ASYMPTOTIC EXPANSIONS FOR THE DISTRIBUTIONS OF LATENT ROOTS OF $S_hS_r^{-1}$ AND OF CERTAIN TEST STATISTICS IN MANOVA

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Abstract

$S_r$ and $S_h$ are independent central and noncentral Wishart matrices having Wishart distributions $W_p(n_r, \Sigma)$ and $W_p(n_h, \Sigma; \Omega)$ respectively. Asymptotic expansions are given for the distributions of latent roots of $S_hS_r^{-1}$ and of certain test statistics in MANOVA under the assumption that $n=n_r+n_h$ becomes large with a fixed ratio $n_r:n_h=e:h$ ($e+h=1$, $e>0$, $h>0$) and $\Omega=O(n)$.

1. Introduction

In MANOVA model we are often interested in latent roots of $S_hS_r^{-1}$ where $S_r$ and $S_h$ denote independent $p \times p$ matrices distributed as central Wishart $W_p(n_r, \Sigma)$ and noncentral Wishart $W_p(n_h, \Sigma; \Omega)$ respectively, where $n_r$ and $n_h$ are the degrees of freedom for the error and for the hypothesis respectively and $\Omega$ is the noncentrality matrix.

In this paper we shall consider the problems of MANOVA model in the situation that $n_h$ is relatively large, for example, in testing the interactions in multivariate multi-way classification design with relatively large levels and a small sample size. (See Fujikoshi [2]). Therefore we shall assume that $n=n_r+n_h$ becomes large with a fixed ratio $n_r:n_h=e:h$ ($e+h=1$, $e>0$, $h>0$) and $\Omega=O(n)$.

In Section 2 an asymptotic expansion is derived for the joint and marginal distributions of the latent roots of $S_hS_r^{-1}$ when all the roots of $\Omega$ are assumed to be simple.

In Section 3 we shall consider three test statistics, the likelihood ratio, Hotelling's $T^2_r$ and Pillai's $V$ statistics, which are available in tests of dimensionality for MANOVA model, and we shall give asymptotic expansions for the distributions of these three test statistics in

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the nonnull and the null cases.

Our results are expressed in terms of the standard normal distribution function \( \Phi(x) \) and its \( j \)th \( (j=1, 2, \cdots) \) derivatives \( \Phi^{(j)}(x) \), and are given up to order \( m^{-1/2} \) with respect to \( m=\mu^{-1}n-\gamma \) where \( \mu \) and \( \gamma \) are correction terms. We shall obtain the asymptotic expansions for the distributions of traditional statistics by choosing (i) \( \mu^{-1}=(1+e)/2 \), \( \gamma=(p+1)/2 \) for the likelihood ratio statistic, (ii) \( \mu^{-1}=e, \gamma=0 \) for Hotelling's \( T^2 \) statistic and (iii) \( \mu^{-1}=1, \gamma=0 \) for Pillai's \( V \) statistic. In the sequel we may assume without loss of generality that \( \Sigma=I_p \) (the identity matrix of order \( p \)) and \( \Omega=n\Theta=n\text{diag}(\theta_1, \cdots, \theta_p) \) since we treat only latent roots of \( S_nS_r^{-1} \).

2. Asymptotic expansions for the distribution of latent roots

We shall apply a perturbation method in order to get expansions of latent roots of \( S_nS_r^{-1} \) and derive an asymptotic expansion for their joint distribution under the assumption that all \( \theta_i \)'s are simple. Formerly the perturbation formula for latent roots was used by Girshick [5] and Lawley [7] and recently has been given as Taylor's series expansion by Sugiura [8].

Let \( S \) and \( V^{(j)} \) \( (j=0, 1, \cdots) \) be \( p \times p \) real matrices. We assume that \( S \) is expanded in terms of \( V^{(j)} \) in the following way:

\[
S=A+\varepsilon V^{(0)}+\varepsilon^2 V^{(1)}+\cdots,
\]

where \( A=\text{diag}(\lambda_1, \cdots, \lambda_p), \lambda_1>\cdots>\lambda_p \) and \( \varepsilon \) is a real number whose absolute value is small enough. The following lemma can be easily obtained as in Bellman [1], pp. 60–63.

**Lemma 2.1.** For any fixed \( \alpha (\alpha=1, \cdots, p) \) let \( \lambda_\alpha \) be simple. When \( S \) is given in (2.1), the \( \alpha \)th latent root \( \lambda_\alpha \) of \( S \) can be expanded as follows:

\[
\lambda_\alpha = \lambda_\alpha + \varepsilon \lambda_\alpha^{(1)} + \varepsilon^2 \lambda_\alpha^{(2)} + \cdots,
\]

where \( \lambda_\alpha^{(i)} \) \( (i=1, 2, \cdots) \) is a scalar. The first few of them are determined by

\[
\begin{align*}
\lambda_\alpha^{(1)} &= \psi^{(0)}_{a\alpha}, \\
\lambda_\alpha^{(2)} &= \psi^{(1)}_{a\alpha} - \sum_{j\neq a} \psi^{(0)}_{a\alpha} \lambda_{j\alpha} \psi^{(0)}_{j\alpha}, \\
\lambda_\alpha^{(3)} &= \psi^{(2)}_{a\alpha} - \sum_{j\neq a} \lambda_{j\alpha} (\psi^{(1)}_{a\alpha} \psi^{(0)}_{j\alpha} + \psi^{(0)}_{a\alpha} \psi^{(1)}_{j\alpha}) - \psi^{(0)}_{a\alpha} \sum_{j\neq a} \lambda_{j\alpha} \psi^{(0)}_{a\alpha} \psi^{(0)}_{j\alpha} \\
&\quad + \sum_{j\neq a} \lambda_{j\alpha} \lambda_{a\alpha} \psi^{(0)}_{a\alpha} \psi^{(0)}_{j\alpha} \psi^{(0)}_{j\alpha},
\end{align*}
\]
where \( v_{u,v}^{(q)} \) is the \((u,v)\) element of \( V^{(q)} \) and \( \lambda_u = (\lambda_u - \lambda) \). Now we shall consider expansions of latent roots of \( S_nS_e^{-1} \). Let \( T_e \) and \( T_h \) be the statistics defined by

\[
T_e = \sqrt{m} (S_n/m - \mu I_p) \quad \text{and} \quad T_h = \sqrt{m} (S_n/m - \mu (hI_p + 2\Theta)),
\]

for \( m = \mu^{-1}n - \gamma \) where \( \mu \) and \( \gamma \) are correction terms. It is well known that each of \( T_e \) and \( T_h \) converges in law to a \( p(p+1)/2 \) variate normal distribution with mean 0 as \( n \) tends to infinity. From (2.4) it follows that

\[
\frac{1}{m} S_e = \mu I_p + \frac{1}{\sqrt{m}} T_e \quad \text{and} \quad \frac{1}{m} S_h = \mu (hI_p + 2\Theta) + \frac{1}{\sqrt{m}} T_h.
\]

As the latent roots of \( S_nS_e^{-1/2} \) are the same as those of \( S_e^{-1/2}S_nS_e^{-1/2} \), we shall consider an expansion of the latter as follows:

\[
S_e^{-1/2}S_nS_e^{-1/2} = \left( \frac{1}{m} S_e^{-1/2} \right) \left( \frac{1}{m} S_n \right) \left( \frac{1}{m} S_e^{-1/2} \right)^{-1/2} = \left( \mu I_p + \frac{1}{\sqrt{m}} T_e \right)^{-1/2} \left( \mu (hI_p + 2\Theta) + \frac{1}{\sqrt{m}} T_h \right) \left( \mu I_p + \frac{1}{\sqrt{m}} T_e \right)^{-1/2} = A + \frac{1}{m} V^{(0)} + \frac{1}{m} V^{(1)} + \cdots,
\]

where

\[
A = \frac{h}{e} I_p + \frac{2}{e} \Theta,
\]

\[
V^{(0)} = \frac{1}{\mu e} \left( T_h - \frac{1}{2} (T_e A + AT_e) \right),
\]

\[
V^{(1)} = \frac{1}{(\mu e)^2} \left[ \frac{3}{8} (T_e^2 A + AT_e^2) - \frac{1}{2} (T_e T_h + T_h T_e) + \frac{1}{4} T_e^2 A T_e \right].
\]

Applying Lemma 2.1 to the matrix \( S_e^{-1/2}S_nS_e^{-1/2} \) in (2.6), we have the following:

**Theorem 2.1.** If \( \theta_1 > \cdots > \theta_p \), an asymptotic expansion for the joint distribution of the latent roots \( \lambda_1 > \cdots > \lambda_p \) of \( S_nS_e^{-1} \) is given by

\[
P \left\{ \left( \frac{1}{\mu e k^2 m/2} \right)^{1/2} \left[ l_s - \left( \frac{h}{e} + \frac{2}{e} \theta_s \right) \right] < x_s \right\} = \prod_{s=1}^p \Phi(x_s) \left( 1 - \frac{1}{\sqrt{m}} K_s \right) + O(m^{-1}),
\]
where

\[ k_s = e \left( (1+2\theta_s)^2 - e \right)^{-1/2} \]

\[ K_i = \frac{1}{\sqrt{2\mu e}} \sum_{s=1}^{p} k_s \left( \frac{1}{2} R_s \phi^{(1)}(x_s) + \frac{2}{3} S_s \phi^{(3)}(x_s) \right) \]

\[ (2.9) \quad R_s = 2(p+1) \{(1+2\theta_s) e^{-1} - 1 \} + \sum_{j \neq s} \left( \frac{(1+2\theta_j)(1+2\theta_s) - e}{e(\theta_s - \theta_j)} \right) \]

\[ S_s = k_s^2 e^{-1} \{2(1+2\theta_s)^2 - 3e(1+2\theta_s) + e^2 \} \]

\[ \phi^{(j)}(x) = \Phi^{(j)}(x) / \Phi(x) \]  

**Proof.** An asymptotic expansion for the joint distribution of the latent roots \( l_1 > \cdots > l_p \) of the matrix \( S^{-1/2} T S^{-1/2} \) given in (2.6) can be obtained by inverting their characteristic function

\[ (2.10) \quad \phi(t_1, \cdots, t_p) = \mathbb{E} \left[ \exp i \sqrt{m} \sum_{s=1}^{p} t_s (l_s - \lambda_s) \right] = \mathbb{E} \left[ \exp i \sum_{s=1}^{p} t_s \lambda_s^{(1)} \left( 1 + \frac{i}{\sqrt{m}} \sum_{s=1}^{p} t_s \lambda_s^{(2)} + O(m^{-1}) \right) \right] \]

where the expectation \( \mathbb{E} \) is taken with respect to the random matrices \( T \) and \( S \). Useful integral formulas for calculating this type of expectation are given in Fujikoshi [2]. We can get the following expectations.

\[ (2.11) \quad \mathbb{E} \left[ \exp i \sum_{s=1}^{p} t_s \lambda_s^{(1)} \right] = \left\{ 1 - \frac{4t^2}{3\sqrt{m}} g_s + O(m^{-1}) \right\} \exp (-g_s) \]

\[ (2.12) \quad \mathbb{E} \left[ t_s \lambda_s^{(2)} \exp i \sum_{s=1}^{p} t_s \lambda_s^{(1)} \right] = \left[ \frac{t_s}{2\mu e} R_s + \frac{4t^2 t_s^2}{\mu e^2} (h + 2\theta_s) \right] \times \left\{ (1+2\theta_s)^2 - e \right\} + O(m^{-1/2}) \exp (-g_s) \]

where

\[ g_s = \frac{1}{\mu e^2} \sum_{s=1}^{p} \{(1+2\theta_s)^2 - e\} t_s^2 \]

\[ (2.13) \quad g_s = \frac{1}{\mu e^2} \sum_{s=1}^{p} \{(1+2\theta_s)^2 - 3e(1+2\theta_s)^2 + 2e^2\} t_s^2 \]

The inversion of the characteristic function \( \phi(t_1, \cdots, t_p) \) yields the result.

Q.E.D.

**Corollary 2.1.** When \( \theta_s \) is simple for fixed \( \alpha \), an asymptotic expansion for the marginal distribution of \( l_s \) is given by
(2.14) \[ P \left\{ (\mu e k^2 m / 2)^{1/4} \left[ l - \left( \frac{h}{e} + \frac{2}{e} \theta_* \right) \right] < x \right\} = \Phi(x) - \frac{1}{\sqrt{m}} \left\{ a_1 \Phi^{(1)}(x) + a_2 \Phi^{(2)}(x) \right\} + O(m^{-1}), \]

where

(2.15) \[ a_1 = \frac{1}{2\sqrt{2\mu e}} k_* R_* \quad \text{and} \quad a_2 = \frac{2}{3\sqrt{2\mu e}} k_* S_* , \]

and \( k_* , R_* \) and \( S_* \) are the same as in Theorem 2.1.

**Proof.** Let \( x_i = +\infty \) for \( i = 1, \ldots, p \), except \( \alpha \), in Theorem 2.1.

3. Asymptotic expansions for the distributions of three test statistics

We shall consider the following multivariate linear model:

(3.1) \[ X = A\xi + E \]

where \( A \) is an \( N \times w \) known matrix with rank \((A) = w \), \( \xi \) is a \( w \times p \) unknown matrix and each row of the random matrix \( E \) is independently distributed as \( N_p(0, \Sigma) \).

Consider the hypothesis \( H_0 : \text{rank} (B\xi) = k \), and the alternative \( H_1 : \text{rank} (B\xi) > k \), where \( B \) is a \( b \times w \) known matrix with rank \((B) = b \). Then we have three typical test statistics,

1. The likelihood ratio statistic

\[ \Lambda_k = \sum_{\alpha = k+1}^p (1 + l_\alpha)^{-q/2} \quad (q = N - w), \]

2. Hotelling's \( T^2 \) type statistic

\[ T_k = \sum_{\alpha = k+1}^p l_\alpha , \]

3. Pillai's \( V \) type statistic

\[ V_k = \sum_{\alpha = k+1}^p l_\alpha / (1 + l_\alpha) , \]

where \( l_1, \ldots, l_p \) (\( l_1 > \cdots > l_p \)) are the latent roots of \( S_*^{-1/2} S \xi S_*^{-1/2} \) and \( S_* \) and \( S_h \) are independent \( p \times p \) random matrices distributed as central Wishart \( W_p(q, I) \) (\( q = N - w \)) and noncentral Wishart \( W_p(b, I; \Omega) \) respectively and \( \Omega \) is a diagonal matrix. Assuming that \( \Omega = O(q) \) and \( b \) is a fixed constant, asymptotic expansions for the distributions of these three statistics have been obtained up to order \( m^{-1/2} \) by Fujikoshi [3].

Now we shall assume that \( q = n_*, b = n_h, n = n_* + n_h \) with a fixed ratio \( n_* : n_h = e : h \) (\( e + h = 1, e > 0, h > 0 \)) and \( \Omega = n\Theta, \Theta = \text{diag} (\theta_1, \ldots, \theta_p), \Lambda = \)
diag (λ₁, · · · , λₙ) = (h/e)Fₙ + (2/e)θ. Under these assumptions asymptotic expansions for the distributions of those statistics are given in both the nonnull and the null cases.

1° (Nonnull case)

Suppose that the latent roots θ_{k+1}, · · · , θ_{p} are simple. Based on Lemma 2.1, we have the following expansion of the modified likelihood ratio statistic

\[ A^*_k = \prod_{s=k+1}^{p} (1+\lambda_s)^{-m/s} \]  

with \( m \) replacing \( q \) in \( A_k \).

(3.2) \[ \bar{A}_k = \sqrt{m} \left( -\frac{2}{m} \log A^*_k - \sum_{s=k+1}^{p} \log (1+\lambda_s) \right) \]

\[ = \sum_{s=k+1}^{p} \frac{\lambda_s^{(3)}}{1+\lambda_s} + \frac{1}{\sqrt{m}} \sum_{s=k+1}^{p} \left( \frac{\lambda_s^{(3)}}{1+\lambda_s} - \frac{\lambda_s^{(3)^2}}{2(1+\lambda_s)^2} \right) + O(m^{-1}). \]

The characteristic function of \( \bar{A}_k \) is written as

(3.3) \[ E \left[ \exp it \sum_{s=k+1}^{p} \frac{\lambda_s^{(3)}}{1+\lambda_s} \right] \left[ 1 + \frac{it}{\sqrt{m}} \sum_{s=k+1}^{p} \left( \frac{\lambda_s^{(3)}}{1+\lambda_s} - \frac{\lambda_s^{(3)^2}}{2(1+\lambda_s)^2} \right) + O(m^{-1}) \right]. \]

Utilizing the formulas due to Fujikoshi [2], the expectation of each term in (3.3) can be given in the following:

E \left[ \exp it \sum_{s=k+1}^{p} \frac{\lambda_s^{(3)}}{1+\lambda_s} \right] = \left[ 1 - \frac{4(it)^3}{3\sqrt{m}} d_s + O(m^{-1}) \right] \exp (-d_s t^2), \]

where

E \left[ \sum_{s=k+1}^{p} \frac{\lambda_s^{(3)}}{1+\lambda_s} \right] \left[ \frac{1}{2\mu} \sum_{s=k+1}^{p} \frac{R_s}{1+2\theta_s} + \frac{4(it)^3}{(\mu e)^3} \sum_{s=k+1}^{p} (h+2\theta_s)((1+2\theta_s)^{-1} \right.

(3.4) \left. \left( -e(1+2\theta_s)^{-1} + O(m^{-1}) \right) \exp (-d_s t^2), \right]

where

E \left[ \sum_{s=k+1}^{p} \frac{\lambda_s^{(3)^2}}{(1+\lambda_s)^2} \right] \left[ \frac{2}{\mu e} \sum_{s=k+1}^{p} \left[ 1-e(1+2\theta_s)^{-1} \right] + \frac{4(it)^3}{(\mu e)^3} \sum_{s=k+1}^{p} \left[ 1-e(1+2\theta_s)^{-1} \right] \right.

\left. + O(m^{-1/2}) \right] \exp (-d_s t^2), \]

where

\[ d_s = \frac{1}{\mu e} \sum_{s=k+1}^{p} \left[ 1-e(1+2\theta_s)^{-1} \right], \]

(3.5) \[ d_s = \frac{1}{(\mu e)^2} \sum_{s=k+1}^{p} \left[ 1-3e(1+2\theta_s)^{-1} + 2e^3(1+2\theta_s)^{-1} \right]. \]
and $R_\ast$ is given by (2.9). Combining these terms and inverting the characteristic function of $\tilde{A}_\ast$ we have the following result:

**Theorem 3.1.** If $\theta_{k+1}, \cdots, \theta_p$ are simple, the following asymptotic expansion for the likelihood ratio statistic $\Lambda_\ast$ can be derived

\[
P \left\{ \frac{\sqrt{m}}{\tau_1} \left[ -\frac{2}{m} \log \Lambda_\ast - \sum_{a=k+1}^{p} \log \left( \frac{1+2\theta_a}{e} \right) \right] < x \right\} 
= \Phi(x) - \frac{1}{\sqrt{m}} \left\{ \frac{a_1}{\tau_1} \Phi^{(1)}(x) + \frac{a_3}{\tau_1} \Phi^{(3)}(x) \right\} + O(m^{-1}),
\]

where

\[
\tau_1^2 = \frac{2}{\mu e} \{ (p-k) - es_1 \},
\]

\[
a_1 = \frac{1}{2\mu} \left( (p-k)(p+k+1)e^{-1} - 2(p+1)s_i + s_i^2 \right) + \sum_{a=k+1}^{p} \sum_{j=1}^{k} \left( \frac{1+2\theta_a}{e} \frac{(1+2\theta_j)}{e} - e \right),
\]

\[
a_3 = \frac{2}{3(\mu e)^2} \left( (p-k) + 2s_i - 3s_i e \right),
\]

\[
s_i = \sum_{a=k+1}^{p} (1+2\theta_a)^{-i}, \quad i = 1, 2, \cdots.
\]

By similar procedures we can get asymptotic expansions for the $T_\ast$ and $V_\ast$ statistics.

**Theorem 3.2.** If $\theta_{k+1}, \cdots, \theta_p$ are simple, the following asymptotic expansions for Hotelling's statistic $T_\ast$ and Pillai's statistic $V_\ast$ can be derived

\[
P \left\{ \frac{\sqrt{m}}{\tau_2} \left[ T_\ast - \sum_{a=k+1}^{p} (h+2\theta_a)e^{-1} \right] < x \right\}
= \Phi(x) - \frac{1}{\sqrt{m}} \left\{ \frac{b_1}{\tau_2} \Phi^{(1)}(x) + \frac{b_3}{\tau_2} \Phi^{(3)}(x) \right\} + O(m^{-1}),
\]

where

\[
\tau_2^2 = \frac{2}{\mu e^3} \{ t_2 - (p-k)e \},
\]

\[
b_1 = \frac{1}{2\mu e^3} \left( 2(p+1)t_1 - 2(p+1)(p-k)e \right) + \sum_{a=k+1}^{p} \sum_{j=1}^{k} \left( \frac{1+2\theta_a}{\theta_a - \theta_j} - e \right),
\]

\[
\theta_i = \sum_{a=k+1}^{p} (1+2\theta_a)^{-i}, \quad i = 1, 2, \cdots.
\]
\[ b_i = \frac{4}{3\mu^2 e^i} \{(p-k)e^i - 3et_i + 2t_i\}, \]

\[ t_i = \sum_{a=k+1}^{p} (1+2\theta_a^i), \quad i=1, 2, \ldots, \]

and

\[ P \left\{ \frac{\sqrt{m}}{\tau_i} \left( V_x - \sum_{a=k+1}^{p} \frac{h+2\theta_a}{1+2\theta_a} \right) < x \right\} \]

\[ = \Phi(x) - \frac{1}{\sqrt{m}} \left\{ \frac{c_1}{\tau_i} \phi^{(1)}(x) + \frac{c_2}{\tau_i^2} \phi^{(2)}(x) \right\} + O(m^{-1}), \]

where

\[ \tau_i^2 = \frac{2\mu}{\sigma} (s_i - es_i), \]

(3.11)

\[ c_1 = \frac{e}{\mu} \left\{ k\sigma e_i - (p+1)s_i + s_i s_1 + s_2 \right\} + \frac{1}{2\mu} \sum_{a=k+1}^{p} \sum_{j=1}^{a} \left( \frac{1+2\theta_a}{1+2\theta_a} - e \right) \left( \theta_a - \theta_j \right) \left( 1+2\theta_a \right)^2, \]

\[ c_2 = \frac{4e}{3\mu^2} \left\{ -s_i + 3es_i + e^2 s_2 - 3e^2 s_3 \right\}, \]

and the \( s_i, i=1, 2, \ldots, \) are defined by (3.7).

2° (Null case)

Assume that \( \theta_1 > \theta_2 > \cdots > \theta_k > \theta_{k+1} = \cdots = \theta_p = 0. \) We have an asymptotic expansion for the likelihood ratio statistic by a similar procedure to that in the nonnull case. We have only to be careful for the number of multiplicity.

A perturbation formula for multiple roots was obtained by Lawley [7]. In this paper we shall utilize the following perturbation formula given by Fujikoshi [4].

**Lemma 3.1.** When \( S = \Lambda + \epsilon V^{(0)} + \epsilon^2 V^{(1)} + \cdots, \) where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k, \lambda_{k+1}I_{(p-k+1)}), \) with \( \lambda_1 > \cdots > \lambda_k > \lambda_{k+1} \) and \( I_{(p-k)}, \) the identity matrix of order \( (p-k), \) and \( V^{(i)} \) is a \( p \times p \) symmetric matrix partitioned similarly to \( \Lambda; \) namely

\[ V^{(i)} = \begin{pmatrix} V_{\lambda_1}^{(i)} & \cdots & V_{\lambda_{k+1}}^{(i)} \\ \vdots & \ddots & \vdots \\ V_{\lambda_{k+1},1}^{(i)} & \cdots & V_{\lambda_{k+1},k+1}^{(i)} \end{pmatrix}, \]

the \( i \)th \( (i=k+1, \cdots, p) \) latent root of \( S \) is the \((i-k)\)th latent root of \( Z \) which is defined by
(3.12) \[ Z = \lambda_{k+1} I_{(p-k)} + \varepsilon V_{k+1,k+1}^{(0)} + \varepsilon^2 \left[ V_{k+1,k+1}^{(1)} + \sum_{j=1}^{k} \lambda_{k+1,j} V_{k+1,j}^{(0)} V_{j,k+1}^{(0)} \right] \\
+ \varepsilon^3 \left[ V_{k+1,k+1}^{(2)} + \sum_{j=1}^{k} \lambda_{k+1,j} \left( V_{k+1,j}^{(0)} V_{j,k+1}^{(0)} + V_{k+1,j}^{(0)} V_{j,k+1}^{(0)} \right) \right] \\
+ \frac{1}{2} \left( \sum_{j=1}^{k} \lambda_{k+1,j} \right) \left( V_{k+1,j}^{(0)} V_{j,k+1}^{(0)} \right) V_{k+1,k+1}^{(0)} \\
- \frac{1}{2} \left( \sum_{j=1}^{k} \lambda_{k+1,j} \right) \left( V_{k+1,j}^{(0)} V_{j,k+1}^{(0)} \right) \left. \right] + O(\varepsilon^4). \\

Based on Lemma 3.1, an expansion for the likelihood ratio \( L^* \) under the null case is given by

(3.13) \[ \tilde{\lambda}_k = \sqrt{m} \left[ -\frac{2}{m} \log \lambda_{k+1}^{(p-k)} \log (1 + \lambda_{k+1}) \right] \\
= \frac{\text{tr} Z^{(1)}}{1+\lambda_{k+1}} + \frac{1}{\sqrt{m}} \left( \frac{\text{tr} Z^{(1)}}{1+\lambda_{k+1}} - \frac{\text{tr} Z^{(1)^2}}{2(1+\lambda_{k+1})} \right) + O(m^{-1}), \]

where \[ Z^{(1)} = V_{k+1,k+1}^{(0)} \] and \[ Z^{(1)} = V_{k+1,k+1}^{(1)} + \sum_{j=1}^{k} \lambda_{k+1,j} V_{k+1,j}^{(0)} V_{j,k+1}^{(0)}. \]

Therefore the characteristic function of \( \tilde{\lambda}_k \) is written as

(3.14) \[ \mathbb{E} \exp \left( \frac{i t}{1+\lambda_{k+1}} \text{tr} Z^{(1)} \right) \left[ \frac{i t}{\sqrt{m}} \left( \frac{\text{tr} Z^{(1)}}{1+\lambda_{k+1}} - \frac{\text{tr} Z^{(1)^2}}{2(1+\lambda_{k+1})} \right) + O(m^{-1}) \right]. \]

Each expectation can be evaluated as follows:

\[ \mathbb{E} \left[ \exp \left( \frac{i t}{1+\lambda_{k+1}} \text{tr} Z^{(1)} \right) \right] = \left[ 1 - \frac{4(i t)^2}{3\sqrt{m}} \tilde{d}_3 + O(m^{-1}) \right] \exp (-\tilde{d}_3 t^2), \]

\[ \mathbb{E} \left[ \frac{\text{tr} Z^{(1)}}{1+\lambda_{k+1}} \exp \left( \frac{i t}{1+\lambda_{k+1}} \text{tr} Z^{(1)} \right) \right] \]

(3.15) \[ = \left[ \frac{(p-k)}{2 \mu} \tilde{p}_{k+1} + \frac{4(p-k)(i t)^2}{(\mu e)^2} \tilde{h}^2 + O(m^{-1}) \right] \exp (-\tilde{d}_3 t^2), \]

\[ \mathbb{E} \left[ \frac{\text{tr} Z^{(1)^2}}{1+\lambda_{k+1}} \exp \left( \frac{i t}{1+\lambda_{k+1}} \text{tr} Z^{(1)} \right) \right] \]

\[ = \left[ \frac{2(p-k)h}{\mu e} + \frac{4(p-k)(i t)^2}{(\mu e)^2} + O(m^{-1/2}) \right] \exp (-\tilde{d}_3 t^2), \]

where
\[ \tilde{a}_k = \frac{(p-k)h}{\mu e}, \quad \tilde{a}_k = \frac{(p-k)(1-2e)h}{(\mu e)^2}, \]

\[ \tilde{R}_{k+1} = \frac{1}{e} \left\{ 2(p+1)h - 2k - h \sum_{j=1}^{k} \frac{1}{\theta_j} \right\}. \]

Inverting the characteristic function of \( \tilde{A}_k \), we have an asymptotic expansion for the distribution of \( \tilde{A}_k \).

**Theorem 3.3.** If \( \theta_1 > \cdots > \theta_k > \theta_{k+1} = \cdots = \theta_p = 0 \), an asymptotic expansion for the distribution of the likelihood ratio statistic \( \Lambda^*_k \) is given by

\[ \begin{align*}
\mathbb{P} \left\{ \frac{\sqrt{m}}{\tau_1} \left[ -\frac{2}{m} \log \Lambda^*_k + (p-k) \log e \right] < x \right\} &= \Phi(x) - \frac{1}{\sqrt{m}} \left\{ \tilde{a}_1 \frac{\phi^{(1)}(x)}{\tau_1} + \tilde{a}_2 \frac{\phi^{(3)}(x)}{\tau_1^3} \right\} + O(m^{-1}),
\end{align*} \]

where

\[ \tau_1 = \frac{2}{\mu e} (p-k)h, \]

\[ \begin{align*}
\tilde{a}_1 &= \frac{(p-k)}{2\mu e} \left( (h-e)p-k+1 - h \sum_{j=1}^{k} \frac{1}{\theta_j} \right), \\
\tilde{a}_2 &= \frac{2}{3(\mu e)^2} (p-k)(1+e)h.
\end{align*} \]

Similar results can be given for Hotelling’s \( T^2_9 \) type and Pillai’s \( V \) type statistics.

**Theorem 3.4.** If \( \theta_1 > \cdots > \theta_k > \theta_{k+1} = \cdots = \theta_p = 0 \), asymptotic expansions for distributions of Hotelling’s \( T_k \) and Pillai’s \( V_k \) statistics can be given by

\[ \begin{align*}
\mathbb{P} \left\{ \frac{\sqrt{m}}{\tau_2} \left[ T_k - \frac{(p-k)h}{e} \right] < x \right\} &= \Phi(x) - \frac{1}{\sqrt{m}} \left\{ \tilde{b}_1 \frac{\phi^{(1)}(x)}{\tau_2} + \tilde{b}_3 \frac{\phi^{(3)}(x)}{\tau_2^3} \right\} + O(m^{-1}),
\end{align*} \]

where

\[ \tau_2 = \frac{2}{\mu e^3} (p-k)h, \]

\[ \begin{align*}
\tilde{b}_1 &= \frac{(p-k)}{2\mu e^2} \left( 2(p+1)h - 2k - h \sum_{j=1}^{k} \frac{1}{\theta_j} \right), \\
\tilde{b}_3 &= \frac{2}{3(\mu e)^3} (p-k)(1+e)h.
\end{align*} \]
\[ \bar{b}_s = \frac{4}{3 \mu e} (p-k)(2-e)h, \]

and

\begin{equation}
\begin{aligned}
P \left\{ \sqrt{\frac{m}{\bar{\tau}_1}} \left[ V_s - (p-k)h \right] < x \right\} &= \Phi(x) - \frac{1}{\sqrt{m}} \left\{ \frac{\bar{c}_1}{\bar{\tau}_1} \Phi^{(1)}(x) + \frac{\bar{c}_2}{\bar{\tau}_2} \Phi^{(2)}(x) \right\} + O(m^{-1}),
\end{aligned}
\end{equation}

where

\[ \bar{\tau}_1 = \frac{2e}{\mu} (p-k)eh, \]

\begin{equation}
\begin{aligned}
\bar{c}_1 &= \frac{(p-k)}{2\mu} \left\{ 2kh - 2k - h \sum_{j=1}^{k} \frac{1}{\theta_j} \right\},
\end{aligned}
\end{equation}

\[ \bar{c}_2 = \frac{4(p-k)}{3\mu^2} (e-h)eh. \]

It is noted that the result for the null case can be also obtained by substituting \( \theta_{k+1} = \cdots = \theta_p = 0 \) into the formulas for the nonnull case. When we set \( k=0 \) in Theorems 3.1–3.4, each coefficient of order \( m^{-1/2} \) corresponds to that obtained in Fujikoshi [2]. Coefficients of order \( m^{-1} \) are omitted here because of their complexity but are given in the author's master thesis [6].

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REFERENCES


