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ON THE INDEPENDENCE OF INTERDEPARTURE INTERVALS FROM SINGLE SERVER QUEUEING SYSTEMS

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1. Introduction

An important question to determine in studying tandem queues is the distribution of output from one channel, which then comprises the input into the subsequent channel.

Burke [1] showed the independence of time intervals between departures for the steady state in the case of $M/M/C$. In his system, he allowed an infinite queue before the channels.

Instead we consider here the queueing systems with limited queue. We shall prove that the interdeparture times for $M/G/1(0)$ and $GI/M/1(0)$ are independent, while the interdeparture times for $E_2/E_2/1(0)$ and $M/M/1(1)$ are not independent. In the notations used above the parenthesized number denote the maximum queue size allowed before channels. Thus, the systems $M/G/1(0)$, $GI/M/1(0)$ and $E_2/E_2/1(0)$ are zero queue systems, and $M/M/1(1)$ denotes a system whose maximum queue size equals to one.

Using Burke's method, we are going to investigate whether the independence of the interdeparture times can be established. First we introduce the similar notations to the ones which are defined in his paper.

λ; Arrival rate
μ; Service rate
$\rho=\lambda/\mu$; Utilization factor
L; Length of an arbitrary interdeparture interval
$n(t)$; State at a time t after the last previous departure
$n(L)=n(L+0)$; State at a time immediately after last departure
$F_k(t)=\mathrm{P}\{n(t)=k,\ L>t\}$; Probability that $n(t)=k$ and jointly that $L>t$
Λ; An arbitrary set which is written as $\{L_{i_1}>t_1,\cdots,L_{i_m}>t_m\}$ for some m and some $L_{i_1},\cdots,L_{i_m}$ which are interdeparture intervals subsequent to arbitrary interval of length L
$\mathrm{P}(L>t)=\sum_{k=0}^{\infty}F_k(t)=F(t)$; Interdeparture time distribution

It should be noted the following theorem by Burke. That is, if the systems have the properties,

The Markovian property ;

$$P(\Lambda | n(L)) = P(\Lambda | n(L), L)$$

The independence of $n(L)$ and L ;

$$P(n(L), L) = P(n(L))\, P(L)\ ,$$

then the probability $P(\Lambda, L)$ may be expressed as

$$(1)\qquad \begin{aligned} P(\Lambda, L) &= \sum_{n(L)} P(\Lambda, L, n(L)) = \sum_{n(L)} P(\Lambda | L, n(L))\, P(L, n(L)) \\ &= \sum_{n(L)} P(\Lambda | n(L))\, P(L)\, P(n(L)) = \sum_{n(L)} P(\Lambda, n(L))\, P(L) \\ &= P(\Lambda)\, P(L)\ . \end{aligned}$$

From this result follows that mutual independence of all interdeparture intervals.

2. The independence of the interdeparture intervals for the systems $M/G/1(0)$ and $GI/M/1(0)$

In the case of $M/G/1(0)$, it is clear that

$$P\{t < L < t+dt,\ n(L)=k\} = P\{t < L < t+dt\}\, P\{n(L+0)=k\}\ ,$$

which implies the independence of L and $n(L)$, since the probabilities $P\{n(L)=k\}$ can be written as

$$P\{n(L)=k\} = \begin{cases} 1 & (\text{for } k=0) \\ 0 & (\text{for } k=1)\ . \end{cases}$$

Moreover for the system $M/G/1(0)$, the Markovian property

$$P(\Lambda | n(L)) = P(\Lambda | n(L), L)$$

exists. Therefore, we have $P(\Lambda, L) = P(\Lambda)\, P(L)$ using the formula (1). The independence of L and Λ is proved by this result.

For the case of $GI/M/1(0)$ in similar way to consider the argument for the case of $M/G/1(0)$, it can be shown that

$$P(\Lambda | n(L)) = P(\Lambda | n(L), L) \qquad P(n(L), L) = P(n(L))\, P(L)\ .$$

It is clear that the system has the property of the independence of $n(L)$ and L. So, we shall prove only the Markovian property.

In order to prove the relation

$$P(\Lambda | n(L)) = P(\Lambda | n(L), L)$$

let

L_n; Interdeparture interval between the $(n-1)$th customer and nth customer

S_n; The service time of nth customer

τ_n; nth customer's arrival time

and let

$$A_n = \tau_n - \tau_{n-1}.$$

Then, the relation

$$L_n = S_n + (A_n - S_{n-1} | A_n - S_{n-1} > 0)$$

holds.

Thus we can see the Markovian property exists for the system $GI/M/1(0)$. Therefore, the formula (1) holds in case of $GI/M/1(0)$ also, which concludes the proof.

3. The dependence of the interdeparture intervals for the system $M/M/1(1)$

In the case of $M/M/1(1)$, the Markovian property exists. But there is not independence of L and $n(L)$, and so, one may expect intuitively that the independence of the interdeparture intervals cannot be established for the case of $M/M/1(1)$.

Now, we shall try to support the statement by analysis. To do this, we set up the following differential equations.

$$\begin{aligned} &F_0'(t) = -\lambda \cdot F_0(t) \\ (2)\qquad &F_1'(t) = \lambda \cdot F_0(t) - (\lambda+\mu) F_1(t) \\ &F_2'(t) = \lambda \cdot F_1(t) - \mu \cdot F_2(t), \end{aligned}$$

and subject to the initial condition

$$F_k(0) = p_k^{(+)},$$

where $p_k^{(+)}$ denotes the equilibrium probability of the system being in the state k at the time immediately after the previous departure. It is easy to see that $p_k^{(+)}$ follows

$$p_0^{(+)} = \frac{1}{1+\rho}, \qquad p_1^{(+)} = \frac{\rho}{1+\rho}, \qquad p_k^{(+)} = 0 \quad (\text{for } k \geqq 2).$$

Equations (2) can be solved to yield

$$(3) \qquad F_0(t)=Ce^{-\lambda t}\,,\quad F_1(t)=C\rho e^{-\lambda t}\,,\quad F_2(t)=C\frac{\rho^2}{1-\rho}(e^{-\lambda t}-e^{-\mu t})\,,$$

where

$$C=\frac{1}{1+\rho}\,.$$

From the results (3) we get

$$(4) \qquad \mathrm{P}\,(L>t)=F_0(t)+F_1(t)+F_2(t)=\frac{1}{1-\rho^2}(e^{-\lambda t}-\rho^2 e^{-\mu t})\,.$$

Now, let L_1 be an interdeparture interval subsequent to the arbitrary interval of length L. Then in the case of $M/M/1(1)$, we get

$$\mathrm{P}\,(L,L_1)=\sum_{n(L)=0}^{1}\mathrm{P}\,(L,n(L),L_1)=\sum_{n(L)=0}^{1}\mathrm{P}\,(L_1|L,n(L))\,\mathrm{P}\,(L,n(L))$$
$$=\sum_{n(L)=0}^{1}\mathrm{P}\,(L_1|n(L))\,\mathrm{P}\,(L,n(L))\,.$$

Hence, if the following relation,

$$(5) \qquad \sum_{n(L)=0}^{1}\mathrm{P}\,(L_1|n(L))\,\mathrm{P}\,(L,n(L))\neq\mathrm{P}\,(L_1)\,\mathrm{P}\,(L)$$

can be shown, it is found that

$$\mathrm{P}\,(L,\Lambda)\neq\mathrm{P}\,(L)\,\mathrm{P}\,(\Lambda)\,.$$

With respect to the left side of (5), it should be expressed

$$(6) \qquad \begin{aligned} &\mathrm{P}\,(L_1>\tau\,|\,n(L)=0)=\frac{1}{\mu-\lambda}(\mu e^{-\lambda\tau}-\lambda e^{-\mu\tau}) \\ &\mathrm{P}\,(L>t,\,n(L)=0)=\int_t^\infty F_1(t)\mu dt=Ce^{-\lambda t} \\ &\mathrm{P}\,(L_1>\tau\,|\,n(L)=1)=e^{-\mu\tau} \\ &\mathrm{P}\,(L>t,\,n(L)=1)=\int_t^\infty F_2(t)\mu dt=\frac{C\rho}{1-\rho}(e^{-\lambda t}-\rho e^{-\mu t})\,. \end{aligned}$$

Then we get

$$(7) \qquad \sum_{n(L)=0}^{1}\mathrm{P}\,(L_1|n(L))\,\mathrm{P}\,(L,n(L))$$
$$=\frac{1}{\mu-\lambda}(\mu e^{-\lambda\tau}-\lambda e^{-\mu\tau})Ce^{-\lambda t}+e^{-\mu\tau}\frac{\rho}{1-\rho}C(e^{-\lambda t}-\rho e^{-\mu t})\,.$$

On the other hand, with respect to the right side of (5) we have

$$(8) \qquad P(L_1>\tau)\,P(L>t)=\left\{\frac{1}{1-\rho^2}(e^{-\lambda t}-\rho^2 e^{-\mu t})\right\}\left\{\frac{1}{1-\rho^2}(e^{-\lambda\tau}-\rho^2 e^{-\mu\tau})\right\}.$$

Using (7) and (8) we obtain (5), which implies the dependence of interdeparture intervals.

4. The dependence of the interdeparture intervals for the system $E_2/E_2/1(0)$

We had the independence of the interdeparture intervals in the case of $M/G/1(0)$ and $GI/M/1(0)$, but we have no further the property in the case of $GI/G/1(0)$.

For the statement above mentioned, we shall show the dependence of the interdeparture intervals in the system $E_2/E_2/1(0)$. First, using the model shown on Fig. 1, let's specify the states as Table 1.

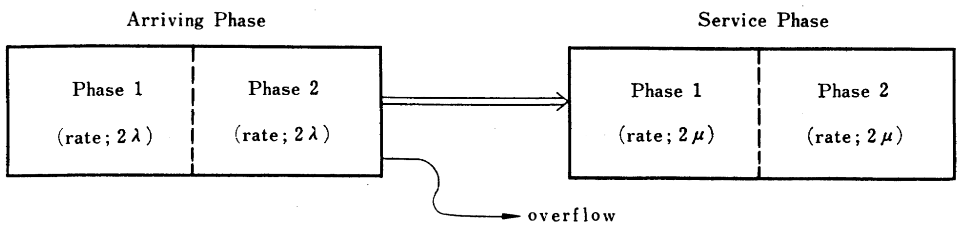

Fig. 1

Table 1

State	Arriving Phase	Service Phase
1	1	0
2	2	0
3	1	1
4	2	1
5	1	2
6	2	2

(the number 0 denotes that there is no customer in the system)

Then, we can set up the differential equations on

$$F_k(t)=P\{n(t)=k,\ L>t\} \qquad (\text{for } k=1, 2, \cdots, 6)$$

as following;

$$
(9)\quad
\begin{aligned}
&F_1'(t)=-2\lambda F_1(t)\\
&F_2'(t)=2\lambda F_1(t)-2\lambda F_2(t)\\
&F_3'(t)=2\lambda F_4(t)+2\lambda F_2(t)-(2\lambda+2\mu)F_3(t)\\
&F_4'(t)=2\lambda F_3(t)-(2\lambda+2\mu)F_4(t)\\
&F_5'(t)=2\lambda F_6(t)+2\mu F_3(t)-(2\lambda+2\mu)F_5(t)\\
&F_6'(t)=2\lambda F_5(t)+2\mu F_4(t)-(2\lambda+2\mu)F_6(t)\,.
\end{aligned}
$$

Taking the Laplace transform

$$\int_0^\infty e^{-st}F_k(t)dt=F_k^*(s)$$

and noting

$$F_k(0)=p_k^{(+)}$$

we obtain the equations

$$
(10)\quad
\begin{aligned}
&sF_1^*(s)-p_1^{(+)}=-2\lambda F_1^*(s)\\
&sF_2^*(s)-p_2^{(+)}=2\lambda F_1^*(s)-2\lambda F_2^*(s)\\
&sF_3^*(s)-p_3^{(+)}=2\lambda F_4^*(s)+2\lambda F_2^*(s)-(2\lambda+2\mu)F_3^*(s)\\
&sF_4^*(s)-p_4^{(+)}=2\lambda F_3^*(s)-(2\lambda+2\mu)F_4^*(s)\\
&sF_5^*(s)-p_5^{(+)}=2\lambda F_6^*(s)+2\mu F_3^*(s)-(2\lambda+2\mu)F_5^*(s)\\
&sF_6^*(s)-p_6^{(+)}=2\lambda F_5^*(s)+2\mu F_4^*(s)-(2\lambda+2\mu)F_6^*(s)\,.
\end{aligned}
$$

Now, let's consider the probability

$$\mathrm{P}\{t<L<t+dt,\ n(L+0)=k\}\,.$$

Then, we can see

$$
\begin{aligned}
&\mathrm{P}\{t<L<t+dt,\ n(L+0)=1\}=F_5(t)2\mu dt\\
&\mathrm{P}\{t<L<t+dt,\ n(L+0)=2\}=F_6(t)2\mu dt\\
&\mathrm{P}\{t<L<t+dt,\ n(L+0)=k\}=0 \qquad (\text{for } k=3,4,5,6)\,.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\mathrm{P}(L, L_1)=&\mathrm{P}(L_1\,|\,n(L)=1)\,\mathrm{P}(L,\ n(L)=1)\\
&+\mathrm{P}(L_1\,|\,n(L)=2)\,\mathrm{P}(L,\ n(L)=2)\,.
\end{aligned}
$$

In order to prove the dependence of the interdeparture intervals, we shall show the following;

$$\mathrm{P}(L, L_1) \neq \mathrm{P}(L)\,\mathrm{P}(L_1)\,.$$

In other words, if we can show the expression

$$\mathrm{P}(L_1 | n(L)=1)\,\mathrm{P}(L, n(L)=1)+\mathrm{P}(L_1 | n(L)=2)\,\mathrm{P}(L, n(L)=2)$$

is unequal to

$$\mathrm{P}(L)\,\mathrm{P}(L_1)\,,$$

it is sufficient to see the dependence of the interdeparture intervals. We then define

A_i=Holding time in phase i of arriving channel (i=1, 2)

S=Service time

It is assumed that A_i's are exponentially distributed with average $1/2\lambda$, and S has the Erlang distribution of phase 2 with average $1/\mu$.

Next, let $g(\tau)$ and $h(\tau)$ be the probability density functions for (A_1+A_2+S) and (A_2+S), respectively. Then, we shall try to get the following;

$$(g(\tau)d\tau)\,\mathrm{P}(L, n(L)=1)+(h(\tau)d\tau)\,\mathrm{P}(L, n(L)=2) \neq \mathrm{P}(\tau<L_1<\tau+d\tau)\,\mathrm{P}(L)\,.$$

Because the following relations

$$\mathrm{P}(\tau<L_1<\tau+d\tau | n(L)=1)=\mathrm{P}(\tau<A_1+A_2+S<\tau+d\tau)\,,$$

$$\mathrm{P}(\tau<L_1<\tau+d\tau | n(L)=2)=\mathrm{P}(\tau<A_2+S<\tau+d\tau)$$

hold.

Thus, we shall show

(11) $$(g(\tau)d\tau)(F_5(t)2\mu dt)+(h(\tau)d\tau)(F_6(t)2\mu dt) \neq \mathrm{P}(\tau<L_1<\tau+d\tau)\,\mathrm{P}(t<L<t+dt)\,.$$

Taking the Laplace transform of $g(\tau)$ and $h(\tau)$,

$$\int_0^\infty g(\tau)e^{-s_1\tau}d\tau=g^*(s_1)\,, \qquad \int_0^\infty h(\tau)e^{-s_1\tau}d\tau=h^*(s_1)\,,$$

let's prove the following relation to get the expression (11)

(12) $$2\mu g^*(s_1)F_5^*(s)+2\mu h^*(s_1)F_6^*(s) \neq \left\{\left(\frac{2\lambda}{2\lambda+s_1}\right)^2\left(\frac{2\mu}{2\mu+s_1}\right)^2 p_1^{(+)}+\left(\frac{2\lambda}{2\lambda+s_1}\right)\left(\frac{2\mu}{2\mu+s_1}\right)^2 p_2^{(+)}\right\}F^*(s)$$

where

$$F^*(s)=\sum_{k=1}^{6} F_k^*(s) .$$

In regard to $p_1^{(+)}$, $p_2^{(+)}$ and $F^*(s)$ in (12), we get the following results after some simple calculations.

$$p_1^{(+)}=\frac{1+2\rho+2\rho^2}{(1+2\rho)^2} \qquad p_2^{(+)}=\frac{2\rho+2\rho^2}{(1+\rho)^2}$$

$$F^*(s)=\left(\frac{2\lambda}{2\lambda+s}\right)^2\left(\frac{2\mu}{2\mu+s}\right)^2 p_1^{(+)}+\left(\frac{2\lambda}{2\lambda+s}\right)\left(\frac{2\mu}{2\mu+s}\right)^2 p_2^{(+)} .$$

In order to find that the relation (12) is true, noting

$$g^*(s_1)=\left(\frac{2\lambda}{2\lambda+s_1}\right)^2\left(\frac{2\mu}{2\mu+s_1}\right)^2 \qquad h^*(s_1)=\left(\frac{2\lambda}{2\lambda+s_1}\right)\left(\frac{2\mu}{2\mu+s_1}\right)^2 ,$$

one may establish the fact that the following

$$(13) \qquad \left(\frac{2\lambda}{2\lambda+s_1}\right)F_5^*(s)2\mu+F_6^*(s)2\mu=\left(\frac{2\lambda}{2\lambda+s_1}\right)F^*(s)p_1^{(+)}+F^*(s)p_2^{(+)}$$

does not exist.

So, one may examine whether (13) cannot be satisfied, for example, in the case of $\lambda=1/2$, $\mu=1$.

In this case, we get

$$2\mu F_5^*(0)=\frac{101}{168} \qquad 2\mu F_6^*(0)=\frac{67}{168}$$

$$p_1^{(+)}=\frac{5}{8} \qquad p_2^{(+)}=\frac{3}{8} .$$

Using the results, we can conclude that the relation (13) is not given.

Consequently, we have

$$\mathrm{P}(L, L_1)\neq\mathrm{P}(L)\,\mathrm{P}(L_1) .$$

This implies that the interdeparture intervals for $E_2/E_2/1(0)$ are dependent.

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References

[1] Burke, P. J. (1956). The output of a queueing system, *Operat. Res.*, 4, 699-704.
[2] Makino, T. (1964). On the mean passage time concerning some queueing problems of the tandem type, *J. Operat. Res. Soc. Japan*, 7, 17-47.
[3] Makino, T. (1966). On a study of output distribution, *J. Operat. Res. Soc. Japan*, 8, 109-133.

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CORRECTIONS TO
" ON THE INDEPENDENCE OF INTERDEPARTURE INTERVALS FROM SINGLE SERVER QUEUEING SYSTEMS "

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In the above titled paper (this Annals Vol. 29, No. 2, A, (1977), pp. 307–315), we claimed that the interdeparture interval for GI/M/1(0) is independent. However, the assertion is incorrect and the following amendments should be made accordingly:

(i) On page 307, lines 8–9 of Section 1, " M/G/1(0) and GI/M/1(0) are " should be " M/G/1(0) is ".
(ii) On page 308, in the title of Section 2 and on page 311, line 3 of Section 4, " and GI/M/1(0) " should be removed.
(iii) On page 309, lines 9–14, the two statements, " Then, the relation . . . " and " Thus we can . . . " should be removed.

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