i+1]}\\ &=\frac{b_K}{\Gamma(K-1)\cdot u}\sum_{j=K}^{\infty}\Gamma(j)(u+1)^{-[j+1]}\\ &=\frac{b_K}{\Gamma(K-1)\cdot u}\left\{\frac{1}{(u+1)^2}-\sum_{j=1}^{K-1}\Gamma(j)(u+1)^{-[j+1]}\right\}\\ &=\frac{b_K}{\Gamma(K-1)\cdot u}\,\frac{\Gamma(K)}{(u+1)(u+1)^{[K]}}=\underline{S}_K(u)\,,\end{aligned}$$

and

$$\begin{aligned}S_K(u)&<\frac{b_K}{\Gamma(K)}\sum_{i=K+1}^{\infty}\Gamma(i)u^{-[i+1]}\\ &=\frac{b_K}{\Gamma(K)}\left\{\frac{1}{u^2}-\sum_{i=1}^{K}\Gamma(i)u^{-[i+1]}\right\}\end{aligned}$$

$$=\frac{b_K}{\Gamma(K)}\frac{\Gamma(K+1)}{uu^{[K+1]}}=\bar{S}_K(u) ,$$

which completes the proof of the lemma.

LEMMA 2.2. (i) *For $x>-1$ and for any positive integer $M\geqq 3$,*

$$\ln\Gamma(x+1)=\frac{1}{2}\ln 2\pi+\left(x+\frac{1}{2}\right)\ln(x+1)-(x+1) -\sum_{i=1}^{M}a_i(x+1)^{-[i]}-R_M(x+1) , \tag{2.11}$$

with

$$\underline{R}_M(y)<R_M(y)<\bar{R}_M(y) , \tag{2.12}$$

where

$$R_M(y)=\sum_{i=M+1}^{\infty}a_iy^{-[i]} ,\qquad (y>0) , \tag{2.13}$$

$$\underline{R}_M(y)=\frac{M}{4}\left[(M-1)(M+4)\left(\frac{1}{y+1}+\frac{1}{M+1}\frac{y}{y+2}\right) -(M-2)\frac{y+M}{y+1}\right]a_My^{-[M+1]} , \tag{2.14}$$

$$\bar{R}_M(y)=(M-1)\left[Ma_M\left(\frac{1}{y+1}+\frac{1}{M-1}\frac{y}{y+2}\right)-MW(M)\left(\frac{1}{y+1} +\frac{1}{M+1}\frac{y}{y+2}\right)+W(M)\frac{y+M}{y}\right]y^{-[M+1]} , \tag{2.15}$$

with

$$W(M)=\sum_{j=0}^{M-2}(-1)^{M+j-2}d_{j,M-2}\left(\frac{1}{2}\right)^{M-j+1}\left(\frac{1}{M-j}-\frac{1}{M-j+1}\right) , \tag{2.16}$$

and $a_1=-1/12$, $a_2=0$, $a_3=1/360$, $a_4=1/120$, $a_5=5/168$, $a_6=11/84$, $a_7=3499/5040$, $a_8=1039/240$, $a_9=369689/11880$, $a_{10}=83711/330,\cdots$; *in general*

$$a_i=\frac{1}{2i}\sum_{j=0}^{i-1}(-1)^{i+j-1}d_{j,i-1}\left(\frac{1}{i-j+1}-\frac{2}{i-j+2}\right) ,\qquad (i\geqq 1) . \tag{2.17}$$

(ii) *For $x>-1/2$ and any positive integers $M\geqq 3$, $K\geqq 2$,*

$$\ln\Gamma(x+1)=\frac{1}{2}\ln 2\pi+\left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right)-\left(x+\frac{1}{2}\right) -\sum_{i=1}^{M}a_i(x+1)^{-[i]}-\frac{1}{2}\sum_{i=1}^{K}b_i(2(x+1))^{-[i]}-R^*_{M,K}(x) , \tag{2.18}$$

where a's and b's are the same as before and

$$\underline{R}^*_{M,K}(x)<R^*_{M,K}(x)<\bar{R}^*_{M,K}(x)\ , \tag{2.19}$$

where

$$\underline{R}^*_{M,K}(x)=\underline{R}_M(x+1)+(x+1/2)\underline{S}_K(2x+1)\ , \tag{2.20}$$

$$\bar{R}^*_{M,K}(x)=\bar{R}_M(x+1)+(x+1/2)\bar{S}_K(2x+1)\ . \tag{2.21}$$

PROOF. (i) The R.H.S. expression of (2.11) can be derived from the result given in Theorem 2.1 in [5], that is, for $y>0$ it holds that

$$\ln\Gamma(y+1)=\frac{1}{2}\ln 2\pi+\left(y+\frac{1}{2}\right)\ln y-y+\frac{1}{12y}-R(y)\ , \tag{2.22}$$

where

$$R(y)=\sum_{i=2}^{\infty}a_{i+1}y^{-[i+1]}=\sum_{i=3}^{\infty}a_i y^{-[i]} \tag{2.23}$$

$$a_i=\frac{1}{i}\int_0^1 t(1-t)(2-t)\cdots(i-1-t)\left(\frac{1}{2}-t\right)dt\ ,\qquad (i\geqq 2)\ , \tag{2.24}$$

which is equivalent to (2.17). Then, putting $y=x+1$ $(x>-1)$ in (2.22) and noticing the fact $\ln\Gamma(x+1)=\ln\Gamma(x+2)-\ln(x+1)$, it follows that

$$\ln\Gamma(x+1)=\frac{1}{2}\ln 2\pi+\left(x+\frac{1}{2}\right)\ln(x+1)-(x+1)+\frac{1}{12(x+1)}-R(x+1)\ ,$$

which is nothing but (2.11) with (2.13).

It remains to prove the inequality (2.12). For $i\geqq M+1$ and $M\geqq 3$ we can evaluate a_i as

$$\begin{aligned}a_i&>\frac{1}{i}\prod_{j=M-1}^{i-2}\left(j+\frac{1}{2}\right)\cdot\int_0^{1/2}\left(\frac{1}{2}-y\right)\{(-1)^{M-1}y^{(M)}-(1-y)y^{[M-1]}\}dy\\&=\frac{1}{i}\prod_{j=M-1}^{i-2}j\cdot\prod_{j=M-1}^{i-2}\left(1+\frac{1}{2j}\right)\cdot Ma_M\ ,\end{aligned}$$

from which we have

$$a_i>\left\{\frac{3}{2}\cdot\frac{\Gamma(i-1)}{\Gamma(M-1)}-\frac{1}{2}\cdot\frac{\Gamma(i-2)}{\Gamma(M-2)}\right\}\cdot\frac{M}{i}a_M\ ,\qquad (i\geqq M+1,\ M\geqq 3)\ . \tag{2.25}$$

On the other hand, it can be seen that

$$\begin{aligned}a_i&<\frac{1}{i}\prod_{j=M-1}^{i-2}j\cdot\int_0^{1/2}\left(\frac{1}{2}-y\right)\{(-1)^{M-2}y^{(M-1)}(\overline{M-1-y}+\overline{i-M})-(1-y)y^{[M-1]}\}dy\\&=\frac{1}{i}\frac{\Gamma(i-1)}{\Gamma(M-1)}\left\{Ma_M+(i-M)(-1)^{M-2}\int_0^{1/2}\left(\frac{1}{2}-y\right)y^{(M-1)}dy\right\}\ ,\end{aligned}$$

that is,

$$(2.26)\quad a_i < \frac{M}{i}\,\frac{\Gamma(i-1)}{\Gamma(M-1)}\cdot\left\{a_M+\left(\frac{i}{M}-1\right)W(M)\right\}\,, \qquad (i\geqq M+1,\ M\geqq 3)\,,$$

where $W(M)$ is the quantity defined in (2.16). Thus, applying these inequalities (2.25) and (2.26) to (2.13), we can prove the inequality (2.12) in the same manner as those in the proof of Lemma 2.1.

(ii) By using (2.1) to the part of $\ln(x+1)$ in (2.11), the R.H.S. expression of (2.18) follows at once, so the inequality (2.19) can be directly derived.

Remark 2.1. Lemma 2.1 is also true for $u<-1$ with exclusions $u=-2, -3, \cdots$, although our later discussions need not to consider such cases. As for Lemma 2.2, (2.11) and (2.18) give wider applicability than the usual Stirling formula for $\ln n!$ which is used for sufficiently large n, whereas our results are also valid for small values of n and even when $n=0$ with fair accuracy. Though the above lemmas are prepared for developing an approximation theory in the subsequent sections, some numerical computations show that (2.1), (2.11) and (2.18) are useful in the practical purposes and that give considerably close bounds for various combinations of (u, K), (x, M) and (x, M, K).

Remark 2.2. It should be also remarked that

$$S_K(u)\to 0 \quad \text{as} \quad K\to\infty\,, \qquad \text{for any } u>0\,,$$

and that

$$R_M(y)\to 0 \quad \text{as} \quad M\to\infty\,, \qquad \text{for any } y>0\,,$$

because by Lemma 3.2 and Lemma 3.3 of [5] we see that

$$0<S(u)\equiv\sum_{i=1}^{\infty} b_i u^{-[i+1]} < \frac{1}{2u(u+1)}+\frac{1}{6u^2(u+1)}$$

and

$$0<R(y)\equiv\sum_{i=3}^{\infty} a_i y^{-[i]} < \frac{1}{64}\left\{\frac{1}{y^2}-\frac{1}{y(y+1)}\right\}\,,$$

thus, the series of positive terms $S(u)$ and $R(y)$ are absolutely *convergent* for any $u>0$ and $y>0$, respectively. Therefore, the more terms of the series we use, the more accurate results we get, in case of approximating $\ln\Gamma(x+1)$ by making use of the inverse factorial series. On the contrary, the usual Stirling series does not have the convergent property;

$$\ln\Gamma(x+1)=\frac{1}{2}\ln 2\pi+\left(x+\frac{1}{2}\right)\ln x-x+\frac{1}{12x}-\mu(x)\,, \qquad (x>0)\,,$$

where

$$\mu(x)=\sum_{n=2}^{\infty}\frac{(-1)^{n-1}B_n}{2n(2n-1)x^{2n-1}}\equiv\sum_{n=2}^{\infty}(-1)^{n-1}c_n x^{-(2n-1)},$$

and B_n denotes the Bernoulli number defined by

$$B_n=\frac{2(2n)!}{(2^{2n}-1)\pi^{2n}}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^{2n}}.$$

Since $C_{n+1}/C_n\sim B_{n+1}/B_n\sim(n/\pi)^2$ as $n\to\infty$, the remainder part $\mu(x)$ is *divergent* series for any *fixed* $x>0$.

The following lemma is a partial improvement of Feller's result [2] which is useful to replace sums by integrals in our approximation methods in Section 5. The proof of the lemma can be done by the similar manner as that in [2], so it will be omitted here.

LEMMA 2.3. *For* $0<h<1$ *and* $|xh|\leqq\sqrt{5}$,

$$J\equiv\int_{x-h/2}^{x+h/2}e^{-u^2/2}du=h\cdot\exp\left\{-\frac{x^2}{2}+\frac{(x^2-1)h^2}{24}+\omega(x,h)\right\},\tag{2.27}$$

where

$$-\frac{x^4h^4}{960}<\omega(x,h)<\frac{h^4}{288-24h^2}.\tag{2.28}$$

3. Approximations to binomial distribution

Let S_n be a random variable distributed according to the binomial distribution $B(n, p)$ whose probability function is given by

$$P_r(S_n=k)=\binom{n}{k}p^kq^{n-k},\qquad k=0,1,\cdots,n,\tag{3.1}$$

where $0<p<1$ and $q=1-p$. First, let us apply the improved Stirling formulae to the binomial coefficient in the manner such that we use (2.11) to $n!$ in the numerator but (2.18) to the factorials in the denominator, then it can be represented as

$$\binom{n}{k}=\frac{1}{\sqrt{2\pi(n+1)}}\left\{\left(\frac{k+1/2}{n+1}\right)^{(k+1/2)/(n+1)}\cdot\left(1-\frac{k+1/2}{n+1}\right)^{1-[(k+1/2)/(n+1)]}\right\}^{-(n+1)}\exp(r_{M,K}(k;n)),\tag{3.2}$$

where for any positive integers M ($\geqq3$) and K ($\geqq2$)

$$(3.3) \qquad r_{M,K}(k;\, n)=\sum_{i=1}^{M} a_i\{(k+1)^{-[i]}+(n-k+1)^{-[i]}-(n+1)^{-[i]}\}$$
$$+\frac{1}{2}\sum_{i=1}^{K} b_i\{(2k+2)^{-[i]}+(2(n-k)+2)^{-[i]}\}$$
$$+R^*_{M,K}(k)+R^*_{M,K}(n-k)-R_M(n+1)\,,$$

and where a's, b's, $R_M(\cdot)$ and $R^*_{M,K}(\cdot)$ are the same ones as those defined in the preceding section. Thus, by applying the inequalities (2.12) and (2.19) to (3.3), we get the lower and upper bounds for $r_{M,K}(k;\, n)$;

$$(3.4) \qquad \underline{r}_{M,K}(k;\, n)<r_{M,K}(k;\, n)<\bar{r}_{M,K}(k;\, n)\,,$$

where

$$(3.5) \qquad \underline{r}_{M,K}(k;\, n)=\Sigma_{M,K}+\underline{R}^*_{M,K}(k)+\underline{R}^*_{M,K}(n-k)-\bar{R}_M(n+1)\,,$$

$$(3.6) \qquad \bar{r}_{M,K}(k;\, n)=\Sigma_{M,K}+\bar{R}^*_{M,K}(k)+\bar{R}^*_{M,K}(n-k)-\underline{R}_M(n+1)\,,$$

and where $\Sigma_{M,K}$ denotes the sum of the first two terms of the R.H.S. in (3.3). Assume now that n and k are positive integers such that

$$(3.7) \qquad \delta\equiv\delta(k;\, p)=\frac{k+1/2}{n+1}-p>0$$

for a given p, then $0<p+\delta<1$, $0<q-\delta<1$, $p/\delta>0$, $q/\delta>1$, and then it can be represented as

$$(3.8) \qquad \binom{n}{k}p^k q^{n-k}=\frac{1}{\sqrt{2\pi(n+1)pq}}\exp\{-(n+1)L_{M,K}(k;\, n,\, p)\}\,,$$

where

$$(3.9) \qquad L_{M,K}(k;\, n,\, p)=l(\delta;\, p)-\frac{1}{n+1}r_{M,K}(k;\, n)\,,$$

with

$$(3.10) \qquad l(\delta;\, p)=(p+\delta)\ln\left(1+\frac{\delta}{p}\right)+(q-\delta)\ln\left(1-\frac{\delta}{q}\right).$$

Here, making use of Lemma 2.1 and the remarkable inequalities given by Okamoto [6] and Kraft [4], the R.H.S. of (3.10) can be evaluated as

$$(3.11) \qquad \underline{l}(\delta;\, p)<l(\delta;\, p)<\bar{l}(\delta;\, p)\,,$$

where

$$(3.12) \qquad \underline{l}(\delta;\, p)=\max\{\underline{l}_M,\, \underline{l}_O,\, \underline{l}_K\}\,,$$

with for any integer K ($\geqq 2$)

$$(3.13)\quad \underline{l}_M=\frac{\delta^2}{2pq}-\frac{\delta^2}{pq}\sum_{i=2}^{K}b_i\left\{q\left(\frac{p}{\delta}+2\right)^{-[i-1]}-p\left(\frac{q}{\delta}+1\right)^{-[i-1]}\right\}$$
$$-\frac{b_K\delta^3}{p^2q^2}\left\{Kq^2\left(\frac{p}{\delta}+2\right)^{-[K-1]}-(K-1)p^2\left(\frac{q}{\delta}+1\right)^{-[K-1]}\right\},$$
$$\left(\geqq\frac{\delta^2}{2pq}-\frac{(q-p)\delta^3}{6p^2q^2},\ \text{for } K=2\right),$$

$$(3.14)\quad \underline{l}_K=2\delta^2+(4/9)\delta^4,$$

$$(3.15)\quad \underline{l}_O=2(\sqrt{p+\delta}-\sqrt{p})^2,$$

and where for any positive integer K ($\geqq 2$)

$$(3.16)\quad \bar{l}(\delta;p)=\frac{\delta^2}{2pq}-\frac{\delta^2}{pq}\sum_{i=2}^{K}b_i\left\{q\left(\frac{p}{\delta}+2\right)^{-[i-1]}-p\left(\frac{q}{\delta}+1\right)^{-[i-1]}\right\}$$
$$-\frac{b_K\delta^3}{pq(p+\delta)(q-\delta)}\left\{(K-1)q(q-\delta)\left(\frac{p}{\delta}+2\right)^{-[K-1]}\right.$$
$$\left.-Kp(p+\delta)\left(\frac{q}{\delta}+1\right)^{-[K-1]}\right\}$$
$$\left(\leqq\frac{\delta^2}{2pq}-\frac{(q-p-\delta)\delta^2}{6pq(p+\delta)(q-\delta)},\ \text{for } K=2\right).$$

Thus, from (3.4), (3.9) and (3.11), we have for any positive integers M ($\geqq 3$) and K ($\geqq 2$),

$$(3.17)\quad \underline{L}_{M,K}(k;n,p)<L_{M,K}(k;n,p)<\bar{L}_{M,K}(k;n,p)$$

where

$$(3.18)\quad \underline{L}_{M,K}(k;n,p)=\underline{l}(\delta;p)-\frac{1}{n+1}\bar{r}_{M,K}(k;n)$$

and

$$(3.19)\quad \bar{L}_{M,K}(k;n,p)=\bar{l}(\delta;p)-\frac{1}{n+1}\underline{r}_{M,K}(k,n).$$

Summarizing the above result we obtain the following estimations for binomial individual probabilities:

THEOREM 3.1. *For non-negative integers n and k such that $(n+1)p-1/2<k\leqq n$, it holds that*

$$(3.20)\quad \underline{b}(k;n,p)<\binom{n}{k}p^kq^{n-k}<\bar{b}(k;n,p),$$

where

$$(3.21) \qquad \underline{b}(k;n,p)=\frac{1}{\sqrt{2\pi(n+1)pq}}\exp\{-(n+1)\bar{L}_{M,K}(k;n,p)\},$$

and

$$(3.22) \qquad \bar{b}(k;n,p)=\frac{1}{\sqrt{2\pi(n+1)pq}}\exp\{-(n+1)\underline{L}_{M,K}(k;n,p)\}.$$

Next, let us consider to estimate extreme tail probabilities of the binomial distribution. Usually, this problem is treated asymptotically by resorting to certain approximate distributions such as normal approximations to the binomial tails. In what follows, however, we shall try to give some computable bounds on the probabilities without using other approximate distributions. To this end the following inequalities due to Hodges-Lehmann [3] and Bahadur [1] are useful:

LEMMA 3.1. *Under the same conditions for n and k as those in Theorem 3.1, it holds that*

$$(3.23) \qquad \underline{\eta}(k;n,p)\leqq\frac{P_r(S_n\geqq k)}{P_r(S_n=k)}\leqq\bar{\eta}(k;n,p),$$

where

$$(3.24) \qquad \underline{\eta}(k;n,p)=\max\left\{\frac{1-(1/n\cdot p/q)^{n+1}}{1-1/n\cdot p/q},\ \frac{\alpha+1-z}{\alpha+(1-z)^2}q\right\},$$

$$(3.25) \qquad \bar{\eta}(k;n,p)=\min\left\{\frac{1-((n-k)/(k+1)\cdot p/q)^{n-1}}{1-((n-k)/(k+1)\cdot p/q)},\ \frac{\alpha+1}{\alpha+1-z}q\right\},$$

and where

$$(3.26) \qquad z=z(k;n,p)=\frac{n+1}{k+1}p,$$

$$(3.27) \qquad \alpha=\alpha(k;n,p)=\left(\frac{1}{k+1}-\frac{1}{n+1}\right)\left(\frac{k+1}{k+2}\right)z.$$

In view of Theorem 3.1 and the above lemma, we can immediately state the following estimations on the binomial (upper) tail probability:

THEOREM 3.2. *Let $0<p<1$, $q=1-p$, and k be non-negative integer such that $(n+1)p-1/2<k\leqq n$, then it holds that*

$$(3.28) \qquad \underline{c}(k;n,p)<P_r(S_n\geqq k)<\bar{c}(k;n,p),$$

where

$$(3.29) \qquad \underline{c}(k;n,p)=\underline{\eta}(k;n,p)\cdot\underline{b}(k;n,p),$$

and

$$\bar{c}(k;n,p)=\bar{\eta}(k;n,p)\cdot\bar{b}(k;n,p)\ . \tag{3.30}$$

4. Approximations to multinomial distribution

In this section we give some estimates for multinomial probabilities along the line of the preceding section. The results given below will become basically important in the subsequent section where χ^2-approximations related to multinomial random variables will be investigated.

Let $X_{(k)}=(X_1,\cdots,X_k)$ be a k $(\geqq 2)$-dimensional random variable which takes values $x_{(k)}=(x_1,\cdots,x_k)=(m_1/N,\cdots,m_k/N)$, where

$$N=n+k/2\ ,\qquad m_i=n_i+1/2\quad (i=1,\cdots,k)\ , \tag{4.1}$$

and where n_i's are non-negative integers satisfying $\sum_{i=1}^{k} n_i=n$. Let, further, the probability function of $X_{(k)}$ be given by

$$P_r(X_{(k)}=x_{(k)})=\frac{n!}{\prod_{i=1}^{k} n_i!}\prod_{i=1}^{k} p_i^{n_i}\equiv P_n(x_{(k)}\,|\,p_{(k)})\ , \tag{4.2}$$

where $p_{(k)}=(p_1,\cdots,p_k)$ is the parameter vector of a multinomial distribution and is any point in the simplex

$$\Omega=\left\{(z_1,\cdots,z_k)\,|\,z_i\geqq 0,\ i=1,\cdots,k;\ \sum_{i=1}^{k} z_i=1\right\}\ .$$

Then, taking a convention $p_i^{n_i}=1$ if $p_i=n_i=0$, we can rewrite (4.2) as

$$P_n(x_{(k)}\,|\,p_{(k)})=\prod_{i=1}^{k}(x_i/p_i)^{1/2}\cdot P_n(x_{(k)}\,|\,x_{(k)})\exp\{-N\cdot I(x_{(k)};p_{(k)})\}\ , \tag{4.3}$$

where

$$I(x_{(k)};p_{(k)})=\sum_{i=1}^{k} x_i\ln(x_i/p_i)\ . \tag{4.4}$$

Now, let us again apply the improved Stirling formulae (2.22) and (2.18) given in Lemma 2.2 to the multinomial coefficient in $P_n(x_{(k)}\,|\,x_{(k)})$ to $n!$ in the numerator and to the factorials in the denominator, respectively. Then, it can be further represented that

$$P_n(x_{(k)}\,|\,p_{(k)})=\frac{1}{(\sqrt{2\pi N})^{k-1}\sqrt{\prod_{i=1}^{k} p_i}}\exp\{-N\cdot I(x_{(k)};p_{(k)})\}\cdot\exp\{R^{*}_{M,K}(n,k;n_i)\}\ , \tag{4.5}$$

where for any positive integers $M(\geqq 3)$ and $K(\geqq 2)$

$$(4.6)\quad R^{*}_{M,K}(n, k; n_i)=-\sum_{j=1}^{M} a_j n^{-[j]}+\left(\frac{1}{2}+\frac{1}{4n}\right)\sum_{j=1}^{K} b_j k\left(\frac{2n}{k}+1\right)^{-[j]}$$
$$-\frac{k}{4n}-R_M(n)+\left(n+\frac{1}{2}\right)S_K\left(\frac{2n}{k}\right)+\frac{1}{24(n_i+1)}$$
$$+\sum_{i=1}^{k}\left[\sum_{j=3}^{M} a_j(n_i+1)^{-[j]}+\frac{1}{2}\sum_{j=2}^{K} b_j(2(n_i+1))^{-[j]}\right.$$
$$\left.+R^{*}_{M,K}(n_i)\right],$$

and where a's, b's, $R_M(\cdot)$, $S_K(\cdot)$ and $R^{*}_{M,K}(\cdot)$ are the same quantities as defined in Lemma 2.1 and Lemma 2.2. We thus have the estimations for (4.6) as

$$(4.7)\quad \underline{R}^{*}_{M,K}(n, k; n_i)<R^{*}_{M,K}(n, k; n_i)<\bar{R}^{*}_{M,K}(n, k; n_i)\,,$$

by applying the inequalities (2.2), (2.12) and (2.19) to $S_K(\cdot)$, $R_M(\cdot)$ and $R^{*}_{M,K}(\cdot)$ in (4.6), respectively.

Next, we proceed to evaluate $I(x_{(k)}; p_{(k)})$. Let us put

$$(4.8)\quad \delta_i=\delta(n_i; p_i)=\frac{n_i+1/2}{n+k/2}-p_i=x_i-p_i\,,$$

for $i=1,\cdots, k$, then $0\leqq p_i+\delta_i\ (=x_i)<1$ for each i and $\sum_{i=1}^{k}\delta_i=0$, and then it follows that

$$(4.9)\quad I(x_{(k)}; p_{(k)})=\sum_{i=1}^{k} t(\delta_i; p_i)\,,$$

where

$$(4.10)\quad t(\delta_i; p_i)=(p_i+\delta_i)\ln\left(1+\frac{\delta_i}{p_i}\right).$$

Using Lemma 2.1 it can be represented as

$$(4.11)\quad t(\delta_i; p_i)=\begin{cases} 0\,, & \text{if } \delta_i=0\,,\\ \delta_i+\delta_i^2/2p_i-w_K^{+}(\delta_i; p_i)\,, & \text{if } \delta_i>0\,,\\ \delta_i+\delta_i^2/2p_i+w_K^{-}(\delta_i; p_i)\,, & \text{if } \delta_i<0\,,\end{cases}$$

where for any positive integer $K\,(\geqq 2)$

$$(4.12)\quad w_K^{+}(\delta_i; p_i)=(p_i+\delta_i)\left\{\sum_{j=1}^{K} b_j\left(\frac{p_i}{\delta_i}\right)^{-[j+1]}+S_K\left(\frac{p_i}{\delta_i}\right)\right\}$$

and

$$(4.13)\quad w_K^{-}(\delta_i; p_i)=(p_i+\delta_i)\left\{\sum_{j=1}^{K} b_j\left(\frac{p_i+\delta_i}{-\delta_i}\right)^{-[j+1]}+S_K\left(\frac{p_i+\delta_i}{-\delta_i}\right)\right\}.$$

Thus, applying the inequalities (2.2) to $S_K(\cdot)$ in (4.12) and (4.13), we have estimates

$$(4.14)\qquad 0\leqq \underline{w}^+_{K,i}<w^+_K(\delta_i;p_i)<\overline{w}^+_{K,i}\qquad 0\leqq \underline{w}^-_{K,i}<w^-_K(\delta_i;p_i)<\overline{w}^-_{K,i}\,,$$

where the concrete forms of the above bounds are easily obtained and will be omitted to avoid redundancy. In passing, the following bounds are less accurate but simpler than those obtained by (4.11) and (4.14);

$$(4.15)\qquad \delta_i+\frac{\delta_i^2}{2p_i}-\frac{\delta_i^3}{6p_i^2}\leqq t(\delta_i;p_i)\leqq \delta_i+\frac{\delta_i^2}{2p_i}-\frac{\delta_i^3}{6p_i(p_i+\delta_i)}\,,$$

which are valid independently of the sign of δ_i, $i=1,\cdots,k$.

Noticing the fact $\sum_{i=1}^{k}\delta_i=0$, we have from (4.9)–(4.14)

$$(4.16)\qquad \underline{I}(x_{(k)};p_{(k)})\leqq I(x_{(k)};p_{(k)})\leqq \overline{I}(x_{(k)};p_{(k)})\,,$$

where

$$(4.17)\qquad \underline{I}(x_{(k)};p_{(k)})=\frac{1}{2}\sum_{i=1}^{k}\frac{\delta_i^2}{p_i}-\textstyle\sum_+\overline{w}^+_{K,i}+\sum_-\underline{w}^-_{K,i}$$
$$\left(\geqq\frac{1}{2}\sum_{i=1}^{k}\frac{\delta_i^2}{p_i}-\frac{1}{6}\sum_{i=1}^{k}\frac{\delta_i^3}{p_i^2}\right),$$

and

$$(4.18)\qquad \overline{I}(x_{(k)};p_{(k)})=\frac{1}{2}\sum_{i=1}^{k}\frac{\delta_i^2}{p_i}-\textstyle\sum_+\underline{w}^+_{K,i}+\sum_-\overline{w}^-_{K,i}$$
$$\left(\leqq\frac{1}{2}\sum_{i=1}^{k}\frac{\delta_i^2}{p_i}-\frac{1}{6}\sum_{i=1}^{k}\frac{\delta_i^3}{p_i(p_i+\delta_i)}\right),$$

where $\sum_+$ and $\sum_-$ denote the summations over the sets $\{i;\delta_i>0\}$ and $\{i;\delta_i<0\}$, respectively.

Consequently, we have the following local approximation to the multinomial probabilities:

THEOREM 4.1. *For any point $p_{(k)}\in\Omega$ it holds that*

$$(4.19)\qquad \underline{m}(k;n,p_{(k)})<P_r(X_{(k)}=x_{(k)})<\overline{m}(k;n,p_{(k)})\,,$$

where

$$(4.20)\qquad \underline{m}(k;n,p_{(k)})=C_0\exp\{-N\cdot\overline{I}(x_{(k)};p_{(k)})+\underline{R}^*_{M,K}(n,k;n_i)\}\,,$$

$$(4.21)\qquad \overline{m}(k;n,p_{(k)})=C_0\exp\{-N\cdot\underline{I}(x_{(k)};p_{(k)})+\overline{R}^*_{M,K}(n,k;n_i)\}\,,$$

$$(4.22)\qquad C_0=\frac{1}{(\sqrt{2\pi N})^{k-1}\sqrt{\prod_{i=1}^{k}p_i}}\,,$$

and where $\underline{I}(\cdot)$, $\bar{I}(\cdot)$, $\underline{R}^*_{M,K}(\cdot)$ *and* $\bar{R}^*_{M,K}(\cdot)$ *are the same bounds defined by* (4.17), (4.18) *and* (4.7), *respectively.*

5. Evaluations on the χ^2-approximation

In this section we give bounds to the so-called χ^2-approximation to the distribution functions of chi-square type statistics and the K-L information number related to multinomial random variables.

Let n and k be positive integers such that $2<k<n$, and let, $C_1, \cdots, C_k$ be mutually exclusive cells, each of which having probability $P_r(C_i)=p_i$ (>0, $i=1,\cdots,k$) with $\sum_{i=1}^k p_i=1$. Let, further, n_i be the observed cell-frequency in C_i after n independent trials, then $\sum_{i=1}^k n_i=n$ and the joint probability of $n_1,\cdots,n_k$ is given by the multinomial probability function stated in the preceding section, that is

$$(5.1)\qquad P_n(x_{(k)}\,|\,p_{(k)})=\frac{n!}{\prod_{i=1}^k n_i!}\prod_{i=1}^k p_i^{n_i}\,,$$

where as before $p_{(k)}=(p_1,\cdots,p_k)$ is the parameter vector of the multinomial distribution and $x_{(k)}=(x_1,\cdots,x_k)=(m_1/N,\cdots,m_k/N)$ with $N=n+k/2$ and $m_i=n_i+1/2$ $(i=1,\cdots,k)$.

Assume that the parameter vector $p_{(k)}$ is completely specified. Under this situation, K. Pearson [7] introduced the statistic

$$(5.2)\qquad X_P^2=\sum_{i=1}^k\frac{(n_i-np_i)^2}{np_i}$$

for the purpose of the goodness of fit test.

When k is fixed independently of n, it is well-known that as $n\to\infty$ the sampling distribution of X_P^2 weakly converges to the distribution function

$$(5.3)\qquad K_{k-1}(x)=\frac{1}{2^{(k-1)/2}\Gamma((k-1)/2)}\int_0^x e^{-u/2}u^{(k-3)/2}du\,,$$

that is, the χ^2-distribution with $k-1$ degrees of freedom, which is completely independent of $p_{(k)}$. Based on this chi-square approximation the so-called χ^2-test has been made. But, as Uspensky [8] pointed out, the lack of information as to the approximation error by using the limiting distribution renders the application of χ^2-test somewhat dubious. Certainly, several literatures on the error evaluation are found, but the almost all of them treat the problem *asymptotically*. From the practical point of view we wish to know the approximation error in the case of moderately small n. Only Vora [9], so far as the author knows, considered such cases by extending Feller's notable techniques for the normal approximation to the binomial distribution in [2] to the k-dimensional case. He gave bounds on the chi-square approximation for the sampling distribution of the statistic

$$X_V^2=\sum_{i=1}^{k}\frac{(m_i-Np_i)^2}{Np_i}=\sum_{i=1}^{k}\frac{\{n_i+1/2-(n+k/2)p_i\}^2}{(n+k/2)p_i}\,. \tag{5.4}$$

At first glance this statistic seems more complicated than X_P^2, but the former statistic also converges in law to the distribution $K_{k-1}(x)$ as $n\to\infty$. Further, as he showed, we can get somewhat sharper bounds for $P_r(X_V^2\leqq c)$ than for $P_r(X_P^2\leqq c)$ and fortunately the bounds for X_V^2 is helpful for obtaining the bounds for X_p^2. This fact would be suggested by the discussions in the preceding section, too. In the subsequent part we intend to improve Vora's result along almost the same line of his. The improvements result from using the bounds stated in Section 2 which are more accurate than almost all his bounds corresponding to ours. Our approach, different from his, has another merit that we can see the behaviors of the following statistic

$$I(X_{(k)};\,p_{(k)})=\sum_{i=1}^{k}X_i\ln(X_i/p_i)\,, \tag{5.5}$$

which is a special case of the K-L information number playing very important roles in the modern statistical theory.

Prior to estimating the χ^2-approximation on X_V^2 statistic we must prepare some lemmas to approximate sums by integrals in k-dimensional case.

Let $\xi_{(k-1)}=(\xi_1,\cdots,\xi_{k-1})'$ be a $(k-1)$-dimensional column vector such that

$$\xi_i=\xi_i(n_i)=\frac{m_i-Np_i}{\sqrt{Np_i(1-p_i)}}\,,\qquad(i=1,\cdots,k-1)\,, \tag{5.6}$$

then we have

$$X_V^2=\xi'_{(k-1)}\Sigma^{-1}\xi_{(k-1)}=\sum_{i,j=1}^{k-1}\sigma^{ij}\xi_i\xi_j\,, \tag{5.7}$$

where $\Sigma^{-1}=(\sigma^{ij})$ with

$$\sigma^{ij}=[p_ip_j(1-p_i)(1-p_j)]^{1/2}\left(\frac{\delta_{ij}}{p_i}+\frac{1}{p_k}\right)\,, \tag{5.8}$$

$$(i,\,j=1,\cdots,k-1;\ \delta_{ij}\text{: Kronecker's delta})\,.$$

Since Σ^{-1} is positive definite, there exists a $(k-1)\times(k-1)$ lower triangular matrix $T=(t_{ij})$, $t_{ij}>0$ such that $\Sigma^{-1}=T'T$. Then according to

[9], let us consider the transformed vector $y_{(k-1)}=(y_1,\cdots,y_{k-1})'$ defined by

$$y_{(k-1)}=\sqrt{d}\,T\xi_{(k-1)}\ , \tag{5.9}$$

where d is a positive constant satisfying some conditions described later in (5.20) and we have put $n_{(k-1)}=(n_1,\cdots,n_{k-1})$. Therefore, it follows that

$$\sum_{i=1}^{k-1} y_i^2=dX_V^2\ . \tag{5.10}$$

Now, let $h_{(k-1)}=(h_1,\cdots,h_{k-1})'$ be a real vector whose components are defined by

$$h_i=\sqrt{\frac{d}{Np_i(1-p_i)}}\,t_{ii}\ (>0)\ ,\qquad (i=1,\cdots,k-1)\ , \tag{5.11}$$

and consider rectangles in the $(k-1)$-dimensional Euclidean space $R_{(k-1)}$ of the following form

$$A(y_{(k-1)};h_{(k-1)})=\{u_{(k-1)}\,|\,y_i-h_i/2<u_i\leqq y_i+h_i/2;\ i=1,\cdots,k-1\}\ , \tag{5.12}$$

where $u_{(k-1)}=(u_1,\cdots,u_{k-1})$ and y_i's are the same ones defined in (5.9). Under the setup the following result due to Vora holds (cf. Lemma 1 in [9]):

LEMMA 5.1. *Let $n_i=0,\pm1,\pm2,\cdots$ $(i=1,\cdots,k-1)$, then the rectangles $A(y_{(k-1)};h_{(k-1)})$ are non-overlapping and cover the entire space $R_{(k-1)}$.*

Now, noticing the definition of T and (5.8), some calculations (cf. [9]) yield that

$$t_{ii}^2=(1-p_i)\left(1+p_i\Big/\sum_{j=i+1}^{k}p_j\right)\ ,\qquad (i=1,\cdots,k-1)\ , \tag{5.13}$$

and

$$\prod_{i=1}^{k-1}t_{ii}^2=p_k^{-1}\prod_{i=1}^{k-1}(1-p_i)\ . \tag{5.14}$$

Therefore, it follows from (5.11) that

$$h_i^2=\frac{dt_{ii}^2}{Np_i(1-p_i)}=\frac{d}{N}\left\{p_i^{-1}+\left(\sum_{j=i+1}^{k}p_j\right)^{-1}\right\}\equiv\frac{d}{N}b_i\ ,\qquad (i=1,\cdots,k-1)\ , \tag{5.15}$$

and from (5.11) and (5.14) that

$$\frac{1}{(\sqrt{2\pi})^{k-1}}\prod_{i=1}^{k-1}h_i=d^{(k-1)/2}C_0\equiv C_0^*(d)\ , \tag{5.16}$$

where C_0 is the constant defined by (4.22). Let us then evaluate the multiple integral

$$J(y_{(k-1)}; h_{(k-1)}) \equiv \int_{A(y_{(k-1)}; h_{(k-1)})} f(u_{(k-1)}) du_{(k-1)} , \tag{5.17}$$

where

$$f(u_{(k-1)}) = \frac{1}{(\sqrt{2\pi})^{k-1}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{k-1} u_i^2\right\} \tag{5.18}$$

and $du_{(k-1)} = du_1 \cdots du_{k-1}$. We can state the following generalization of Lemma 2.3:

LEMMA 5.2. *Put, for each n*

$$\underline{b} = \min_{1 \leq i \leq k-1} b_i \quad \textit{and} \quad \bar{b} = \max_{1 \leq i \leq k-1} b_i , \tag{5.19}$$

where b_i $(i=1, \cdots, k-1)$ are the quantities defined in (5.15). *If the conditions*

$$\bar{b} \cdot \frac{d}{N} < 1 \quad \textit{and} \quad \bar{b} \cdot X_V^2 \cdot \frac{d^2}{N} < 5 \tag{5.20}$$

are satisfied, then

$$J(y_{(k-1)}; h_{(k-1)}) \geqq C_0^*(d) \exp\left\{-\frac{1}{2}\left(d - \frac{\underline{b}}{12N} d^2\right) X_V^2 - \bar{r}\right\} , \tag{5.21}$$

$$J(y_{(k-1)}; h_{(k-1)}) \leqq C_0^*(d) \exp\left\{-\frac{1}{2}\left(d - \frac{\bar{b}}{12N} d^2\right) X_V^2 - \underline{r}\right\} , \tag{5.22}$$

where $C_0^*(d)$ is the quantity defined in (5.16),

$$\bar{r} = \bar{r}(d, X_V^2) = \frac{d \sum_{i=1}^{k-1} b_i}{24N} + \frac{d^4 \bar{b}^2}{960 N^2} X_V^4 \tag{5.23}$$

and

$$\underline{r} = \underline{r}(d) = \frac{d \sum_{i=1}^{k-1} b_i}{24N} - \frac{d^2 \sum_{i=1}^{k-1} b_i^2}{264 N^2} . \tag{5.24}$$

PROOF. From (5.15) and (5.20) we see that

$$h_i^2 \leqq \bar{b} \frac{d}{N} < 1 \quad \text{and} \quad \sum_{i=1}^{k-1} h_i^2 y_i^2 \leqq \bar{b} X_V^2 \frac{d}{N} < 5 , \tag{5.25}$$

hence, $0 < h_i < 1$ and $|h_i y_i| < \sqrt{5}$ $(i=1, \cdots, k)$. Thus, by Lemmas 2.3 and 5.1, it can be represented that

$$J(y_{(k-1)}\,;\,h_{(k-1)})=\frac{\prod_{i=1}^{k-1}h_i}{(\sqrt{2\pi})^{k-1}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{k-1}\left(1-\frac{h_i^2}{12}\right)y_i^2-\frac{1}{24}\sum_{i=1}^{k-1}h_i^2+\omega\right\}, \tag{5.26}$$

where

$$-\frac{1}{960}\sum_{i=1}^{k-1}h_i^4y_i^4<\omega<\frac{1}{264}\sum_{i=1}^{k-1}h_i^4\,. \tag{5.27}$$

Remembering (5.10) and (5.15) we have

$$\frac{d^2}{N}\underline{b}X_V^2\leqq\sum_{i=1}^{k-1}h_i^2y_i^2\leqq\frac{d^2}{N}\bar{b}X_V^2 \tag{5.28}$$

$$\frac{d^4\underline{b}^2}{N^2(k-1)}X_V^4\leqq\sum_{i=1}^{k-1}h_i^4y_i^4\leqq\frac{d^4\bar{b}^2}{N^2}X_V^4\,. \tag{5.29}$$

Thus, from (5.26)–(5.29), we immediately obtain the target results (5.21) and (5.22).

With the help of Lemma 5.2 we can evaluate the multinomial probabilities (4.2) by certain multiple integrals related to $J(y_{(k-1)}\,;\,h_{(k-1)})$. For the later applications let us assume that

$$X_V^2\leqq c \qquad (c\text{: a non-negative constant})\,. \tag{5.30}$$

Assume also that

$$\bar{b}/N\leqq 3\,, \tag{5.31}$$

then we can set the numbers

$$\underline{D}=6\left\{1-\left(1-\frac{\underline{b}}{3N}\right)^{1/2}\right\}\left(\frac{\underline{b}}{N}\right)^{-1} \tag{5.32}$$

and

$$\bar{D}=6\left\{1-\left(1-\frac{\bar{b}}{3N}\right)^{1/2}\right\}\left(\frac{\bar{b}}{N}\right)^{-1} \tag{5.33}$$

as two special values of d in (5.9). The above values have been so chosen that the coefficients of X_V^2 in (5.21) and (5.22) are equal to 1/2, respectively. These facts will be used in the later proof of Lemma 5.3. Further, let us define $(k-1)$-dimensional vectors analogously to (5.9) and (5.11):

$$\underline{y}_{(k-1)}=(\underline{y}_1,\cdots,\underline{y}_{k-1})\,;\qquad \underline{y}_i=\bar{D}^{1/2}\sum_{r=1}^{i}t_{ri}\xi_r(n_r)\,,$$

$$\bar{y}_{(k-1)}=(\bar{y}_1,\cdots,\bar{y}_{k-1})\,;\qquad \bar{y}_i=\underline{D}^{1/2}\sum_{r=1}^{i}t_{ri}\xi_r(n_r)\,,$$

$$\underline{h}_{(k-1)}=(\underline{h}_1,\cdots,\underline{h}_{k-1})\,;\qquad \underline{h}_i=(b_i\bar{D}/N)^{1/2}\,,$$

$$\bar{h}_{(k-1)}=(\bar{h}_1,\cdots,\bar{h}_{k-1})\,;\qquad \bar{h}_i=(b_i\underline{D}/N)^{1/2}\,.$$

We can now state the following

LEMMA 5.3. *If the conditions*

$$\bar{b}/N\leqq 3\,,\qquad \bar{b}\bar{D}/N<1\,,\qquad c\bar{b}\bar{D}^2/N<5 \tag{5.34}$$

are satisfied, then

$$P_r(X_{(k)}=x_{(k)})>J(\underline{y}_{(k-1)};\underline{h}_{(k-1)})\exp\{\underline{Q}_{M,K}(n,k;n_i)\}\bar{D}^{-(k-1)/2}\,, \tag{5.35}$$

$$P_r(X_{(k)}=x_{(k)})<J(\bar{y}_{(k-1)};\bar{h}_{(k-1)})\exp\{\bar{Q}_{M,K}(n,k;n_i)\}\underline{D}^{-(k-1)/2}\,, \tag{5.36}$$

where for any positive integers $M(\geqq 3)$ *and* $K(\geqq 2)$

$$\underline{Q}_{M,K}(n,k;n_i)=\underline{R}^{*}_{M,K}(n,k;n_i)+\underline{r}(\bar{D})+N\{\textstyle\sum_+\underline{w}^{+}_{K,i}-\sum_-\bar{w}^{-}_{K,i}\}\,, \tag{5.37}$$

$$\bar{Q}_{M,K}(n,k;n_i)=\bar{R}^{*}_{M,K}(n,k;n_i)+\bar{r}(\underline{D},c)+N\{\textstyle\sum_+\bar{w}^{+}_{K,i}-\sum_-\underline{w}^{-}_{K,i}\} \tag{5.38}$$

and where $R^{*}_{M,K}(\cdot)$, $\sum_+\underline{w}^{+}_{K,i}$, $\sum_-\underline{w}_{K,i}$ *etc. and* $\underline{r}$, $\bar{r}$ *are the same definitions as those in* (4.6), (4.17), (4.18), (5.23) *and* (5.24), *respectively.*

PROOF. From Theorem 4.1 we have

$$\begin{aligned}C_0\exp(-X_V^2/2)\cdot\exp(\underline{r}^{*}_{M,K})&<P_r(X_{(k)}=x_{(k)})\\&<C_0\exp(-X_V^2/2)\cdot\exp(\bar{r}^{*}_{M,K})\,,\end{aligned} \tag{5.39}$$

where

$$\underline{r}^{*}_{M,K}=\underline{R}^{*}_{M,K}(n,k;n_i)+N\{\textstyle\sum_+\underline{w}^{+}_{K,i}-\sum_-\bar{w}^{-}_{K,i}\}\,, \tag{5.40}$$

$$\bar{r}^{*}_{M,K}=\bar{R}^{*}_{M,K}(n,k;n_i)+N\{\textstyle\sum_+\bar{w}^{+}_{K,i}-\sum_-\underline{w}^{-}_{K,i}\}\,. \tag{5.41}$$

On the other hand from Lemma 5.2, (5.32) and (5.33) we have

$$J(\bar{y}_{(k-1)};\bar{h}_{(k-1)})\geqq C_0^{*}(\underline{D})\exp\left\{-\frac{1}{2}\left(\underline{D}-\frac{b}{12N}\underline{D}^2\right)X_V^2-\bar{r}(\underline{D},c)\right\}\,, \tag{5.42}$$

$$J(\underline{y}_{(k-1)};\underline{h}_{(k-1)})\leqq C_0^{*}(\bar{D})\exp\left\{-\frac{1}{2}\left(\bar{D}-\frac{\bar{b}}{12N}\bar{D}^2\right)X_V^2-\underline{r}(\bar{D})\right\}\,. \tag{5.43}$$

Thus, noticing (5.32) and (5.16) and combining (5.39), (5.42) and (5.43), we get the desired inequalities (5.35) and (5.36), which completes the proof of the lemma.

Remark 5.1. In the above proof we have used Theorem 4.1 which are similar but of different forms from Vora's, so the differences are kept in the present lemma. Apart from them, however, Lemma 5.3 gives more accurate evaluations under weaker conditions than those in [9].

Now, we are in a position to evaluate the χ^2-approximation to the sampling distribution of the X_V^2 statistic. We can prove the following

THEOREM 5.1. *Under the same conditions as those in Lemma* 5.3, *it holds that*

$$(5.44)\qquad P_r(X_V^2\leqq c)\geqq\exp\{\underline{Q}_{M,K}(n,k;\underline{n}_i)\}\cdot\bar{D}^{-(k-1)/2}\times\left[K_{k-1}(\underline{c})+\left\{V_R(A^*;\underline{h}_{(k-1)})-\frac{(\underline{c}\pi)^{(k-1)/2}}{\Gamma((k+1)/2)}\right\}\times\frac{e^{-\underline{c}'/2}}{(\sqrt{2\pi})^{k-1}}\right],$$

$$(5.45)\qquad P_r(X_V^2\leqq c)\leqq\exp\{\bar{Q}_{M,K}(n,k;\bar{n}_i)\}\cdot\underline{D}^{-(k-1)/2}\times\left[K_{k-1}(\bar{c})-\left\{\frac{(\bar{c}\pi)^{(k-1)/2}}{\Gamma((k+1)/2)}-V_R(A^*;\bar{h}_{(k-1)})\right\}\times\frac{e^{-\bar{c}/2}}{(\sqrt{2\pi})^{k-1}}\right].$$

where $K_{k-1}(\cdot)$ *is the distribution function of the* χ^2*-distribution with* $k-1$ *degrees of freedom,* $V_R(A^*;h_{(k-1)})$ *is the quantity defined by* (5.58),

$$(5.46)\qquad \underline{n}_i=(1-\alpha_i)Np_i-1/2\,,\qquad (i=1,\cdots,k)$$

$$(5.47)\qquad \bar{n}_i=(1+\alpha_i)Np_i-1/2\,,\qquad (i=1,\cdots,k)$$

where α_{ni}*'s are some constants satisfying later* (5.57) *and*

$$(5.48)\qquad \alpha_i\leqq\sqrt{\frac{c}{N}\left(\frac{1}{p_i}-1\right)}\,,\qquad (i=1,\cdots,k)\,,$$

$$(5.49)\qquad \underline{c}=\begin{cases}\bar{D}\left\{c^{1/2}-\left(\frac{1}{4N}\sum_{i=1}^{k-1}b_i\right)^{1/2}\right\}^2, & \textit{if } c\geqq\frac{1}{4N}\sum_{i=1}^{k-1}b_i\\ 0\,, & \textit{otherwise}\,,\end{cases}$$

$$(5.50)\qquad \bar{c}=\underline{D}\left\{c^{1/2}+\left(\frac{1}{4N}\sum_{i=1}^{k-1}b_i\right)^{1/2}\right\}^2,$$

and

$$(5.51)\qquad \underline{c}'=\left\{\underline{c}^{1/2}+3\left(\frac{1}{4N}\sum_{i=1}^{k-1}b_i\right)^{1/2}\right\}^2.$$

PROOF. Let

$$A^*(y_{(k-1)}; h_{(k-1)}) = \left\{ u_{(k-1)} \middle| \begin{array}{l} y_i - \dfrac{h_i}{2} < u_i \leqq y_i + \dfrac{h_i}{2}, \qquad i=1, \cdots, k-1, \\ \text{for all } n_1, \cdots, n_{k-1} \text{ such that } \sum_{i=1}^{k-1} y_i^2(n_{(k-1)}) \leqq Dc \end{array} \right\}$$

and define

$$J^*(y_{(k-1)}; h_{(k-1)}) = \int_{A^*(y_{(k-1)}; h_{(k-1)})} f(u_{(k-1)}) du_{(k-1)} . \tag{5.52}$$

Since

$$\begin{aligned} P_r(X_V^2 \leqq c) &= \sum_{\{x_{(k)};\, X_V^2 \leqq c\}} P_r(X_{(k)} = x_{(k)}) \\ &= \sum_{\{n_{(k-1)};\, \sum_{i=1}^{k-1} y_i^2(n_{(k-1)}) \leqq Dc\}} P_r(X_{(k)} = x_{(k)}) , \end{aligned}$$

we have by Lemma 5.3 that

$$P_r(X_V^2 \leqq c) > J^*(\underline{y}_{(k-1)}; \underline{h}_{(k-1)}) \exp\{\underline{Q}_{M,K}(n, k; \underline{n}_i)\} \bar{D}^{-(k-1)/2} , \tag{5.53}$$

and

$$P_r(X_V^2 \leqq c) < J^*(\bar{y}_{(k-1)}; \bar{h}_{(k-1)}) \exp\{\bar{Q}_{M,K}(n, k; \bar{n}_i)\} \underline{D}^{-(k-1)/2} . \tag{5.54}$$

In the above we have used the following facts that, from (4.6), (4.12) and (4.13), $\underline{Q}_{M,K}(n, k; n_i)$ and $\bar{Q}_{M,K}(n, k; n_i)$ defined by (5.37) and (5.38), respectively, are monotone decreasing functions with respect to each n_i for given n, k and $p_{(k)}$, and that for each i

$$\underline{n}_i \geqq \inf \left\{ n_i ;\ \sum_{i=1}^{k} \delta_i = 0,\ N \sum_{i=1}^{k} \delta_i^2 / p_i = X_V^2 \leqq c \right\} , \tag{5.55}$$

and

$$\bar{n}_i \leqq \sup \left\{ n_i ;\ \sum_{i=1}^{k} \delta_i = 0,\ N \sum_{i=1}^{k} \delta_i^2 / p_i = X_V^2 \leqq c \right\} , \tag{5.56}$$

which are seen from the estimations (cf. [9])

$$\left| \frac{\delta_i}{p_i} \right| \leqq \sqrt{\frac{X_V^2}{N} \left(\frac{1}{p_i} - 1 \right)} \leqq \alpha_i , \qquad (i=1, \cdots, k) . \tag{5.57}$$

Here, let $V_R(A^*; h_{(k-1)})$ and $V_S(B; c)$ be the volumes of $A^*(y_{(k-1)}; h_{(k-1)})$ and $B_{(k-1)} = \{u_{(k-1)};\ \sum_{i=1}^{k-1} u_i^2 \leqq c\}$, respectively, then

$$V_R(A^*; h_{(k-1)}) = \# \left\{ n_{(k-1)};\ \sum_{i=1}^{k-1} y_i^2(n_{(k-1)}) \leqq Dc \right\} \prod_{i=1}^{k-1} h_i , \tag{5.58}$$

provided that $c \geqq \sum_{i=1}^{k-1} b_i/4N$. Otherwise, let us put conventionally $V_R(A^*; h_{(k-1)})=0$.

Let $(\underline{y}_1+\underline{\theta}_1, \cdots, \underline{y}_{k-1}+\underline{\theta}_{k-1})$ be any point outside $A^*(\underline{y}_{(k-1)}; \underline{h}_{(k-1)})$, then

$$\sum_{i=1}^{k-1} \underline{y}_i^2 > \bar{D}c \qquad \text{and} \qquad \underline{\theta}_i^2 \leqq (\underline{h}_i/2)^2 = \bar{D}b_i/4N,$$

and then, by the Cauchy-Schwartz inequality, we have

$$\sum_{i=1}^{k-1} (\underline{y}_i+\underline{\theta}_i)^2 \geqq \left\{\left(\sum_{i=1}^{k-1} \underline{y}_i^2\right)^{1/2} - \left(\sum_{i=1}^{k-1} \underline{\theta}_i^2\right)^{1/2}\right\}^2 \geqq \left\{(\bar{D}c)^{1/2} - \left(\bar{D}\sum_{i=1}^{k-1} b_i/4N\right)^{1/2}\right\}^2 = \underline{c},$$

if $c \geqq \sum_{i=1}^{k-1} b_i/4N$. That is, $A^*(\underline{y}_{(k-1)}; \underline{h}_{(k-1)})$ contains the sphere $\underline{B}_{(k-1)} = \{u_{(k-1)}; \sum_{i=1}^{k-1} u_i^2 \leqq \underline{c}\}$. Thus,

$$\begin{aligned}(5.59) \quad J^*(\underline{y}_{(k-1)}; \underline{h}_{(k-1)}) &\geqq \int_{\underline{B}(k-1)} f(u_{(k-1)})du_{(k-1)} \\ &\quad + \{V_R(A^*; \underline{h}_{(k-1)}) - V_S(\underline{B}; \underline{c})\} \inf_{A^*-\underline{B}} f(u_{(k-1)}) \\ &= K_{(k-1)}(\underline{c}) + \left\{V_R(A^*; h_{(k-1)}) - \frac{(\pi\underline{c})^{(k-1)/2}}{\Gamma((k+1)/2)}\right\} \\ &\quad \times \frac{e^{-\underline{c}'/2}}{(\sqrt{2\pi})^{k-1}}.\end{aligned}$$

Combining (5.53) and (5.59) we have the desired inequality (5.44).

On the other hand, we can prove (5.45) in the similar manner as the above.

Remark 5.2. In the bounds in (5.44) and (5.45) we have additional terms of the form

$$D^{-(k-1)/2} e^{Q_{M,K}} \{V_R(A^*; h_{(k-1)}) - V_S(B; c)\} \frac{e^{-c/2}}{(\sqrt{2\pi})^{k-1}}$$

which were not appeared in [9]. Related to these, Yarnold [10], [11] obtained a similar term of the order of magnitude $O(n^{-(k-1)/k})$ in his new asymptotic expansion, whereas his term is a little different from ours. It should be also remarked that he gave further higher order terms of the magnitude $O(n^{-1})$. However, it seems difficult to give exact bounds on the underlying approximation by the multivariate Edgeworth expansion.

Next, let us consider to evaluate the sampling distribution of the K-L information number defined by (5.5). From (4.9) and (4.10) the statistic can be also represented as

(5.60) $$I(x_{(k)}; p_{(k)}) = \frac{X_V^2}{2N} - \sum_{i=1}^{k} G\left(\frac{\delta_i}{p_i}\right)\frac{\delta_i^2}{p_i},$$

where

(5.61) $$G(z) = z + 1/2 - (1+z)\ln(1+z)/z^2, \quad (z \neq 0) \text{ and } G(0) = 0,$$

which is a monotone increasing function of z. Put here that

(5.62) $$\underline{G} = \max_{1 \leq i \leq k} G(-\alpha_i)$$

and

(5.63) $$\bar{G} = \min_{1 \leq i \leq k} G(\alpha_i),$$

then by the estimates (5.57) it holds that

(5.64) $$\underline{G} \leqq G(\delta_i/p_i) \leqq \bar{G}.$$

Thus, from (5.60) and (5.64), we have

(5.65) $$2N \cdot I(x_{(k)}; p_{(k)}) \leqq 2N \cdot \bar{I}(x_{(k)}; p_{(k)}) \leqq (1-\underline{G})X_V^2,$$

(5.66) $$2N \cdot I(x_{(k)}; p_{(k)}) \geqq 2N \cdot \underline{I}(x_{(k)}; p_{(k)}) \geqq (1-\bar{G})X_V^2.$$

Thus, for $c^\sharp \geqq 0$

(5.67) $$P_r(2N \cdot I(X_{(k)}; p_{(k)}) \leqq c^\sharp) \geqq P_r(2N \cdot \bar{I}(X_{(k)}; p_{(k)}) \leqq c^\sharp) \geqq P_r(X_V^2 \leqq c^\sharp(1-\underline{G})^{-1}) \equiv P_r(X_V^2 \leqq \underline{c}^*),$$

(5.68) $$P_r(2N \cdot I(X_{(k)}; p_{(k)}) \leqq c^\sharp) \leqq P_r(2N \cdot \underline{I}(X_{(k)}; p_{(k)}) \leqq c^\sharp) \leqq P_r(X_V^2 \leqq c^\sharp(1-\bar{G})^{-1}) \equiv P_r(X_V^2 \leqq \bar{c}^*),$$

where we have put

(5.69) $$\underline{c}^* = c^\sharp(1-\underline{G})^{-1} \quad \text{and} \quad \bar{c}^* = c^\sharp(1-\bar{G})^{-1}.$$

Consequently, from (5.67), (5.68) and Theorem 5.1, we get the following

THEOREM 5.2. *Under the same conditions as those in Theorem* 5.1, *it holds that*

(5.70) $$P_r(2N \cdot I(X_{(k)}; p_{(k)}) \leqq c^\sharp) \geqq \underline{L}^\sharp(c^\sharp(1-\underline{G})^{-1})$$

and

(5.71) $$P_r(2N \cdot I(X_{(k)}; p_{(k)}) \leqq c^\sharp) \leqq \bar{L}^\sharp(c^\sharp(1-\bar{G})^{-1}),$$

where $\underline{L}^\sharp(\underline{c}^*)$ *and* $\bar{L}^\sharp(\bar{c}^*)$ *denote the R.H.S. members in* (5.44) *and* (5.45), *respectively.*

In the final place of this section we shall give some numerical results on the evaluations for the sampling distribution function of X_V^2. For brevity let us denote the exact probability and its lower bounds and upper ones as follows:

$$E\equiv P_r(X_V^2\leqq c)\qquad \text{(the exact probability)},$$

$$L1\equiv\exp\{\underline{Q}_{M,K}(n,k;\underline{n}_i)\}\cdot\bar{D}^{-(k-1)/2}K_{k-1}(\underline{c})\equiv\underline{\mu}K_{k-1}(\underline{c})$$

$$L2\equiv L1+\underline{\mu}\left\{V_R(A^*;\underline{h}_{(k-1)})-\frac{(\underline{c}\pi)^{(k-1)/2}}{\Gamma((k+1)/2)}\right\}\frac{e^{-\underline{c}/2}}{(\sqrt{2\pi})^{k-1}},$$

$$U_1\equiv\exp\{\bar{Q}_{M,K}(n,k;\bar{n}_i)\}\cdot\underline{D}^{-(k-1)/2}K_{k-1}(\bar{c})\equiv\bar{\mu}K_{k-1}(\bar{c}),$$

$$U_2\equiv U_1-\bar{\mu}\left\{\frac{(\bar{c}\pi)^{(k-1)/2}}{\Gamma((k+1)/2)}-V_R(A^*;\bar{h}_{(k-1)})\right\}\frac{e^{-\bar{c}/2}}{(\sqrt{2\pi})^{k-1}},$$

$$U1\equiv\min(U_1,1)\qquad\text{and}\qquad U2\equiv\min(U_2,1),$$

then under the conditions of Theorem 5.1 the inequality $L1\leqq L2\leqq E\leqq U2\leqq U1$ must theoretically hold. From the practical point of view it is interesting to know the numerical accurracy of the bounds. In the following examples we shall treat only the case where the underlying multinomial distributions have equal cell probabilities;

$$(*)\qquad p_1=p_2=\cdots=p_k=1/k=p\qquad\text{(say)}.$$

In addition, to avoid trivial bounds, let us set that

$$(\text{S}\cdot 1)\qquad \alpha_1=\alpha_2=\cdots=\alpha_k\equiv\alpha=\min\left(\sqrt{\frac{c}{N}(k-1)},\ 0.3\right)$$

corresponding to (5.48), and that

$$(\text{S}\cdot 2)\qquad \frac{1}{N}\max\left\{\left|\sum_{i=1}^{k}\frac{(m_i-Np_i)^3}{Np_i^2}\right|,\left|\sum_{i=1}^{k}\frac{(m_i-Np_i)^3}{m_ip_i}\right|\right\}\equiv\beta\leqq\frac{1}{20}$$

related to the last terms in (5.37) and (5.38). It should be remarked that $-\underline{Q}_{M,K}(n,k;\underline{n}_i)$ in (5.44) and $\bar{Q}_{M,K}(n,k;\bar{n}_i)$ in (5.45) are monotone decreasing function of α. So, larger choices of the values of α and β give us poor bounds. The above values $\alpha=0.3$ and $\beta=1/20$ were chosen by some trial computations, although the last setting seems to be a little conservative in practical situations. Other setting is put on the occasion of using inverse factorial series of the forms in Lemma 2.1 and Lemma 2.2;

$$(\text{S}\cdot 3)\qquad M=K=10.$$

Under the above set-up computations were carried out to give

Table 5.1 Bounds for $P_r(X_V^2 \leqq c)$ for small samples

c		4.0000	5.0000	6.0000	7.0000	8.0000	9.0000	10.0000	11.0000
$k=3,\ p=1/3$									
$K_{k-1}(c)$		0.8647	0.9179	0.9502	0.9698	0.9817	0.9889	0.9933	0.9959
n		Exact prob., Lower and Upper bounds							
10	$L1$	0.6872							
	$L2$	0.7520							
	E	*0.9069*							
	$U2$	0.9410							
	$U1$	0.9914							
12	$L1$	0.7037	0.7972						
	$L2$	0.8048	0.8335						
	E	*0.9147*	*0.9296*						
	$U2$	0.9476	0.9296						
	$U1$	0.9828	1.0000						
15	$L1$	0.7218	0.8114	0.8715					
	$L2$	0.7854	0.8811	0.9080					
	E	*0.8752*	*0.9571*	*0.9697*					
	$U2$	0.9164	0.9773	0.9972					
	$U1$	0.9729	0.9988	1.0000					
20	$L1$	0.7420	0.8270	0.8833	0.9201	0.9440	0.9594		
	$L2$	0.8327	0.8933	0.9193	0.9487	0.9577	0.9700		
	E	*0.8979*	*0.9444*	*0.9624*	*0.9778*	*0.9864*	*0.9928*		
	$U2$	0.9182	0.9655	0.9877	1.0000	1.0000	1.0000		
	$U1$	0.9610	0.9894	1.0000	1.0000	1.0000	1.0000		
25	$L1$	0.7555	0.8373	0.8910	0.9258	0.9483	0.9627	0.9719	0.9777
	$L2$	0.8557	0.9012	0.9360	0.9510	0.9640	0.9724	0.9797	0.9821
	E	*0.9022*	*0'9262*	*0.9699*	*0.9779*	*0.9881*	*0.9927*	*0.9944*	*0.9978*
	$U2$	0.9167	0.9588	0.9863	0.9999	1.0000	1.0000	1.0000	1.0000
	$U1$	0.9526	0.9829	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
30	$L1$	0.7654	0.8447	0.8964	0.9298	0.9513	0.9650	0.9737	0.9792
	$L2$	0.8750	0.8874	0.9407	0.9577	0.9656	0.9739	0.9815	0.9833
	E	*0.9068*	*0.9203*	*0.9669*	*0.9711*	*0.9860*	*0.9901*	*0.9969*	*0.9974*
	$U2$	0.9181	0.9454	0.9827	0.9977	1.0000	1.0000	1.0000	1.0000
	$U1$	0.9463	0.9780	0.9960	1.0000	1.0000	1.0000	1.0000	1.0000
$k=4,\ p=1/4$									
$K_{k-1}(c)$		0.7385	0.8282	0.8884	0.9281	0.9540			
n		Exact prob., Lower and Upper bounds							
15	$L1$	0.4658							
	$L2$	0.5553							
	E	*0.8089*							
	$U2$	0.8719							
	$U1$	0.9575							
20	$L1$	0.5011	0.6303	0.7307					
	$L2$	0.6129	0.7160	0.7934					
	E	*0.7908*	*0.8788*	*0.9117*					
	$U2$	0.8484	0.9257	0.9672					
	$U1$	0.9337	0.9796	1.0000					
25	$L1$	0.5258	0.6522	0.7491	0.8205				
	$L2$	0.6534	0.7414	0.8109	0.8638				
	E	*0.7883*	*0.8629*	*0.9103*	*0.9484*				
	$U2$	0.8350	0.9089	0.9576	0.9894				
	$U1$	0.9166	0.9660	0.9954	1.0000				
30	$L1$	0.5442	0.6683	0.7624	0.8311	0.8801			
	$L2$	0.6566	0.7423	0.8360	0.8799	0.9169			
	E	*0.7595*	*0.8353*	*0.9056*	*0.9410*	*0.9641*			
	$U2$	0.8133	0.8898	0.9471	0.9800	1.0000			
	$U1$	0.9034	0.9557	0.9870	1.0000	1.0000			

bounds for $P_r(X_V^2 \leqq c)$ for small samples. We shall now give a table of the bounds for a number of combinations of k, n and c. In Table 5.1 the exact probability E, the bounds $L1$, $L2$, $U1$, $U2$, and the theoretical probability $K_{k-1}(c)$ are presented for $k=3, 4$, some selected n's$\leqq 30$ and for $c \geqq 4$.

Acknowledgements

The author is deeply grateful to Professor M. Okamoto for his valuable comments, advice and helpful support in completing this paper. He also wishes to express his gratitude to Dr. R. Shimizu for his useful comments on the paper and many stimulating discussions. Appreciation is expressed to Professor S. Ikeda, Professor K. Matusita, Dr. K. Takahasi and Dr. N. Inagaki for their encouragement.

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