Ann. Inst. Statist. Math.
29 (1977), Part A, 235-246

ASYMPTOTIC EXPANSIONS FOR THE DISTRIBUTIONS OF LATENT ROOTS OF $S_hS_e^{-1}$ AND OF CERTAIN TEST STATISTICS IN MANOVA

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(Received Nov. 12, 1975; revised Mar. 9, 1977)

Abstract

S_e and S_h are independent central and noncentral Wishart matrices having Wishart distributions $W_p(n_e, \Sigma)$ and $W_p(n_h, \Sigma; \Omega)$ respectively. Asymptotic expansions are given for the distributions of latent roots of $S_hS_e^{-1}$ and of certain test statistics in MANOVA under the assumption that $n=n_e+n_h$ becomes large with a fixed ratio $n_e:n_h=e:h$ $(e+h=1,\ e>0,\ h>0)$ and $\Omega=O(n)$.

1. Introduction

In MANOVA model we are often interested in latent roots of $S_hS_e^{-1}$ where S_e and S_h denote independent $p\times p$ matrices distributed as central Wishart $W_p(n_e, \Sigma)$ and noncentral Wishart $W_p(n_h, \Sigma; \Omega)$ respectively, where n_e and n_h are the degrees of freedom for the error and for the hypothesis respectively and Ω is the noncentrality matrix.

In this paper we shall consider the problems of MANOVA model in the situation that n_h is relatively large, for example, in testing the interactions in multivariate multi-way classification design with relatively large levels and a small sample size. (See Fujikoshi [2]). Therefore we shall assume that $n=n_e+n_h$ becomes large with a fixed ratio $n_e:n_h=e:h$ $(e+h=1,\ e>0,\ h>0)$ and $\Omega=O(n)$.

In Section 2 an asymptotic expansion is derived for the joint and marginal distributions of the latent roots of $S_hS_e^{-1}$ when all the roots of Ω are assumed to be simple.

In Section 3 we shall consider three test statistics, the likelihood ratio, Hotelling's T_0^2 and Pillai's V statistics, which are available in tests of dimensionality for MANOVA model, and we shall give asymptotic expansions for the distributions of these three test statistics in

Key Words and Phrases: asymptotic expansion, distribution of latent roots, Wishart, MANOVA, perturbation, test for dimensionality.

the nonnull and the null cases.

Our results are expressed in terms of the standard normal distribution function $\Phi(x)$ and its jth ($j=1, 2, \cdots$) derivatives $\Phi^{(j)}(x)$, and are given up to order $m^{-1/2}$ with respect to $m=\mu^{-1}n-\gamma$ where μ and γ are correction terms. We shall obtain the asymptotic expansions for the distributions of traditional statistics by choosing (i) $\mu^{-1}=(1+e)/2$, $\gamma=(p+1)/2$ for the likelihood ratio statistic, (ii) $\mu^{-1}=e$, $\gamma=0$ for Hotelling's T_0^2 statistic and (iii) $\mu^{-1}=1$, $\gamma=0$ for Pillai's V statistic. In the sequel we may assume without loss of generality that $\Sigma=I_p$ (the identity matrix of order p) and $\Omega=n\Theta=n\,\mathrm{diag}(\theta_1, \cdots, \theta_p)$ since we treat only latent roots of $S_hS_e^{-1}$.

2. Asymptotic expansions for the distribution of latent roots

We shall apply a perturbation method in order to get expansions of latent roots of $S_hS_e^{-1}$ and derive an asymptotic expansion for their joint distribution under the assumption that all θ_i's are simple. Formerly the perturbation formula for latent roots was used by Girshick [5] and Lawley [7] and recently has been given as Taylor's series expansion by Sugiura [8].

Let S and $V^{(j)}$ ($j=0, 1, \cdots$) be $p\times p$ real matrices. We assume that S is expanded in terms of $V^{(j)}$ in the following way:

$$S=\Lambda+\varepsilon V^{(0)}+\varepsilon^2V^{(1)}+\cdots, \tag{2.1}$$

where $\Lambda=\mathrm{diag}(\lambda_1, \cdots, \lambda_p)$, $\lambda_1>\cdots>\lambda_p$ and ε is a real number whose absolute value is small enough. The following lemma can be easily obtained as in Bellman [1], pp. 60–63.

LEMMA 2.1. *For any fixed α ($\alpha=1, \cdots, p$) let λ_α be simple. When S is given in* (2.1), *the αth latent root l_α of S can be expanded as follows*:

$$l_\alpha=\lambda_\alpha+\varepsilon\lambda_\alpha^{(1)}+\varepsilon^2\lambda_\alpha^{(2)}+\cdots, \tag{2.2}$$

where $\lambda_\alpha^{(i)}$ ($i=1, 2, \cdots$) is a scalar. The first few of them are determined by

$$\begin{aligned}
&\lambda_\alpha^{(1)}=v_{\alpha\alpha}^{(0)},\\
&\lambda_\alpha^{(2)}=v_{\alpha\alpha}^{(1)}-\sum_{j\neq\alpha}^{p} v_{\alpha j}^{(0)}\lambda_{j\alpha}v_{j\alpha}^{(0)},\\
&\lambda_\alpha^{(3)}=v_{\alpha\alpha}^{(2)}-\sum_{j\neq\alpha}^{p}\lambda_{j\alpha}(v_{\alpha j}^{(1)}v_{j\alpha}^{(0)}+v_{\alpha j}^{(0)}v_{j\alpha}^{(1)})-v_{\alpha\alpha}^{(0)}\sum_{j\neq\alpha}^{p}\lambda_{j\alpha}^2v_{\alpha j}^{(0)}v_{j\alpha}^{(0)}\\
&\qquad+\sum_{j\neq\alpha}^{p}\sum_{k\neq\alpha}^{p}\lambda_{j\alpha}\lambda_{k\alpha}v_{\alpha k}^{(0)}v_{kj}^{(0)}v_{j\alpha}^{(0)},
\end{aligned} \tag{2.3}$$

where $v_{us}^{(q)}$ *is the* (u, s) *element of* $V^{(q)}$ *and* $\lambda_{us}=(\lambda_u-\lambda_s)^{-1}$.

Now we shall consider expansions of latent roots of $S_hS_e^{-1}$. Let T_e and T_h be the statistics defined by

$$(2.4)\qquad T_e=\sqrt{m}\,(S_e/m-\mu eI_p) \quad\text{and}\quad T_h=\sqrt{m}\,\{S_h/m-\mu(hI_p+2\Theta)\}\ ,$$

for $m=\mu^{-1}n-\gamma$ where μ and γ are correction terms. It is well known that each of T_e and T_h converges in law to a $p(p+1)/2$ variate normal distribution with mean 0 as n tends to infinity. From (2.4) it follows that

$$(2.5)\qquad \frac{1}{m}S_e=\mu eI_p+\frac{1}{\sqrt{m}}T_e \quad\text{and}\quad \frac{1}{m}S_h=\mu(hI_p+2\Theta)+\frac{1}{\sqrt{m}}T_h\ .$$

As the latent roots of $S_hS_e^{-1}$ are the same as those of $S_e^{-1/2}S_hS_e^{-1/2}$, we shall consider an expansion of the latter as follows:

$$\begin{aligned}(2.6)\qquad S_e^{-1/2}S_hS_e^{-1/2}&=\Big(\frac{1}{m}S_e\Big)^{-1/2}\Big(\frac{1}{m}S_h\Big)\Big(\frac{1}{m}S_e\Big)^{-1/2}\\ &=\Big(\mu eI_p+\frac{1}{\sqrt{m}}T_e\Big)^{-1/2}\Big\{\mu(hI_p+2\Theta)+\frac{1}{\sqrt{m}}T_h\Big\}\\ &\quad\cdot\Big(\mu eI_p+\frac{1}{\sqrt{m}}T_e\Big)^{-1/2}\\ &=\Lambda+\frac{1}{\sqrt{m}}V^{(0)}+\frac{1}{m}V^{(1)}+\cdots\ ,\end{aligned}$$

where

$$\Lambda=\frac{h}{e}I_p+\frac{2}{e}\Theta\ ,$$

$$(2.7)\qquad V^{(0)}=\frac{1}{\mu e}\Big\{T_h-\frac{1}{2}(T_e\Lambda+\Lambda T_e)\Big\}\ ,$$

$$V^{(1)}=\frac{1}{(\mu e)^2}\Big\{\frac{3}{8}(T_e^2\Lambda+\Lambda T_e^2)-\frac{1}{2}(T_eT_h+T_hT_e)+\frac{1}{4}T_e\Lambda T_e\Big\}\ .$$

Applying Lemma 2.1 to the matrix $S_e^{-1/2}S_hS_e^{-1/2}$ in (2.6), we have the following:

THEOREM 2.1. *If* $\theta_1>\cdots>\theta_p$, *an asymptotic expansion for the joint distribution of the latent roots* $l_1>\cdots>l_p$ *of* $S_hS_e^{-1}$ *is given by*

$$\begin{aligned}(2.8)\qquad &\mathrm{P}\Big\{\bigcap_{\alpha=1}^{p}(\mu ek_\alpha^2m/2)^{1/2}\Big[l_\alpha-\Big(\frac{h}{e}+\frac{2}{e}\theta_\alpha\Big)\Big]<x_\alpha\Big\}\\ &=\prod_{\alpha=1}^{p}\Phi(x_\alpha)\Big(1-\frac{1}{\sqrt{m}}K_1\Big)+O(m^{-1})\ ,\end{aligned}$$

where

$$k_\alpha = e\{(1+2\theta_\alpha)^2 - e\}^{-1/2},$$

$$K_1 = \frac{1}{\sqrt{2\mu e}} \sum_{\alpha=1}^{p} k_\alpha \left\{ \frac{1}{2} R_\alpha \phi^{(1)}(x_\alpha) + \frac{2}{3} S_\alpha \phi^{(3)}(x_\alpha) \right\},$$

$$R_\alpha = 2(p+1)\{(1+2\theta_\alpha)e^{-1} - 1\} + \sum_{j \neq \alpha}^{p} \frac{(1+2\theta_\alpha)(1+2\theta_j) - e}{e(\theta_\alpha - \theta_j)}, \tag{2.9}$$

$$S_\alpha = k_\alpha^2 e^{-3}\{2(1+2\theta_\alpha)^3 - 3e(1+2\theta_\alpha) + e^2\},$$

$$\phi^{(j)}(x) = \Phi^{(j)}(x)/\Phi(x).$$

PROOF. An asymptotic expansion for the joint distribution of the latent roots $l_1 > \cdots > l_p$ of the matrix $S_e^{-1/2} S_h S_e^{-1/2}$ given in (2.6) can be obtained by inverting their characteristic function

$$\begin{aligned} \phi(t_1, \cdots, t_p) &= \mathrm{E}\left[\exp i\sqrt{m} \sum_{\alpha=1}^{p} t_\alpha (l_\alpha - \lambda_\alpha)\right] \\ &= \mathrm{E}\left[\exp i \sum_{\alpha=1}^{p} t_\alpha \lambda_\alpha^{(1)} \left\{1 + \frac{i}{\sqrt{m}} \sum_{\alpha=1}^{p} t_\alpha \lambda_\alpha^{(2)} + O(m^{-1})\right\}\right], \end{aligned} \tag{2.10}$$

where the expectation E is taken with respect to the random matrices T_e and T_h. Useful integral formulas for calculating this type of expectation are given in Fujikoshi [2]. We can get the following expectations.

$$\mathrm{E}\left[\exp i \sum_{\alpha=1}^{p} t_\alpha \lambda_\alpha^{(1)}\right] = \left\{1 - \frac{4i^3}{3\sqrt{m}} g_3 + O(m^{-1})\right\} \exp(-g_2), \tag{2.11}$$

$$\begin{aligned} \mathrm{E}\left[t_\alpha \lambda_\alpha^{(2)} \exp i \sum_{\alpha=1}^{p} t_\alpha \lambda_\alpha^{(1)}\right] = \Big[&\frac{t_\alpha}{2\mu e} R_\alpha + \frac{4i^2 t_\alpha^3}{\mu^2 e^5}(h+2\theta_\alpha) \\ &\times \{(1+2\theta_\alpha)^2 - e\} + O(m^{-1/2})\Big] \exp(-g_2), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} g_2 &= \frac{1}{\mu e^3} \sum_{\alpha=1}^{p} \{(1+2\theta_\alpha)^2 - e\} t_\alpha^2, \\ g_3 &= \frac{1}{\mu^2 e^5} \sum_{\alpha=1}^{p} \{(1+2\theta_\alpha)^3 - 3e(1+2\theta_\alpha)^2 + 2e^2\} t_\alpha^3. \end{aligned} \tag{2.13}$$

The inversion of the characteristic function $\phi(t_1, \cdots, t_p)$ yields the result.
Q.E.D.

COROLLARY 2.1. *When θ_α is simple for fixed α, an asymptotic expansion for the marginal distribution of l_α is given by*

$$\text{(2.14)}\qquad \mathrm{P}\left\{(\mu e k_\alpha^2 m/2)^{1/2}\left[l_\alpha-\left(\frac{h}{e}+\frac{2}{e}\theta_\alpha\right)\right]<x\right\}$$
$$=\Phi(x)-\frac{1}{\sqrt{m}}\{a_1\Phi^{(1)}(x)+a_3\Phi^{(3)}(x)\}+O(m^{-1}) ,$$

where

$$\text{(2.15)}\qquad a_1=\frac{1}{2\sqrt{2\mu e}}k_\alpha R_\alpha \quad \text{and} \quad a_3=\frac{2}{3\sqrt{2\mu e}}k_\alpha S_\alpha ,$$

and k_α, R_α *and* S_α *are the same as in Theorem* 2.1.

PROOF. Let $x_i=+\infty$ for $i=1,\cdots,p$, except α, in Theorem 2.1.

3. Asymptotic expansions for the distributions of three test statistics

We shall consider the following multivariate linear model:

$$\text{(3.1)}\qquad X=A\Xi+E$$

where A is an $N\times w$ known matrix with rank$(A)=w$, Ξ is a $w\times p$ unknown matrix and each row of the random matrix E is independently distributed as $N_p(0, \Sigma)$.

Consider the hypothesis H_0 : rank$(B\Xi)=k$, and the alternative H_1 : rank$(B\Xi)>k$, where B is a $b\times w$ known matrix with rank$(B)=b$. Then we have three typical test statistics,

(1) The likelihood ratio statistic

$$\Lambda_k=\sum_{\alpha=k+1}^{p}(1+l_\alpha)^{-q/2} \qquad (q=N-w) ,$$

(2) Hotelling's T_0^2 type statistic

$$T_k=\sum_{\alpha=k+1}^{p} l_\alpha ,$$

(3) Pillai's V type statistic

$$V_k=\sum_{\alpha=k+1}^{p} l_\alpha/(1+l_\alpha) ,$$

where $l_1,\cdots,l_p$ $(l_1>\cdots>l_p)$ are the latent roots of $S_e^{-1/2}S_hS_e^{-1/2}$ and S_e and S_h are independent $p\times p$ random matrices distributed as central Wishart $W_p(q, I)$ $(q=N-w)$ and noncentral Wishart $W_p(b, I; \Omega)$ respectively and Ω is a diagonal matrix. Assuming that $\Omega=O(q)$ and b is a fixed constant, asymptotic expansions for the distributions of these three statistics have been obtained up to order $m^{-1/2}$ by Fujikoshi [3]. Now we shall assume that $q=n_e$, $b=n_h$, $n=n_e+n_h$ with a fixed ratio $n_e:n_h=e:h$ $(e+h=1,\ e>0,\ h>0)$ and $\Omega=n\Theta$, $\Theta=\mathrm{diag}(\theta_1,\cdots,\theta_p)$, $\Lambda=$

diag $(\lambda_1, \cdots, \lambda_p)=(h/e)I_p+(2/e)\Theta$. Under these assumptions asymptotic expansions for the distributions of those statistics are given in both the nonnull and the null cases.

1° (Nonnull case)

Suppose that the latent roots $\theta_{k+1}, \cdots, \theta_p$ are simple. Based on Lemma 2.1, we have the following expansion of the modified likelihood ratio statistic $\Lambda_k^*=\prod_{\alpha=k+1}^{p}(1+l_\alpha)^{-m/2}$ with m replacing q in Λ_k.

$$
\begin{aligned}
(3.2)\qquad \tilde{\Lambda}_k &= \sqrt{m}\left(-\frac{2}{m}\log \Lambda_k^* - \sum_{\alpha=k+1}^{p}\log(1+\lambda_\alpha)\right)\\
&= \sum_{\alpha=k+1}^{p}\frac{\lambda_\alpha^{(1)}}{1+\lambda_\alpha}+\frac{1}{\sqrt{m}}\sum_{\alpha=k+1}^{p}\left(\frac{\lambda_\alpha^{(2)}}{1+\lambda_\alpha}-\frac{\lambda_\alpha^{(1)^2}}{2(1+\lambda_\alpha)^2}\right)+O(m^{-1})\,.
\end{aligned}
$$

The characteristic function of $\tilde{\Lambda}_k$ is written as

$$
(3.3)\qquad \mathrm{E}\left[\exp it\sum_{\alpha=k+1}^{p}\frac{\lambda_\alpha^{(1)}}{1+\lambda_\alpha}\left\{1+\frac{it}{\sqrt{m}}\sum_{\alpha=k+1}^{p}\left(\frac{\lambda_\alpha^{(2)}}{1+\lambda_\alpha}-\frac{\lambda_\alpha^{(1)^2}}{2(1+\lambda_\alpha)^2}\right)+O(m^{-1})\right\}\right]\,.
$$

Utilizing the formulas due to Fujikoshi [2], the expectation of each term in (3.3) can be given in the following:

$$
\begin{aligned}
&\mathrm{E}\left[\exp it\sum_{\alpha=k+1}^{p}\frac{\lambda_\alpha^{(1)}}{1+\lambda_\alpha}\right]=\left[1-\frac{4(it)^3}{3\sqrt{m}}d_3+O(m^{-1})\right]\exp(-d_2t^2)\,,\\
&\mathrm{E}\left[\left(\sum_{\alpha=k+1}^{p}\frac{\lambda_\alpha^{(2)}}{1+\lambda_\alpha}\right)\exp it\sum_{\alpha=k+1}^{p}\frac{\lambda_\alpha^{(1)}}{1+\lambda_\alpha}\right]\\
&\quad=\left\{\frac{1}{2\mu}\sum_{\alpha=k+1}^{p}\frac{R_\alpha}{1+2\theta_\alpha}+\frac{4(it)^2}{(\mu e)^2}\sum_{\alpha=k+1}^{p}(h+2\theta_\alpha)[(1+2\theta_\alpha)^{-1}\right.\\
(3.4)\qquad&\qquad\left.-e(1+2\theta_\alpha)^{-3}]+O(m^{-1})\right\}\exp(-d_2t^2)\,,\\
&\mathrm{E}\left[\left(\sum_{\alpha=k+1}^{p}\frac{\lambda_\alpha^{(1)^2}}{(1+\lambda_\alpha)^2}\right)\exp it\sum_{\alpha=k+1}^{p}\frac{\lambda_\alpha^{(1)}}{1+\lambda_\alpha}\right]\\
&\quad=\left\{\frac{2}{\mu e}\sum_{\alpha=k+1}^{p}[1-e(1+2\theta_\alpha)^{-2}]+\frac{4(it)^2}{(\mu e)^2}\sum_{\alpha=k+1}^{p}[1-e(1+2\theta_\alpha)^{-2}]^2\right.\\
&\qquad\left.+O(m^{-1/2})\right\}\exp(-d_2t^2)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
&d_2=\frac{1}{\mu e}\sum_{\alpha=k+1}^{p}\{1-e(1+2\theta_\alpha)^{-2}\}\,,\\
(3.5)\qquad &d_3=\frac{1}{(\mu e)^2}\sum_{\alpha=k+1}^{p}\{1-3e(1+2\theta_\alpha)^{-1}+2e^2(1+2\theta_\alpha)^{-3}\}\,,
\end{aligned}
$$

and R_α is given by (2.9). Combining these terms and inverting the characteristic function of $\tilde{\Lambda}_k$ we have the following result:

THEOREM 3.1. *If $\theta_{k+1},\cdots,\theta_p$ are simple, the following asymptotic expansion for the likelihood ratio statistic Λ_k^* can be derived*

$$\text{(3.6)}\qquad \begin{aligned} &P\left\{\frac{\sqrt{m}}{\tau_1}\left[-\frac{2}{m}\log\Lambda_k^*-\sum_{\alpha=k+1}^{p}\log\left(\frac{1+2\theta_\alpha}{e}\right)\right]<x\right\}\\ &=\Phi(x)-\frac{1}{\sqrt{m}}\left\{\frac{a_1}{\tau_1}\Phi^{(1)}(x)+\frac{a_3}{\tau_1^3}\Phi^{(3)}(x)\right\}+O(m^{-1}),\end{aligned}$$

where

$$\text{(3.7)}\qquad \begin{aligned} &\tau_1^2=\frac{2}{\mu e}\{(p-k)-es_2\},\\ &a_1=\frac{1}{2\mu}\left\{(p-k)(p+k+1)e^{-1}-2(p+1)s_1+s_1^2+s_2\right.\\ &\qquad\left.+\sum_{\alpha=k+1}^{p}\sum_{j=1}^{k}\frac{(1+2\theta_\alpha)(1+2\theta_j)-e}{e(\theta_\alpha-\theta_j)(1+2\theta_\alpha)}\right\},\\ &a_3=\frac{2}{3(\mu e)^2}\{(p-k)+2e^2s_3-3e^2s_4\},\\ &s_i=\sum_{\alpha=k+1}^{p}(1+2\theta_\alpha)^{-i},\qquad i=1,2,\cdots.\end{aligned}$$

By similar procedures we can get asymptotic expansions for the T_k and V_k statistics.

THEOREM 3.2. *If $\theta_{k+1},\cdots,\theta_p$ are simple, the following asymptotic expansions for Hotelling's statistic T_k and Pillai's statistic V_k can be derived*

$$\text{(3.8)}\qquad \begin{aligned} &P\left\{\frac{\sqrt{m}}{\tau_2}\left[T_k-\sum_{\alpha=k+1}^{p}(h+2\theta_\alpha)e^{-1}\right]<x\right\}\\ &=\Phi(x)-\frac{1}{\sqrt{m}}\left\{\frac{b_1}{\tau_2}\Phi^{(1)}(x)+\frac{b_3}{\tau_2^3}\Phi^{(3)}(x)\right\}+O(m^{-1}),\end{aligned}$$

where

$$\text{(3.9)}\qquad \begin{aligned} &\tau_2^2=\frac{2}{\mu e^3}\{t_2-(p-k)e\},\\ &b_1=\frac{1}{2\mu e^2}\left\{2(p+1)t_1-2(p+1)(p-k)e\right.\\ &\qquad\left.+\sum_{\alpha=k+1}^{p}\sum_{j=1}^{k}\frac{(1+2\theta_\alpha)(1+2\theta_j)-e}{\theta_\alpha-\theta_j}\right\},\end{aligned}$$

$$b_3=\frac{4}{3\mu^2e^5}\{(p-k)e^2-3et_1+2t_3\}\ ,$$

$$t_i=\sum_{\alpha=k+1}^{p}(1+2\theta_\alpha)^i\ ,\qquad i=1,2,\cdots,$$

and

$$\text{(3.10)}\qquad \mathrm{P}\left\{\frac{\sqrt{m}}{\tau_3}\left(V_k-\sum_{\alpha=k+1}^{p}\frac{h+2\theta_\alpha}{1+2\theta_\alpha}\right)<x\right\}$$
$$=\Phi(x)-\frac{1}{\sqrt{m}}\left\{\frac{c_1}{\tau_3}\Phi^{(1)}(x)+\frac{c_3}{\tau_3^3}\Phi^{(3)}(x)\right\}+O(m^{-1})\ ,$$

where

$$\tau_3^2=\frac{2e}{\mu}(s_2-es_4)\ ,$$

$$\text{(3.11)}\qquad c_1=\frac{e}{\mu}\left\{ke^{-1}s_1-(p+1)s_2+s_1s_2+s_3\right.$$
$$\left.+\frac{1}{2e}\sum_{\alpha=k+1}^{p}\sum_{j=1}^{k}\frac{(1+2\theta_\alpha)(1+2\theta_j)-e}{(\theta_\alpha-\theta_j)(1+2\theta_\alpha)^2}\right\}\ ,$$

$$c_3=\frac{4e}{3\mu^2}\{-s_3+3es_5+e^2s_6-3e^2s_7\}\ ,$$

and the s_i, $i=1,2,\cdots$, *are defined by* (3.7).

2° (Null case)

Assume that $\theta_1>\theta_2>\cdots>\theta_k>\theta_{k+1}=\cdots=\theta_p=0$. We have an asymptotic expansion for the likelihood ratio statistic by a similar procedure to that in the nonnull case. We have only to be careful for the number of multiplicity.

A perturbation formula for multiple roots was obtained by Lawley [7]. In this paper we shall utilize the following perturbation formula given by Fujikoshi [4].

LEMMA 3.1. *When* $S=\Lambda+\varepsilon V^{(0)}+\varepsilon^2V^{(1)}+\cdots$, *where* $\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_k,\lambda_{k+1}I_{(p-k)})$, *with* $\lambda_1>\cdots>\lambda_k>\lambda_{k+1}$ *and* $I_{(p-k)}$, *the identity matrix of order* $(p-k)$, *and* $V^{(j)}$ *is a* $p\times p$ *symmetric matrix partitioned similarly to* Λ; *namely*

$$V^{(j)}=\begin{pmatrix}V_{1,1}^{(j)} & \cdots & V_{1,k+1}^{(j)}\\ \vdots & & \vdots\\ V_{k+1,1}^{(j)} & \cdots & V_{k+1,k+1}^{(j)}\end{pmatrix},$$

the ith $(i=k+1,\cdots,p)$ *latent root of* S *is the* $(i-k)$*th latent root of* Z *which is defined by*

$$(3.12)\quad Z=\lambda_{k+1}I_{(p-k)}+\varepsilon V^{(0)}_{k+1,k+1}+\varepsilon^2\Big[V^{(1)}_{k+1,k+1}+\sum_{j=1}^{k}\lambda_{k+1,j}V^{(0)}_{k+1,j}V^{(0)}_{j,k+1}\Big]$$
$$+\varepsilon^3\Big[V^{(2)}_{k+1,k+1}+\sum_{j=1}^{k}\lambda_{k+1,j}\{V^{(0)}_{k+1,j}V^{(1)}_{j,k+1}+V^{(1)}_{k+1,j}V^{(0)}_{j,k+1}\}$$
$$+\sum_{j=1}^{k}\sum_{r=1}^{k}\lambda_{k+1,j}\lambda_{k+1,r}V^{(0)}_{k+1,j}V^{(0)}_{j,r}V^{(0)}_{r,k+1}$$
$$-\frac{1}{2}\Big\{\sum_{j=1}^{k}\lambda^2_{k+1,j}V^{(0)}_{k+1,j}V^{(0)}_{j,k+1}\Big\}V^{(0)}_{k+1,k+1}$$
$$-\frac{1}{2}V^{(0)}_{k+1,k+1}\Big\{\sum_{j=1}^{k}\lambda^2_{k+1,j}V^{(0)}_{k+1,j}V^{(0)}_{j,k+1}\Big\}\Big]+O(\varepsilon^4)\,.$$

Based on Lemma 3.1, an expansion for the likelihood ratio Λ_k^* under the null case is given by

$$(3.13)\quad \tilde{\Lambda}_k=\sqrt{m}\Big[-\frac{2}{m}\log\Lambda_k^*-(p-k)\log(1+\lambda_{k+1})\Big]$$
$$=\frac{\operatorname{tr}Z^{(1)}}{1+\lambda_{k+1}}+\frac{1}{\sqrt{m}}\Big\{\frac{\operatorname{tr}Z^{(2)}}{1+\lambda_{k+1}}-\frac{\operatorname{tr}Z^{(1)^2}}{2(1+\lambda_{k+1})^2}\Big\}+O(m^{-1})\,,$$

where

$$Z^{(1)}=V^{(0)}_{k+1,k+1}\,,\qquad Z^{(2)}=V^{(1)}_{k+1,k+1}+\sum_{j=1}^{k}\lambda_{k+1,j}V^{(0)}_{k+1,j}V^{(0)}_{j,k+1}\,.$$

Therefore the characteristic function of $\tilde{\Lambda}_k$ is written as

$$(3.14)\quad \mathrm{E}\exp\frac{(it)\operatorname{tr}Z^{(1)}}{1+\lambda_{k+1}}\Big[1+\frac{it}{\sqrt{m}}\Big\{\frac{\operatorname{tr}Z^{(2)}}{1+\lambda_{k+1}}-\frac{\operatorname{tr}Z^{(1)^2}}{2(1+\lambda_{k+1})^2}\Big\}+O(m^{-1})\Big]\,.$$

Each expectation can be evaluated as follows:

$$\mathrm{E}\Big[\exp\frac{(it)\operatorname{tr}Z^{(1)}}{1+\lambda_{k+1}}\Big]=\Big[1-\frac{4(it)^3}{3\sqrt{m}}\tilde{d}_3+O(m^{-1})\Big]\exp(-\tilde{d}_2t^2)\,,$$

$$\mathrm{E}\Big[\frac{\operatorname{tr}Z^{(2)}}{1+\lambda_{k+1}}\exp\frac{(it)\operatorname{tr}Z^{(1)}}{1+\lambda_{k+1}}\Big]$$
$$(3.15)\quad =\Big[\frac{(p-k)}{2\mu}\tilde{R}_{k+1}+\frac{4(p-k)(it)^2}{(\mu e)^2}h^2+O(m^{-1})\Big]\exp(-\tilde{d}_2t^2)\,,$$

$$\mathrm{E}\Big[\frac{\operatorname{tr}Z^{(1)^2}}{(1+\lambda_{k+1})^2}\exp\frac{(it)\operatorname{tr}Z^{(1)}}{1+\lambda_{k+1}}\Big]$$
$$=\Big[\frac{2(p-k)h}{\mu e}+\frac{4(p-k)h^2(it)^2}{(\mu e)^2}+O(m^{-1/2})\Big]\exp(-\tilde{d}_2t^2)\,,$$

where

$$(3.16)\qquad \tilde{d}_2=\frac{(p-k)h}{\mu e}\,,\qquad \tilde{d}_3=\frac{(p-k)(1-2e)h}{(\mu e)^2}\,,$$
$$\tilde{R}_{k+1}=\frac{1}{e}\left\{2(p+1)h-2k-h\sum_{j=1}^{k}\frac{1}{\theta_j}\right\}.$$

Inverting the characteristic function of $\tilde{\Lambda}_k$, we have an asymptotic expansion for the distribution of $\tilde{\Lambda}_k$.

THEOREM 3.3. *If $\theta_1>\cdots>\theta_k>\theta_{k+1}=\cdots=\theta_p=0$, an asymptotic expansion for the distribution of the likelihood ratio statistic Λ_k^* is given by*

$$(3.17)\qquad \mathrm{P}\left\{\frac{\sqrt{m}}{\tilde{\tau}_1}\left[-\frac{2}{m}\log\Lambda_k^*+(p-k)\log e\right]<x\right\}$$
$$=\Phi(x)-\frac{1}{\sqrt{m}}\left\{\frac{\tilde{a}_1}{\tilde{\tau}_1}\Phi^{(1)}(x)+\frac{\tilde{a}_3}{\tilde{\tau}_1^3}\Phi^{(3)}(x)\right\}+O(m^{-1})\,,$$

where

$$\tilde{\tau}_1^2=\frac{2}{\mu e}(p-k)h\,,$$
$$(3.18)\qquad \tilde{a}_1=\frac{(p-k)}{2\mu e}\left\{(h-e)p-k+1-h\sum_{j=1}^{k}\frac{1}{\theta_j}\right\},$$
$$\tilde{a}_3=\frac{2}{3(\mu e)^2}(p-k)(1+e)h\,.$$

Similar results can be given for Hotelling's T_0^2 type and Pillai's V type statistics.

THEOREM 3.4. *If $\theta_1>\cdots>\theta_k>\theta_{k+1}=\cdots=\theta_p=0$, asymptotic expansions for distributions of Hotelling's T_k and Pillai's V_k statistics can be given by*

$$(3.19)\qquad \mathrm{P}\left\{\frac{\sqrt{m}}{\tilde{\tau}_2}\left[T_k-\frac{(p-k)h}{e}\right]<x\right\}$$
$$=\Phi(x)-\frac{1}{\sqrt{m}}\left\{\frac{\tilde{b}_1}{\tilde{\tau}_2}\Phi^{(1)}(x)+\frac{\tilde{b}_3}{\tilde{\tau}_2^3}\Phi^{(3)}(x)\right\}+O(m^{-1})\,,$$

where

$$\tilde{\tau}_2^2=\frac{2}{\mu e^3}(p-k)h\,,$$
$$(3.20)\qquad \tilde{b}_1=\frac{(p-k)}{2\mu e^2}\left\{2(p+1)h-2k-h\sum_{j=1}^{k}\frac{1}{\theta_j}\right\},$$

$$\tilde{b}_3=\frac{4}{3\mu^2 e^5}(p-k)(2-e)h \ ,$$

and

$$\text{(3.21)} \qquad \mathrm{P}\left\{\frac{\sqrt{m}}{\tilde{\tau}_3}[V_k-(p-k)h]<x\right\}$$

$$=\Phi(x)-\frac{1}{\sqrt{m}}\left\{\frac{\tilde{c}_1}{\tilde{\tau}_3}\Phi^{(1)}(x)+\frac{\tilde{c}_3}{\tilde{\tau}_3^3}\Phi^{(3)}(x)\right\}+O(m^{-1}) \ ,$$

where

$$\tilde{\tau}_3^2=\frac{2e}{\mu}(p-k)eh \ ,$$

$$\text{(3.22)} \qquad \tilde{c}_1=\frac{(p-k)}{2\mu}\left\{2kh-2k-h\sum_{j=1}^{k}\frac{1}{\theta_j}\right\} \ ,$$

$$\tilde{c}_3=\frac{4(p-k)}{3\mu^2}(e-h)eh \ .$$

It is noted that the result for the null case can be also obtained by substituting $\theta_{k+1}=\cdots=\theta_p=0$ into the formulas for the nonnull case. When we set $k=0$ in Theorems 3.1–3.4, each coefficient of order $m^{-1/2}$ corresponds to that obtained in Fujikoshi [2]. Coefficients of order m^{-1} are omitted here because of their complexity but are given in the author's master thesis [6].

Acknowledgements

I wish to thank Professor Y. Fujikoshi for his valuable guidance during the preparation of this dissertation and also Professor M. Okamoto for valuable advices and comments on the paper.

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