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ON METHODS FOR GENERATING UNIFORM RANDOM POINTS ON THE SURFACE OF A SPHERE

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Some methods for generating random points uniformly distributed on the surface of an n-sphere have been proposed to simulate spherical processes on computer. A standard method is to normalize random points inside of the sphere, see M. Muller [5]. Improved methods were given by J. M. Cook [1] and G. Marsaglia [4] in three and four dimensions, and computational methods in higher dimensions by J. S. Hicks and R. F. Wheeling [3] and M. Sibuya [6].

In this paper we shall offer direct methods for generating uniform random points on the surface of a unit n-sphere, which can be easily combined with Marsaglia's idea for getting more improved methods. Our method in even dimensions was obtained by M. Sibuya [6], but a differential-geometric view-point will make analyses simpler, even in odd dimensions.

1. Methods

Let I be the unit interval $[0, 1]$ and $y_1, y_2, \cdots, y_n$ Cartesian coordinates in an n-dimensional Euclidean space. We shall denote by X's and Θ's independent random variables in I.

Method in case $n=2p$. Put $Y_p=1$, $Y_0=0$, and define variables Y_1, $Y_2, \cdots, Y_{p-1}$ by the recursive formulas

$$Y_i=Y_{i+1}X_i^{1/i} \qquad (i=p-1, p-2, \cdots, 1)\,, \tag{1.1}$$

and form

$$y_{2i-1}=\sqrt{Y_i-Y_{i-1}}\cos 2\pi\Theta_i\,, \qquad y_{2i}=\sqrt{Y_i-Y_{i-1}}\sin 2\pi\Theta_i \qquad (i=1, 2, \cdots, p)\,. \tag{1.2}$$

as coordinates of a random point on the surface of the unit n-sphere.

The latter half of this method is due to M. Sibuya [6].

Method in case $n=2p+1$. Put $Z_{p+1}=1$ and define variables $Z_1, Z_2, \cdots, Z_p$ by the recursive formulas

$$Z_i=Z_{i+1}X_i^{2/(2i-1)} \qquad (i=p, p-1, \cdots, 1), \tag{1.3}$$

and form

$$\begin{aligned} &y_1=\pm\sqrt{Z_1}, \\ &y_{2i}=\sqrt{Z_{i+1}-Z_i}\cos 2\pi\Theta_i, \qquad y_{2i+1}=\sqrt{Z_{i+1}-Z_i}\sin 2\pi\Theta_i, \\ &\qquad\qquad\qquad\qquad\qquad\qquad (i=1, 2, \cdots, p) \end{aligned} \tag{1.4}$$

as coordinates of a random point on the surface of the unit n-sphere, where $\pm$ is a random sign.

G. Marsaglia [4] used the property that, if (U, V) is uniform in the interior of the unit circle, then $S=U^2+V^2$ is uniform in $I=[0, 1]$ and independent of the point

$$(U/\sqrt{S}, V/\sqrt{S})=(\cos 2\pi\Theta, \sin 2\pi\Theta).$$

Our methods are easily combined with his idea in even dimensions and in low odd dimensions for reducing the number of used random variables.

2. Ordered sets of random variables

For the later use, we first consider a method for generating ordered sets of n random variables $Y_1, Y_2, \cdots, Y_n$ in I such that

$$0\leqq Y_1\leqq Y_2\leqq\cdots\leqq Y_{n-1}\leqq Y_n\leqq 1.$$

If we regard $Y_1, Y_2, \cdots, Y_n$ as coordinates in an n-dimensional Euclidean space, then $(Y_1, Y_2, \cdots, Y_n)$ represents a random point in an $(n+1)$-faced polyhedron Δ. In order to find n independent random variables $X_1, X_2, \cdots, X_n$ representing random points in Δ, it is sufficient to obtain the transformation from the n-dimensional unit cube I^n to the polyhedron Δ which preserves the density to within a constant factor. The volume element of Δ is given by

$$dY_1\, dY_2\cdots dY_n.$$

Now we put $Y_{n+1}=1$ and

$$X_i=(Y_i/Y_{i+1})^i \qquad (i=1, 2, \cdots, n). \tag{2.1}$$

Then, for each i, the variable X_i depends on Y_i and Y_{i+1} only, and varies over the interval I as Y_i varies from 0 to Y_{i+1}. As is easily seen, the Jacobian of the transformation (2.1) is equal to $n!$ and we have the relation

$$dY_1 dY_2 \cdots dY_n = (1/n!) dX_1 dX_2 \cdots dX_n \, .$$

Seeking for the inverse transformation to (2.1), we obtain the following

Method for ordered sets of random variables. Let $X_1, X_2, \cdots, X_n$ be n independent random variables in I. Put $Y_{n+1}=1$, and define $Y_1, Y_2, \cdots, Y_n$ by the recursive formulas

$$Y_i = Y_{i+1} X_i^{1/i} \qquad (i=n, n-1, \cdots, 1) \, . \tag{2.2}$$

Then $Y_1, Y_2, \cdots, Y_n$ form an ordered set of random variables in I, that is, they give a random partition of I into $n+1$ subintervals.

3. Analysis

Let P be the part of the surface of the unit n-sphere defined by

$$y_1^2 + y_2^2 + \cdots + y_n^2 = 1 \, , \qquad y_i \geqq 0 \, . \tag{3.1}$$

For generating random points on the surface of the unit n-sphere, we suffice to find a transformation of the unit $(n-1)$-cube I^{n-1} onto the part P preserving the volume element to within a constant factor, then attach random signs to the coordinates y_i.

If we put $x_0=0$ and

$$x_i = y_1^2 + y_2^2 + \cdots + y_i^2 \qquad (i=1, 2, \cdots, n) \, , \tag{3.2}$$

then the variables x_i form a sequence in I such that

$$0 = x_0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_{n-1} \leqq x_n = 1 \, . \tag{3.3}$$

Conversely, given a sequence (3.3) in I, the point with coordinates

$$y_i = \sqrt{x_i - x_{i-1}} \qquad (i=1, 2, \cdots, n) \tag{3.4}$$

is on P, that is, the equations (3.4) give a representation of P by parameters $x_1, x_2, \cdots, x_{n-1}$ varying in I with the restrictions (3.3). For each i, y_i depends on x_{i-1} and x_i only, and has non-trivial derivatives

$$\frac{\partial y_i}{\partial x_{i-1}} = \frac{-1}{2\sqrt{x_i - x_{i-1}}} \, , \qquad \frac{\partial y_i}{\partial x_i} = \frac{1}{2\sqrt{x_i - x_{i-1}}} \, .$$

Then the induced Riemannian metric tensor of P is defined by components

$$g_{ij}=\sum_{k=1}^{n}\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j}\qquad (i, j=1, 2,\cdots, n-1)\,,$$

see [2], and the non-trivial components are only

$$g_{ii}=\frac{1}{4}\left(\frac{1}{x_i-x_{i-1}}+\frac{1}{x_{i+1}-x_i}\right),\qquad g_{i,i+1}=g_{i+1,i}=\frac{-1}{4(x_{i+1}-x_i)}\,,\qquad (i=1, 2,\cdots, n-1)\,.$$

We can easily verify the following formula on a determinant of order $n-1$:

$$\begin{vmatrix} a_1+a_2 & -a_2 & 0 & \cdots & 0 & 0 \\ -a_2 & a_2+a_3 & -a_3 & \cdots & 0 & 0 \\ 0 & -a_3 & a_3+a_4 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-2}+a_{n-1} & -a_{n-1} \\ 0 & 0 & 0 & \cdots & -a_{n-1} & a_{n-1}+a_n \end{vmatrix}$$
$$=a_1 a_2\cdots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}+\frac{1}{a_n}\right).$$

Applying this formula to the determinant G of the components g_{ij} of the metric tensor of P, we obtain the expression

$$G=[4^{n-1}x_1(x_2-x_1)(x_3-x_2)\cdots(x_{n-1}-x_{n-2})(1-x_{n-1})]^{-1} \tag{3.5}$$

and the volume element dV of P equal to

$$\begin{aligned} dV&=\sqrt{G}\,dx_1dx_2\cdots dx_{n-1} \\ &=[2^{n-1}x_1(x_2-x_1)(x_3-x_2)\cdots(x_{n-1}-x_{n-2})(1-x_{n-1})]^{-1/2} \\ &\quad\times dx_1dx_2\cdots dx_{n-1}\,. \end{aligned} \tag{3.6}$$

First we consider the case of even dimension $n=2p$. Pairing the intervals $[x_{2i-2}, x_{2i-1}]$ and $[x_{2i-1}, x_{2i}]$, $i=1, 2,\cdots, p$, we put

$$U_{2i-1}=\frac{2}{\pi}\operatorname{Arc\,tan}[(x_{2i-1}-x_{2i-2})/(x_{2i}-x_{2i-1})]^{1/2}\,,\qquad U_{2i}=x_{2i}\,,\qquad (i=1, 2,\cdots, p)\,. \tag{3.7}$$

Then, for each i, the variable U_{2i} is identical with x_{2i}. The variable U_{2i-1} is dependent of x_{2i-2}, x_{2i-1} and x_{2i} only, varies over I as x_{2i-1} varies from x_{2i-2} to x_{2i}, and has derivative

$$\frac{\partial U_{2i-1}}{\partial x_{2i-1}}=\frac{1}{\pi}[(x_{2i-1}-x_{2i-2})(x_{2i}-x_{2i-1})]^{-1/2}\,.$$

Hence the Jacobian of the transformation (3.7) is equal to

$$\frac{1}{\pi^p}\left[\prod_{i=1}^{n}(x_i-x_{i-1})\right]^{-1/2}$$

and the volume element dV of the part P is equal to

$$dV=(\pi^p/2^{n-1})dU_1dU_2\cdots dU_{n-1}\ .$$

Renaming Θ_i for U_{2i-1} and Y_i for U_{2i} in (3.7) for simplicity, and solving the equations in x_i, we obtain

$$\text{(3.8)}\qquad x_{2i-1}=Y_{i-1}\cos\frac{\pi}{2}\Theta_i+Y_i\sin\frac{\pi}{2}\Theta_i\ ,\qquad x_{2i}=Y_i\ .$$

Substituting (3.8) into (3.4) and replacing $\pi\Theta_i/2$ by $2\pi\Theta_i$ for attachment of random signs, we obtain the expressions (1.2). Since $Y_1, Y_2, \cdots, Y_{p-1}$ form an ordered set of random variables in I, they can be obtained by the method stated in Section 2. The volume element dV of P is expressed as

$$dV=(\pi^p/2^{n-1})dY_1dY_2\cdots dY_{p-1}d\Theta_1d\Theta_2\cdots d\Theta_p\ .$$

Next we consider the case of odd dimension $n=2p+1$. The expression (3.6) contains $2p+1$ factors of the form x_i-x_{i-1}. Applying the same method as that in the case $n=2p$ to the last p intervals $[x_{2i-1}, x_{2i+1}]$ $(i=1, 2, \cdots, p)$ and renaming Z_1 for x_1 and Z_i for Y_{i-1} $(i\geqq 2)$, we have the expression

$$dV=\frac{\pi^p}{2^{n-1}\sqrt{Z_1}}dZ_1dZ_2\cdots dZ_pd\Theta_1\cdots d\Theta_p\ ,$$

where $Z_1, Z_2, \cdots, Z_p$ are random variables such that

$$0\leqq Z_1\leqq Z_2\leqq\cdots\leqq Z_{p-1}\leqq Z_p\leqq 1\ .$$

Now we put $Z_{p+1}=1$ and

$$\text{(3.10)}\qquad X_i=(Z_i/Z_{i+1})^{(2i-1)/2}\qquad (i=1, 2, \cdots, p)\ .$$

Then, for each i, the variable X_i depends on Z_i and Z_{i+1} only, and varies over I as Z_i varies from 0 to Z_{i+1}. We can see that the Jacobian of the transformation (3.10) is equal to $(2p-1)!!/2^p\sqrt{Z_1}$ and the volume element dV of P is equal to

$$dV=\frac{\pi^p}{2^{p-1}(2p-1)!!}dX_1dX_2\cdots dX_pd\Theta_1\cdots d\Theta_p\ ,$$

where we have put

$$(2p-1)!!=(2p-1)(2p-3)\cdots 3\cdot 1\ .$$

Seeking for inverse transformation of (3.10), we obtain the recursive formulas (1.3) in which $X_1, X_2, \cdots, X_p$ are independent random variables in I.

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REFERENCES

[1] Cook, J. M. (1957). Rational formulae for the production of a spherically symmetric probability distribution, *Math. Comp.*, **11**, 81-82.

[2] Eisenhart, L. P. (1966). *Riemannian Geometry*, sixth print, Princeton Univ. Press.

[3] Hicks, J. S. and Wheeling, R. F. (1959). An efficient method for generating uniformly distributed points on the surface of an n-dimensional sphere, *Commun. Ass. Comput. Math.*, **2**, 17-19.

[4] Marsaglia, G. (1972). Choosing a point from the surface of a sphere, *Ann. Math. Statist.*, **43**, 645-646.

[5] Muller, M. E. (1959). A note on a method for generating points uniformly on N-dimensional spheres, *Commun. Ass. Comput. Math.* **2**, 19-20.

[6] Sibuya, M. (1964). A method for generating uniformly distributed points on N-dimensional spheres, *Ann. Inst. Statist. Math.*, **14**, 81-85.