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DECOMPOSITION OF INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS WITHOUT GAUSSIAN COMPONENT

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Summary

The problem of characterizing the infinitely divisible characteristic functions which have only infinitely divisible factors is considered. Under the assumption that both the absolutely continuous and the singular (or the discrete) components exist in Poisson spectral measures, several necessary conditions for this problem are obtained. These conditions admit partial converses and new examples of infinitely divisible characteristic functions which have only infinitely divisible factors are given.

1. Introduction

Let a characteristic function (abbrev. c.f.) $f(t)$ be the product of two non-degenerate c.f.'s $f(t)=g(t)h(t)$. Here we say that $f(t)$ is decomposable and that $g(t)$ and $h(t)$ are factors of $f(t)$. The class of c.f.'s which admit only infinitely divisible factors is denoted by I_0. It is well known that any $f(t) \in I_0$ is itself infinitely divisible. Therefore $f(t)$ has the Lévy canonical representation

$$(1.1)\qquad f(t)=\exp\left[i\gamma t-\sigma^2t^2+\int_{-\infty}^{-0} K(t, x)dM(x)+\int_{+0}^{+\infty} K(t, x)dN(x)\right],$$

where the kernel function $K(t, x)=e^{itx}-1-itx/(1+x^2)$.

The main theme of the so-called decomposition problem of c.f.'s is the characterization of the class I_0. In the case $\sigma\neq 0$ in (1.1), Linnik gave a complete description of possible forms of Lévy spectral measures $dM(x)$ and $dN(x)$ in (1.1), see Lukacs ([4], Th. 9.3.1, p. 280). On the other hand, in the case $\sigma=0$, Cramér obtained a necessary condition for membership of I_0, see Lukacs ([4], Th. 6.2.3, p. 176). Shimizu, see Lukacs ([4], Th. 6.2.3, p. 179), and Cuppens [1] extended Cramér's result. And the present author [5] obtained the following result, elaborating Cuppens' result and method.

THEOREM 1.1. *Let $f(t) \in I_0$. If $\sigma=0$ and if Lévy spectral measures*

$dM(x)$ and $dN(x)$ in (1.1) *are both absolutely continuous, then there is a positive constant δ such that either $dM(x)\equiv 0$ and* $\mathrm{Supp}(dN)\subset[\delta, 2\delta]$, *or $dN(x)\equiv 0$ and* $\mathrm{Supp}(dM)\subset[-2\delta, -\delta]$.

These conditions are also sufficient for membership of I_0 as the following theorem shows, see Lukacs ([4], Th. 9.2.4, p. 287).

THEOREM (Ostrovskii). *Let $f(t)$ be an infinitely divisible c.f. defined by* (1.1) *with $\sigma=0$. Then we have;*

(1) *If $dM\equiv 0$ and* $\mathrm{Supp}(dN)\subset[\delta, 2\delta]$ *with a positive constant δ, then $f(t)\in I_0$.*

(2) *If $dM\equiv 0$ and* $\mathrm{Supp}(dN)$ *constitutes a bounded set with rationally independent points, then $f(t)\in I_0$.*

Remark (See Gelfand et al. ([2], Chap. 5, Th. 1, p. 188)). A set is called rationally independent if it is linearly independent over the field of rational numbers. Therefore, in the case (2) above, dN can be singular. We shall use this fact in Section 5.

In this paper we shall extend Theorem 1.1 to cases when the discrete or the singular component is present other than the absolutely continuous one in the Lévy spectral measures in (1.1). Also we shall show that thus obtained necessary conditions for membership of I_0 have partial converses by giving several new type c.f.'s of I_0 which are closely related to those of Ostrovskii cited above and the one mentioned in the book of Linnik and Ostrovskii ([3], Th. 6.4.2, p. 257).

2. Preliminaries

We shall list here some notations and notions used throughout the paper. Let dL be a signed measure. dL_{ac}, dL_s and dL_d mean the absolutely continuous, the singular and the discrete component of dL respectively, and let the continuous component $dL_c=dL_{ac}+dL_s$. The right and left extremity of dL are defined as

$$\mathrm{rext}(dL)=\sup \mathrm{Supp}(dL)\,, \qquad \mathrm{lext}(dL)=\sup \mathrm{Supp}(dL)\,.$$

A point γ is called a point of left increase of dL if the restriction of dL to the interval $(\gamma-\varepsilon, \gamma)$ does not vanish for any positive ε. Points of right increase of dL are defined analogously.

The nth iterated convolution of dL is denoted by dL^{*n} and is defined recurently as

$$dL^{*0}=d\varepsilon \qquad \text{(the unit measure at the origin)}\,,$$

$$dL^{*n}(x)=\int dL^{*(n-1)}(x-y)dL(y) \qquad (n\geqq 1)\,.$$

If dL is written as dL_1, we shall denote dL^{*n} by dL_n. Also if $dL_1 = \xi_1 dx$ we denote dL_n by $\xi_n dx$. To simplify notations in proofs, we shall use symbols (though they are abusive) such as

$$\xi_j * L_k(x) = \int \xi_j(x-y) dL_k(y) , \qquad \xi_0 * L_j(x)dx = dL_j(x) , \quad \text{etc.}$$

There will be no misunderstandings. Finally let constants $C(i, j, \cdots; n) = n!/(i!j!\cdots)$.

Next we shall prove auxiliary lemmas which are basic for the rest of the paper.

LEMMA 1. *Let $\xi(x)$ be a bounded Borel measurable function and dL be a continuous measure of bounded variation. Then $\xi * L(x)$ is a continuous function.*

PROOF. This is a slight generalization of a well known theorem due to Lebesgue and can be proved similarly, so we omit the proof.

LEMMA 2. *Let dL_1 be a positive continuous measure and $\xi_1(x)dx$ be a positive absolutely continuous measure both of which are of bounded variation and have compact supports. Put $dP_1 \equiv dL_1 + \xi_1 dx$ and let*

$$\alpha \equiv \operatorname{lext}(dL_1) < \beta \equiv \operatorname{lext}(\xi_1 dx) < \gamma \equiv \operatorname{rext}(\xi_1 dx) .$$

Fix any small positive θ, then, for all large n, $dP_n - dL_n$ has a continuous density function which is positive at least on the interval $(n\alpha + \beta - \alpha + \theta, n\gamma - \theta)$.

PROOF. $x = \alpha + \beta$ is the left extremity of the measure $\xi_1 * L_1 dx$ which has a continuous density function by Lemma 1. So we can choose two points ε and ε' such that $\alpha + \beta < \varepsilon < \varepsilon'$ (respectively two points δ and δ' such that $\delta < \delta' < 2\gamma$) arbitrarily near to $\alpha + \beta$ (resp. 2γ) so that $\xi_1 * L_1(x) > 0$ (resp. $\xi_2(x) > 0$) on the interval $[\varepsilon, \varepsilon']$ (resp. $[\delta, \delta']$). Also fix a point $2\alpha'$ of left increase of dL_2, a point $\alpha + \beta'$ of left increase of $\xi_1 * L_1 dx$ and a point $2\gamma'$ of right increase of $\xi_2 dx$ such that

$$2\alpha' - 2\alpha,\ (\alpha + \beta') - (\alpha + \beta),\ 2\gamma - 2\gamma' < \operatorname{Min}(\varepsilon' - \varepsilon, \delta' - \delta) .$$

Let dG_1 be the restriction of dL_2 to $[2\alpha, 2\alpha']$, $\psi_1(x)$ be the restriction of $\xi_1 * L_1(x)$ to $[\alpha+\beta, \alpha+\beta']$, $\eta_1(x)$ be the restriction of $\xi_1 * L_1(x)$ to $[\varepsilon, \varepsilon']$, $\nu_1(x)$ be the restriction of $\xi_2(x)$ to $[\delta, \delta']$ and $\phi_1(x)$ be the restriction of $\xi_2(x)$ to $[2\gamma', 2\gamma]$. Put $dH_1 \equiv dG_1 + (\psi_1 + \eta_1 + \nu_1 + \phi_1)dx$. Since $dP_2 \geqq dH_1$, we have

$$dP_{2n} \geqq dH_n = dG_n + \sum_{i+j+k+l+m=n}^{i<n} C(i, j, k, l, m; n) G_i * \psi_j * \eta_k * \nu_l * \phi_m dx .$$

Each of summands in the second term with $k+l\geqq 1$ has a continuous density function by Lemma 1 which is positive at least on the interval

$$I_n(i, j, k, l, m)\equiv(2i\alpha+j(\alpha+\beta)+k\varepsilon+l\delta+2m\gamma', \\ 2i\alpha'+j(\alpha+\beta')+k\varepsilon'+l\delta'+2m\gamma)\ .$$

Therefore the density function of $dP_{2n}-dG_n$ is positive at least on the set

$$\begin{aligned} I_n &\equiv \bigcup_{i+j+k+l+m=n}^{k+l>0} I_n(i, j, k, l, m) \\ &= \bigcup_{i+l+k\leq n}^{k+l>0} \bigcup_{j=0}^{n-i-l-k} I_n(i, j, k, l, n-i-j-k-l) \\ &\equiv \bigcup_{i+l+k\leq n}^{k+l>0} I_n^{(1)}(i, k, l)\ . \end{aligned}$$

Calculating distances between adjoining two intervals comprised in $I_n^{(1)}(i, k, l)$, we can see easily that $I_n^{(1)}(i, k, l)$ is an interval for all large n.

Next define the sets $I_n^{(2)}(k, l)$ and $I_n^{(3)}(k)$ as

$$\begin{aligned} I_n &= \bigcup_{i+l+k\leq n}^{l+k>0} I_n^{(1)}(i, k, l) \\ &= \bigcup_{1\leq l+k\leq n} \bigcup_{i=0}^{n-k-l} I_n^{(1)}(i, l, k) \\ &\equiv \bigcup_{1\leq l+k\leq n} I_n^{(2)}(l, k) \\ &= \left[\bigcup_{l=1}^{n} I_n^{(2)}(l, 0)\right] \cup \left[\bigcup_{k=1}^{n} \bigcup_{l=0}^{n-k} I_n^{(2)}(l, k)\right] \\ &\equiv \bigcup_{k=0}^{n} I_n^{(3)}(k)\ . \end{aligned}$$

Then it is easy to show that $I_n^{(2)}(l, k)$, $I_n^{(3)}(k)$ and, finally, I_n itself are intervals for all large n and, consequently, that $I_n=(2(n-1)\alpha+\varepsilon, 2(n-1)\gamma+\delta')$. This proves the lemma for all large even n and odd cases follow at once.

Proofs of theorems in Sections 3 and 4 are based on the following lemma.

LEMMA 3. *Let $f(t)$ be an infinitely divisible c.f. defined by* (1.1). *Suppose that there exists a signed measure dL_1 with the non-null negative variation satisfying the properties*;

(i) *dL_1 is of bounded variation with a compact support,*

(ii) *$dM-dL_1\geqq 0$ on $(-\infty, 0)$ and $dN-dL_1\geqq 0$ on $(0, +\infty)$,*

(iii) $\sum_{n=1}^{\infty}\frac{1}{n!}dL_n\geqq 0$.

Then $f(t)$ does not belong to I_0.

PROOF. An analogous result is proved in Lukacs ([4], Th. 6.2.3, p. 177). Since the method used there is applicable with slight modifications, the proof is omitted.

3. Necessary conditions for membership of the class I_0

In the following, we shall be interested in those c.f.'s defined by (1.1) with absolutely continuous components in their Lévy spectral measures. Let $[\alpha, \beta]$ be the smallest interval containing the support of dN_{ac}. If we take note of Theorem 1.1, it is sufficient to consider the case $0<\alpha<\beta\leqq 2\alpha$. The crucial point in proofs of theorems in Sections 3 and 4 is to construct signed measures which satisfy the assumptions of Lemma 3. Since other properties will be obvious, it is sufficient to check only (iii).

THEOREM 3.1. *Let $f(t)$ be an element of I_0. If $dN_{ac}\neq 0$, then $dM_s=0$.*

PROOF. Assuming that $dM_s\neq 0$, we shall construct a signed measure having properties of Lemma 3. Let $[\alpha, \beta]$ be the smallest interval containing the support of dN_{ac}. We may assume that $0<\alpha<\beta\leqq 2\alpha$. Fix a point $-\gamma$ of right increase of dM_s and any point α', β' such that $\alpha<\alpha'<\beta'<\beta$. Let dF_1 be the restriction of dM_s to $[-\gamma, 0)$ and $\xi_1(x)$ be the restriction of the density function of dN_{ac} to $[\alpha, \alpha')\cup(\beta', \beta]$. By taking factors of $f(t)$ if necessary, we may suppose without loss of generality that dF_1 is of bounded variation and that ξ_1 is bounded.

For a fixed small number ε, define $dL_1=dL_1(\varepsilon)\equiv dF_1+(\xi_1-\varepsilon\eta_1)dx$, where $\eta_1(x)$ is the indicator function of the interval $[\alpha', \beta']$. Easy calculations show that

$$dL_n=\sum_{i+j+k=n}(-\varepsilon)^k C(i, j, k; n)\xi_i * \eta_k * F_j dx\equiv dL_n^{(+)}-dL_n^{(-)},$$

where $dL_n^{(+)}$ is the sum taken over even k's and $dL_n^{(-)}$ is the sum taken over odd k's. The positive measure $dL_n^{(+)}(\varepsilon)$ majorizes $dL_n^{(+)}(0)$ and, according to Lemma 2, $dL_n^{(+)}-dF_n$ has a continuous function which is positive on $(\alpha+\theta-(n-1)\gamma, n\beta-\theta)$ for all large n (say, from n_0 on), where θ is a fixed small positive constant$<\mathrm{Min}(\alpha'-\alpha, \beta-\beta')$. And the density function of $dL_n^{(-)}$ vanishes outside $(\alpha'-(n-1)\gamma, (n-1)\beta+\beta')$ and can be made uniformly small as ε goes to 0. Then $dL_n\geqq 0$ simultaneously for $n=n_0, n_0+1, \cdots, 2n_0-1$ for all sufficiently small values of $\varepsilon>0$. Hence also $dL_n\geqq 0$ for all $n\geqq n_0$. For example, $dL_{2n_0}=dL_{n_0}*dL_{n_0}\geqq 0$. Moreover, on comparing supports, we can easily see that, for all sufficiently small ε,

$$\sum_{n=1}^{n_0}\frac{1}{n!}dL_n\geqq\frac{1}{n_0!}dL_{n_0}^{(+)}-\sum_{n=1}^{n_0}\frac{1}{n!}dL_n^{(-)}\geqq 0 .$$

Consequently, for all small values of $\varepsilon>0$,

$$\sum_{n=1}^{\infty}\frac{1}{n!}dL_n\geqq 0\ .$$

This completes the proof.

THEOREM 3.2. *Let $f(t)$ be an element of I_0. Assume that $dN_{ac}\neq 0$. If $[\alpha,\beta]$ is the smallest interval containing the support of dN_{ac}, then $dN_s=0$ on the interval $(0,\beta-\alpha)$.*

PROOF. We can assume that $0<\alpha<\beta\leqq 2\alpha$. Fix a point γ of right increase of dN_s and any point γ' such that $0<\gamma,\ \gamma'<\beta-\alpha$. Let dF_1 be the restriction of dN_s to $[\gamma,\beta-\alpha]$, $\xi_1(x)$ be the restriction of the density function of dN_{ac} to $[\alpha,\alpha+\gamma]\cup[\alpha+\gamma',\beta)$ and $\eta_1(x)$ be the restriction of $\xi_1*F_1(x)$ to $[\alpha+\gamma,\alpha+\gamma']$. As before, we may suppose that dF_1 is of bounded variation and that $\xi_1(x)$ is bounded.

Set $dL_1\equiv dF_1+(\xi_1-\varepsilon\eta_1)dx$. Then

$$dL_n=\sum_{i+j+k=n}(-\varepsilon)^k C(i,j,k;n)\xi_i*\eta_k*F_j dx\equiv dL_n^{(+)}-dL_n^{(-)}$$

where $dL_n^{(+)}$, $dL_n^{(-)}$ have the same meanings as in the proof of Theorem 3.1. Proceeding as in the preceding proof, we can show that $dL_n\geqq 0$ for all n from some n_0 on if ε is sufficiently small. On the other hand, direct calculations show that

$$\begin{aligned}dH_n&\equiv\frac{1}{2}\frac{1}{(n+1)!}dL_n^{(+)}-\frac{1}{n!}dL_n^{(-)}\\&\geqq\sum_{i+j+2k+1=n}\varepsilon^k\Big[\frac{1}{2}C(i+1,j+1,2k;0)-\varepsilon C(i,j,2k+1;0)\Big]\\&\quad\cdot\xi_i*\eta_{2k+1}*F_j dx\ .\end{aligned}$$

From this inequality, for sufficiently small ε, $dH_n\geqq 0$ for $n=1,2,\cdots,n_0$. Therefore

$$\begin{aligned}\sum_{n=1}^{n_0}\frac{1}{n!}dL_n&\geqq\frac{1}{2}\frac{1}{n_0!}dL_{n_0}^{(+)}-\frac{1}{n_0!}dL_{n_0}^{(-)}+\sum_{n=1}^{n_0-1}\frac{1}{n!}dH_n\\&\geqq\frac{1}{2}\frac{1}{n_0!}dL_{n_0}^{(+)}-\frac{1}{n_0!}dL_{n_0}^{(-)}\ .\end{aligned}$$

Since the right-hand side of this inequality also can be made nonnegative for sufficiently small ε, we have shown that, for sufficiently small positive ε,

$$\sum_{n=1}^{\infty}\frac{1}{n!}dL_n\geqq 0\ .$$

This proves the theorem.

As we have just seen, if $f(t)$ belongs to I_0, the existence of dN_{ac} prohibits the existence of singular spectral measures on the interval $(-\infty, \beta-\alpha]$. But as to the interval $[\beta-\alpha, +\infty)$ there are various possibilities according to the nature and the location of singular measures. The method used in proofs of above theorems still works in this case so long as iterated convolutions of dN_s become to have absolutely continuous components. But conditions are far from simple, so we shall not state them here. It seems that the circumstance is essentially complicated and that the class I_0 cannot be characterized with several simple theorems in this case. In Section 4 we shall give several sufficient conditions for membership of the class I_0 which constitute partial converses of results omitted here.

4. Complement to Section 3

The method used in proofs of theorems in Section 3 still works if we replace dN_s by dN_d. But in this case an analogous result of Lemma 2 does not hold. Therefore, from the very beginning, we must impose some restriction on the density function of dN_{ac}. In this way we can get some further necessary conditions. We shall give only one of such conditions. It is related the results of Shimizu [6]. Before stating the theorem we recall that a point γ is called a point of condensation of a set A if the ratio of Lebesgue measures of the sets $A\cap[\gamma-\varepsilon, \gamma+\varepsilon]$ and $[\gamma-\varepsilon, \gamma+\varepsilon]$ tends to 1 as $\varepsilon\to 0$. A theorem due to Lebesgue shows that a.e. points of a Lebesgue measurable set A are points of condensation of A.

THEOREM 4.1. *If there are three numbers γ, δ, ν such that points γ, $\gamma+2\delta$ and $\gamma+\delta-\nu$ are different points of condensation of the support of dN_{ac} and if points ν, $\delta+\nu$ are different jump points of Lévy spectral measure, then $f(t)$ does not belong to I_0.*

PROOF. We can suppose $\gamma<\gamma+\delta-\nu<\gamma+2\delta$. Fix a sufficiently small θ and ρ such that θ is smaller than saltus of spectral measures at the jump points ν, $\delta+\nu$ and that the sets $[\gamma-\rho, \gamma+\rho]$, $[(\gamma+\delta-\nu)-\rho, (\gamma+\delta-\nu)+\rho]$ and $[(\gamma+2\delta)-\rho, (\gamma+2\delta)+\rho]$ are mutually disjoint. Let $[\alpha, \beta]$ be the smallest interval containing the support of dN_{ac}. We may suppose $0<\alpha<\beta<2\alpha$. Let $\xi(x)$ be the density function of dN_{ac}, $\xi'(x)$ be the restriction of $\xi(x)$ to $[\alpha, \alpha+\rho]\cup[\beta-\rho, \beta]$, $\eta(x)$ be the function $\mathrm{Min}\,[\xi(x+\gamma), \xi(x+\gamma+2\delta), \xi(x+\gamma+\delta-\nu)]$ and $\eta_1(x)$ be the restriction of $\eta(x)$ to $[-\rho, \rho]$. From the assumptions $\eta_1(x)$ is positive with positive Lebesgue measure in any neighbourhoods of the origin. Put $dL_1\equiv dG_1+(\xi_1+\phi_1)dx$, where

$$dG_1(x)\equiv\theta[d\varepsilon(x-\nu)+d\varepsilon(x-\nu-\delta)]\ ,$$

$$\xi_1(x)\equiv\xi'(x)+\eta_1(x-\gamma-\delta+\nu)\ ,$$

$$\phi_1(x)\equiv\eta_1(x-\gamma)-\varepsilon\eta_1(x-\gamma-\delta)+\eta_1(x-\gamma-2\delta)\ .$$

A straightforward calculation shows that

$$\begin{aligned}\theta^{-1}(\phi_n * G_1)(x)&=\phi_n(x-\nu)+\phi_n(x-\nu-\delta)\\ &\geqq \sum_{i+2j+k+1=n}\varepsilon^{2j}[C(i,2j,k+1;n)-\varepsilon C(i,2j+1,k;n)]\\ &\quad\cdot\eta_n(x-\nu-n\gamma-(2j+2k+1)\delta)\\ &\quad+\sum_{i+2j+k+1=n}\varepsilon^{2j}[C(i+1,2j,k;n)-\varepsilon C(i,2j+1,k;n)]\\ &\quad\cdot\eta_n(x-\nu-n\gamma-(2j+2k+1)\delta)\ .\end{aligned}$$

Therefore there is a positive constant $\varepsilon_0=\varepsilon_0(n)$ such that $\phi_n * G_1(x)\geqq 0$ everywhere for $\varepsilon\leqq\varepsilon_0$. On the other hand, if we set $\phi_1(x)\equiv\xi_1(x)+\phi_1(x)$, it follows by Lemma 2 that there is an integer n_0 and a positive constant ε_1 such that $\phi_n(x)$ is positive everywhere for $\varepsilon\leqq\varepsilon_1$ and for $n\geqq n_0$. dL_n can be developed as

$$dL_n=dG_n+\sum_{i=1}^{n_0}C(i;n)\phi_i * G_{n-i}dx+\sum_{i=n_0+1}^{n}C(i;n)\phi_i * G_{n-i}dx\ . \tag{4.1}$$

If $\varepsilon\leqq\text{Min}\,[\varepsilon_0(1),\cdots,\varepsilon_0(n_0),\varepsilon_1]$, the second and the third sum on the right-hand side of (4.2) are positive for $n\geqq 2$. We note also that, as dL_2 contains the term $\theta\eta_1(x-\gamma-\delta)$, $dL_1+dL_2/2\geqq 0$ for sufficiently small $\varepsilon>0$. Summing up all these results we see that for all sufficiently small $\varepsilon>0$

$$\sum_{n=1}^{\infty}\frac{1}{n!}dL_n\geqq 0\ .$$

This completes the proof.

COROLLARY TO THEOREM 4.1. *Suppose that the density function $\xi(t)$ of dN_{ac} satisfies $\xi(t)>0$ on $[\alpha,\beta]$. If there are two jump points γ, δ of spectral measures in $(\alpha-\beta,\beta-\alpha)$ such that $|\gamma-\delta|<(\beta-\alpha)/2$, then $f(t)$ does not belong to I_0.*

PROOF. In this case any points of (α,β) are points of condensation of the support of dN_{ac}. So the proof follows from the theorem at once.

5. Sufficient conditions for membership of the class I_0

Theorems stated in Section 3 admit partial converses and we can construct new examples of c.f.'s of I_0. Consider a decomposition $f(t)$

$=g(t)h(t)$ of an infinitely divisible c.f. $f(t)$ defined by (1.1). According to a theorem due to Ostrovskii, see Lukacs ([4], Th. 9.4.1, p. 282), if $\sigma=0$, $dM\equiv 0$ and $0<\text{lext}(dN)<\text{rext}(dN)<+\infty$, there are two signed measures (not necessarily non-negative) dH, dJ of bounded variation such that $g(t)$ and $h(t)$ are represented as

$$(5.1)\quad g(t)=\exp\left[\int_{+0}^{+\infty} K(t,x)dH(x)\right], \qquad h(t)=\exp\left[\int_{+0}^{+\infty} K(t,x)dJ(x)\right],$$

(for simplicity we neglect factors of the form $e^{i\delta t}$). Here supports of dH and dJ are contained in $[\text{lext}(dN), \text{rext}(dN)]$. We consider two assumptions about dH and dJ.

ASSUMPTION I.
(1) $dH_d=dJ_d=0$,
(2) $dH_{ac}, dJ_{ac}\geqq 0$,
(3) $d(H_s)^{*n}, d(J_s)^{*n}$ are singular for all $n\geqq 1$.

ASSUMPTION II. $dH_c, dJ_c\geqq 0$.

Then we have the following lemma.

LEMMA 4. *Suppose that, for any decomposition of* $f(t)$, *Assumption* I *is satisfied. If* $f_s(t)\equiv\exp\left[\int_{+0}^{+\infty} K(t,x)dN_s(x)\right]$ *belongs to* I_0, *then* $f(t)$ *itself belongs to* I_0. *Suppose that, for any decomposition of* $f(t)$, *Assumption* II *is satisfied. If* $f_d(t)\equiv\exp\left[\int_{+0}^{+\infty} K(t,x)dN_d(x)\right]$ *belongs to* I_0, *then* $f(t)$ *itself belongs to* I_0.

PROOF. Since both cases can be proved similarly, we shall prove the first case only. Fix any decomposition $f(t)=g(t)h(t)$ and their representations (5.1). From Assumption I-(1), $dH=dH_s+dH_{ac}$. Therefore, by Assumption I-(3),

$$dH^{*n}=d(H_s)^{*n}+d(H^{*n})_{ac}, \qquad n\geqq 1.$$

Because $g(t)$ is a c.f., we have (see the proof of Th. 6.2.3 of Lukacs [4])

$$0\leqq\sum_{n=1}^{+\infty}\frac{1}{n!}dH^{*n}=\sum_{n=1}^{+\infty}\frac{1}{n!}d(H_s)^{*n}+\sum_{n=1}^{+\infty}\frac{1}{n!}d(H^{*n})_{ac}.$$

The right-hand side of this equality is the decomposition of the left-hand side into the singular and the absolutely continuous component. Hence

$$0\leqq\sum_{n=1}^{+\infty}\frac{1}{n!}d(H_s)^{*n}$$

and the analogous result holds for dJ. But then, since $dN_s=dH_s+dJ_s$,

$$f_s(t)=\exp\left[\int_{+0}^{+\infty} K(t, x)dH_s(x)\right] \exp\left[\int_{+0}^{+\infty} K(t, x)dJ_s(x)\right]$$

is a decomposition of $f_s(t)$. Since $f_s(t) \in I_0$, we can conclude that both dH_s and dJ_s are non-negative. Taking note of Assumption I-(2), we can prove the non-negativity of dH and dJ. Therefore $g(t)$ and $h(t)$ are infinitely divisible and $f(t) \in I_0$, as is desired.

In the rest of this section we use the following notations for a given infinitely divisible c.f. $f(t)$ defined by (1.1). Let $[\alpha, \beta]$, $[\alpha', \beta']$, $[\gamma, \delta]$ and $[\gamma', \delta']$ be the smallest intervals containing the supports of dN_{ac}, dN_c, dN_s and dN_d respectively. Also define c.f.'s

$$f_s(t)=\exp\left[\int_{+0}^{+\infty} K(t, x)dN_s(x)\right], \qquad f_d(t)=\exp\left[\int_{+0}^{+\infty} K(t, x)dN_d(x)\right].$$

THEOREM 5.1. *Suppose that* $dM=0$, $dN_d=0$, $\sigma=0$, $0<\alpha<\beta<2\alpha$, $\beta-\alpha<\gamma<\alpha$, $\beta\leqq 2\gamma\leqq\delta\leqq\alpha+\gamma$, $\delta\leqq 3\gamma$ *and* $d(N_s)^{*n}$ *is singular for all* $n\geqq 1$. *If* $f_s(t) \in I_0$, *then* $f(t) \in I_0$. *Suppose that* $dM=0$, $\sigma=0$, $0<\alpha'<\beta'<2\alpha'$, $\beta'-\alpha'<\gamma'<\alpha'$, $\beta'\leqq 2\gamma'<\delta'\leqq\gamma'+\alpha'$, $\delta'\leqq 3\gamma'$. *If* $f_d(t) \in I_0$, *then* $f(t) \in I_0$.

PROOF. Let $g(t)$, $h(t)$, dH and dJ have the same meanings as (5.1). We shall prove the first half of the theorem. The last half can be proved similarly. Because $g(t)$ and $h(t)$ are c.f.'s, we see under the conditions of the theorem that in the interval $(-\infty, 2\gamma)$,

$$0\leqq\sum_{n=1}^{+\infty}\frac{1}{n!}dH^{*n}=dH$$

and also $0\leqq dJ$. Hence, in $(-\infty, 2\gamma)$,

$$dH=dH_s+dH_{ac}, \qquad 0\leqq dH_s\leqq dN_s, \qquad 0\leqq dH_{ac}$$

$$dJ=dJ_s+dJ_{ac}, \qquad 0\leqq dJ_s\leqq dN_s, \qquad 0\leqq dJ_{ac}$$

$$dN_s=dH_s+dJ_s, \qquad dN_{ac}=dH_{ac}+dJ_{ac}.$$

From these relations we see especially that dH_s^{*2} and dJ_s^{*2} are singular on $[2\gamma, 3\gamma)$ $(\supset[2\gamma, \delta))$ and $dH_s^{*2}\leqq dN_s^{*2}$, $dJ_s^{*2}\leqq dN_s^{*2}$. While, on $[2\gamma, \delta)$,

$$0\leqq\sum_{n=1}^{+\infty}\frac{1}{n!}dH^{*n}=dH+\frac{1}{2}dH^{*2}=dH+\frac{1}{2}dH_s^{*2}. \tag{5.2}$$

Therefore, on $[2\gamma, \delta)$, both dH_d and dH_{ac} are positive and similarly for dJ_d and dJ_{ac}. As $dN=dH+dJ$ lacks the discrete and the absolutely continuous component on $[2\gamma, \delta)$, dH and dJ must be singular on $[2\gamma, \delta)$.

Consider the Hahn decompositions on $[2\gamma, \delta)$

$$dH=dH_s=dH_s^{(+)}-dH_s^{(-)}, \qquad dJ=dJ_s=dJ_s^{(+)}-dJ_s^{(-)}$$

(i.e. there are two disjoint Lebesgue null sets $A, B\subset[2\gamma, \delta)$ which support positive measures $dJ_s^{(+)}$ and $dJ_s^{(-)}$ respectively etc.). Then, by (5.2),

$$dH_s^{(+)}\leqq\frac{1}{2}dH_s\,, \qquad dJ_s^{(-)}\leqq\frac{1}{2}dJ_s\,,$$

$$0\leqq dN_s=dH_s+dJ_s=[dH_s^{(+)}-dJ_s^{(-)}]+[dJ_s^{(+)}-dH_s^{(-)}]\,.$$

It follows that

$$0\leqq dH_s^{(+)}-dJ_s^{(-)}\leqq dN_s\,, \qquad 0\leqq dJ_s^{(+)}-dH_s^{(-)}\leqq dN_s\,.$$

Therefore

$$dH_s^{(+)}\leqq dN_s+dJ_s^{(-)}\leqq dN_s+\frac{1}{2}dN_s^{*2}\,,$$

$$dJ_s^{(+)}\leqq dN_s+dH_s^{(-)}\leqq dN_s+\frac{1}{2}dN_s^{*2}\,,$$

and dH_s^{*n}, dJ_s^{*n} are singular for all $n\geqq1$. Since supports of dH and dJ are contained in $[\gamma, \delta]$, we have shown all the conditions of Assumption I. So the theorem has proved by Lemma 4.

Remark. In order to construct examples of c.f.'s of I_0 using Theorem 5.1 (also Theorems 5.2, 5.3) we note the following. From Ostrovskii's theorem cited in Section 2 (see its remark) there are c.f.'s of I_0 of the form $f_s(t)$ above. Singular measures whose supports are sets with rationally independent points satisfy Assumption I-(3). Also, from a theorem of Linnik, see Lukacs ([4], Th. 9.2.1, p. 266), there are c.f.'s of the form $f_d(t)$ above.

THEOREM 5.2. *Suppose* $dM=0$, $dN_d=0$, $\sigma=0$, $\alpha/2>\beta/3$, $0<\alpha<\beta\leqq2\alpha$, *then* $f(t)\in I_0$. $\mathrm{Supp}\,(dN_s)\subset[\beta/3, \alpha/2]$ *and* dN_s^{*n} *are singular for all* $n\geqq 1$. *If* $f_s(t)\in I_0$. *Suppose* $dM=0$, $\sigma=0$, $0<\alpha'<\beta'<2\alpha'$, $\alpha'/2>\beta'/3$ *and* $\mathrm{Supp}\,(dN_d)\subset[\alpha'/3, \beta'/2]$. *If* $f_d(t)\in I_0$, *then* $f(t)\in I_0$.

PROOF. Let $g(t)$, $h(t)$, dH and dJ have the same meanings as (5.1). We must check Assumption I (and II). The proof proceeds almost analogously as that of Theorem 5.1. It is sufficient to consider three cases

$$0\leqq\sum_{n=1}^{+\infty}\frac{1}{n!}dH^{*n}=\begin{cases} dH & \text{in } [\beta/3, 2\alpha/3) \\ dH+\dfrac{1}{2}dH^{*2} & \text{in } [2\beta/3, \alpha) \\ dH & \text{in } [\alpha, \beta] \end{cases}$$

and analogously for dJ. We can repeat the same arguments as in the preceding proof.

THEOREM 5.3. *Suppose that* $dM=0$, $dN_d=0$, $\sigma=0$, $0<\alpha<\beta\leqq 2\alpha$, $\beta-\alpha\leqq\beta/k$ $(k\geqq 4)$, $\mathrm{Supp}\,(dN_s^{*n})\subset[\beta/k, \alpha/(k-1)]\cup[\alpha, \alpha+\beta/k]$ *and* dN_s^{*n} *are singular for all* $n\geqq 1$. *If* $f_s(t)\in I_0$, *then* $f(t)\in I_0$. *If we replace* α, β, dN_s *and* $f_s(t)$ *by* α', β', dN_d *and* $f_d(t)$ *respectively above, then the same conclusion holds.*

PROOF. It is sufficient to repeat the same arguments as in proofs of Theorems 5.1 and 5.2.

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