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# ASYMPTOTIC EXPANSIONS FOR THE JOINT AND MARGINAL DISTRIBUTIONS OF THE LATENT ROOTS OF $S_1S_2^{-1}$*

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## 1. Introduction and summary

Let $S_1$ and $S_2$ be respectively the covariance matrices formed from samples of sizes $n_1+1$ and $n_2+1$ drawn from independent $m$-variate normal distributions with covariance matrices $\Sigma_1$ and $\Sigma_2$; then $n_1S_1$ and $n_2S_2$ have independent Wishart distributions $W_m(n_1, \Sigma_1)$ and $W_m(n_2, \Sigma_2)$ respectively. Let $b_1>b_2>\cdots>b_m$ $(>0)$ and $\omega_1\geqq\omega_2\geqq\cdots\geqq\omega_m$ $(>0)$ denote the latent roots of $S_1S_2^{-1}$ and $\Sigma_1\Sigma_2^{-1}$ respectively. Various functions of $b_1, b_2, \cdots, b_m$ have been proposed as statistics suitable for testing the null hypothesis $\Sigma_1=\Sigma_2$ (see e.g. Khatri [12] and Pillai [15]). In this paper we investigate the asymptotic behavior of the distribution of $b_1, b_2, \cdots, b_m$.

The forms of the limiting joint and marginal distributions of the sample roots $b_1, \cdots, b_m$ (for large $n_1$ and $n_2$) depend on whether the population latent roots $\omega_1, \cdots, \omega_m$ are simple or multiple. It is shown that if $\omega_i$ is a simple root then, for $n_1$ and $n_2$ large, $b_i$ is asymptotically independent of all the other sample roots and the limiting distribution of $[n_1n_2/2(n_1+n_2)]^{1/2}(b_i/\omega_i-1)$ is standard normal $N(0, 1)$. Both the asymptotic independence and normality break down if $\omega_i$ is a multiple root. This result is, of course, analogous to the well-known result concerning the limiting normality of the roots of the sample covariance matrix (see e.g. Girshick [8], Anderson [1], [2]).

In Section 2 it is assumed that the smallest latent root of $\Sigma_1\Sigma_2^{-1}$ is multiple. Putting $n=n_1+n_2$ and writing $n_1=k_1n$, $n_2=k_2n$ $(k_1+k_2=1)$, an asymptotic expansion is given, up to and including the term of order $n^{-1}$, for the joint density function of the sample roots in terms of normal density functions and other "linkage" factors which appear due to the multiple root assumption. This then yields an expansion for the marginal density function of $b_i$. There are, of course, two cases to be considered here; the corresponding population root $\omega_i$ can be either a

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simple or a multiple root. In the latter case it appears extremely difficult to obtain the expansion for the marginal density function of $b_i$ when $\omega_i$ has arbitrary multiplicity, $q$ say. For small $q$, however, the expansion can be readily obtained, and is given for $q=2$ and $q=3$. Asymptotic expansions in the cases of distinct roots and equal roots follow from these expansions.

An alternative approach which works for the extreme latent roots $b_1$ and $b_m$ is suggested in Section 3. It is shown that the distribution functions of $b_1$ and $b_m$ can be expressed in terms of the Gaussian hypergeometric function ${}_2F_1$ of matrix argument. A system of partial differential equations satisfied by this function can be used to expand the distribution functions in terms of normal distribution and density functions.

In Section 4 the expansions for the marginal distributions of the largest latent root, obtained in the previous sections, are examined in the bivariate case, $m=2$.

## 2. Expansions for the joint and marginal density functions

In this section an expansion is given for the joint density function of the latent roots $b_1,\cdots,b_m$ of $S_1S_2^{-1}$ in the case when the smallest root of $\Sigma_1\Sigma_2^{-1}$ is multiple. We assume that

$$\omega_1>\cdots>\omega_k>\omega_{k+1}=\cdots=\omega_m=\omega\,(>0)\,, \qquad m=k+q\,,$$

and put $n=n_1+n_2$. It is convenient to introduce some new notation; put $A_1=n_1S_1$, $A_2=n_2S_2$ and let $a_1>a_2>\cdots>a_m\,(>0)$ denote the latent roots of $A_1A_2^{-1}$. Thus $a_i=b_in_1/n_2$ $(i=1,\cdots,m)$. The (exact) joint density function of $a_1,\cdots,a_m$ can be expressed in the form (see James [10])

$$\text{(2.1)}\qquad \pi^{m^2/2}\Gamma_m(n/2)[\Gamma_m(m/2)\Gamma_m(n_1/2)\Gamma_m(n_2/2)]^{-1}$$
$$\cdot\prod_{i=1}^{m}a_i^{n_1/2-p}\omega_i^{-n_1/2}\prod_{i<j=2}^{m}(a_i-a_j)\,{}_1F_0(n/2\,;\,-\Omega^{-1},A)\,,$$

where $p=(m+1)/2$, $\Gamma_m(a)=\pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma(a-(i-1)/2)$, $\Omega=\mathrm{diag}\,(\omega_1,\cdots,\omega_m)$, $A=\mathrm{diag}\,(a_1,\cdots,a_m)$, and ${}_1F_0$ is a hypergeometric function of two argument matrices. The problem of obtaining asymptotic expansions for the ${}_1F_0$ function in (2.1) has been considered by Chang [3] and further developed by Li, Pillai and Chang [13] and Chattopadhyay and Pillai [4]. An asymptotic expansion for this function has been derived for large $n$ in our one multiple root case in [13]. Throughout this section from now on we write $n_1=k_1n$, $n_2=k_2n$ $(k_1+k_2=1)$. In [13] it is shown that the joint density function of $a_1,\cdots,a_m$ can be expressed as

$$(2.2)\qquad K\prod_{i=1}^{k}[\omega_i^{-k_1n/2+p-1}a_i^{k_1n/2-p}(1+a_i/\omega_i)^{-n/2+p-1}]$$
$$\cdot\prod_{i=k+1}^{m}[\omega^{(-k_1n+k)/2}a_i^{k_1n/2-p}(1+a_i/\omega)^{(-n+k)/2}]$$
$$\cdot\prod_{i=1}^{k}\prod_{\substack{j=1\\i<j}}^{m}[(a_i-a_j)/(\omega_i-\omega_j)]^{1/2}\cdot\prod_{\substack{k+1\\i<j}}^{m}(a_i-a_j)\cdot G\ ,$$

where

$$K=\pi^{q/2}(n/2)^{-k(2m-k-1)/4}\prod_{i=1}^{m}\Gamma((n-i+1)/2)\Big/$$
$$\left[\prod_{i=k+1}^{m}\Gamma((m-i+1)/2)\prod_{j=1}^{2}\prod_{i=1}^{m}\Gamma((k_jn-i+1)/2)\right],$$

$$G=1+(2n)^{-1}\left\{\sum_{i=1}^{k}\sum_{\substack{j=1\\i<j}}^{m}\omega_i\omega_j(1+a_i/\omega_i)(1+a_j/\omega_j)/[(\omega_i-\omega_j)(a_i-a_j)]\right.$$
$$\left.+k[(k-1)(4k+1)/12+(m^2-k^2)/2]\right\}+\cdots,$$

and $p=(m+1)/2$.

Now put $x_i=[n_1n_2/2(n_1+n_2)]^{1/2}(n_2a_i/n_1\omega_i-1)=(k_1k_2n/2)^{1/2}(k_2a_i/k_1\omega_i-1)$ $(i=1,\cdots,m)$. From (2.2) the joint density function of $x_1,\cdots,x_m$ can be expressed in the form

$$(2.3)\qquad \prod_{j=1}^{5}G_j\cdot\prod_{\substack{k+1\\i<j}}^{m}(x_i-x_j)\cdot\left\{1+(2n)^{-1}\left[\sum_{i=1}^{k}\sum_{\substack{j=1\\i<j}}^{m}\omega_i\omega_j/k_1k_2(\omega_i-\omega_j)^2\right.\right.$$
$$\left.\left.+k((k-1)(4k+1)/12+(m^2-k^2)/2)\right]+O(n^{-3/2})\right\},$$

where

$$G_1=\pi^{q/2}\prod_{j=1}^{2}k_j^{k_jmn/2-m(m+1)/4}(n/2)^{-m(m+1)/4}\prod_{i=1}^{m}\Gamma((n-i+1)/2)\Big/$$
$$\left[\prod_{i=k+1}^{m}\Gamma((m-i+1)/2)\prod_{j=1}^{2}\prod_{i=1}^{m}\Gamma((k_jn-i+1)/2)\right],$$

$$G_2=\prod_{i=1}^{m}[1+(k_1k_2n/2)^{-1/2}x_i]^{k_1n/2-p},$$

$$G_3=\prod_{i=1}^{k}[1+(k_1k_2n/2)^{-1/2}k_1x_i]^{-n/2+p-1},$$

$$G_4=\prod_{i=k+1}^{m}[1+(k_1k_2n/2)^{-1/2}k_1x_i]^{(-n+k)/2}$$

and

$$G_5=\prod_{i=1}^{k}\prod_{\substack{j=1\\i<j}}^{m}[1+(k_1k_2n/2)^{-1/2}(\omega_ix_i-\omega_jx_j)/(\omega_i-\omega_j)]^{1/2}.$$

It remains to expand $G_1$, $G_2$, $G_3$, $G_4$ and $G_5$ in (2.3) for large $n$. For example, by expanding the gamma functions for large $n$ it follows that

$$G_1=\left[(2\pi)^{k/2}\prod_{i=k+1}^{m}\sqrt{2}\,\Gamma((m-i+1)/2)\right]^{-1}$$
$$\cdot\,[1-(24k_1k_2n)^{-1}(k_1k_2-1)m(2m^2+3m-1)+O(n^{-2})]\,.$$

The functions $G_2$, $G_3$, $G_4$ and $G_5$ can be easily expanded in terms of $n^{-1/2}$; however these expansions, up to and including the terms of order $n^{-1}$, are quite lengthy and are omitted here. Substituting these expansions in (2.3) gives an expansion for the joint density function of $x_1,\cdots,x_m$. The final result is summarized in the following

THEOREM 2.1. *The joint density function of* $x_i=(k_1k_2n/2)^{1/2}(b_i/\omega_i-1)$ $(i=1,\cdots,m)$, *where* $b_1,\cdots,b_m$ *are the latent roots of* $S_1S_2^{-1}$ *and the latent roots* $\omega_1,\cdots,\omega_m$ *of* $\Sigma_1\Sigma_2^{-1}$ *satisfy* $\omega_1>\cdots>\omega_k>\omega_{k+1}=\cdots=\omega_m\ (=\omega>0)$, *may be expanded for large* $n$ *as*

$$(2.4)\quad \prod_{i=1}^{k}\phi(x_i)\prod_{i=k+1}^{m}[\exp(-x_i^2/2)/\sqrt{2}\,\Gamma((m-i+1)/2)]\prod_{\substack{k+1\\ i<j}}^{m}(x_i-x_j)$$
$$\cdot\left\{1+(2/n)^{1/2}\sum_{i=1}^{m}R_{1i}(x_i)+(2/n)\left[\sum_{i=1}^{m}R_{2i}(x_i)+\sum_{i<j=2}^{m}R_{1i}(x_i)R_{1j}(x_j)\right.\right.$$
$$\left.\left.+\sum_{i=1}^{k}\sum_{\substack{j=1\\ i<j}}^{m}x_ix_j\omega_i\omega_j/2k_1k_2(\omega_i-\omega_j)^2\right]+O(n^{-3/2})\right\},$$

*where* $\phi(\cdot)$ *denotes the standard normal density function*, $n=n_1+n_2$, $n_1=k_1n$, $n_2=k_2n$ $(k_1+k_2=1)$,

$$(2.5)\quad R_{1i}(x)=[1/6(k_1k_2)^{1/2}]\{2(1+k_1)H_3(x)+3[(1+m)k_1+A_i]H_1(x)\}\qquad (i=1,\cdots,k)$$

$$(2.6)\quad =[1/6(k_1k_2)^{1/2}]\{2(1+k_1)H_3(x)+3[1-q+(2+m-q)k_1+A_0]H_1(x)\}\qquad (i=k+1,\cdots,m)\,,$$

$$(2.7)\quad R_{2i}(x)=(1/72k_1k_2)\{4(1+k_1)^2H_6(x)+6[3+(11+2m)k_1+(5+2m)k_1^2$$
$$+2(1+k_1)A_i]H_4(x)+9[4(2+m)k_1+(1+m)(3+m)k_1^2$$
$$+2(3+m)k_1A_i+A_i^2-2B_i]H_2(x)\}\qquad (i=1,\cdots,k)$$

$$(2.8)\quad =(1/144k_1k_2)\{8(1+k_1)^2H_6(x)+12[5-2q+(15+2m-4q)k_1$$
$$+(7+2m-2q)k_1^2+2(1+k_1)A_0]H_4(x)+18[(1-q)(3-q)$$
$$+2(10+3m-mq-7q+q^2)k_1+(8+6m+m^2-2mq$$
$$-6q+q^2)k_1^2+2(1-q+4k_1+mk_1-qk_1)A_0+A_0^2-2B_0]H_2(x)$$
$$+3(1-q)[-(1+4q)+(19+12m-14q)k_1+(5+6m-4q)k_1^2$$
$$12k_1A_0-6B_0]H_0(x)\}\qquad (i=k+1,\cdots,m)\,,$$

$$(2.9)\qquad A_i=\sum_{\substack{j=1\\j\neq i}}^{m}\omega_j/(\omega_i-\omega_j)\ ,\quad B_i=\sum_{\substack{j=1\\j\neq i}}^{m}\omega_j^2/(\omega_i-\omega_j)^2\ ,\qquad (i=1,\cdots,k)\ ,$$

$$A_0=\sum_{l=1}^{k}\omega_l/(\omega-\omega_l)\ ,\quad B_0=\sum_{l=1}^{k}\omega_l^2/(\omega-\omega_l)^2\ ,\qquad q=m-k\ ,$$

*and $H_r(x)$ is the Hermite polynomial of degree $r$ (tabulated to $r=10$ in Kendall and Stuart* [11], p. 155).

By integrating out the variables $x_1,\cdots,x_k$ in the expansion (2.4) an expansion for the joint density function of $x_{k+1},\cdots,x_m$ is easily obtained. Here in carrying out the integration we note the following: It is readily seen that we are involved only with integrating functions of the form,

$$\phi_\alpha(x_i)=x_i^\alpha\exp(-x_i^2/2)/\sqrt{2\pi}\qquad(\alpha=0,1,\cdots)\ ,$$

over the integral domain " $-(k_1k_2n/2)^{1/2}<x_i<+\infty$ " ($i=1,2,\cdots,k$). To calculate using integration by parts, we have only to evaluate the values, $\phi_\alpha(-(k_1k_2n/2)^{1/2})$ ($\alpha=0,1,\cdots$) and $\Phi(-(k_1k_2n/2)^{1/2})$, which appear as parts contributing to the integration from the integral domain " $-(k_1k_2n/2)^{1/2}<x_i$". Here $\Phi(\cdot)$ is the standard normal distribution function, and we note that $\phi_\alpha(-\infty)=0$ ($\alpha=0,1,\cdots$) and $\Phi(-\infty)=0$. It is easily shown that

$$\phi_\alpha(-(k_1k_2n/2)^{1/2})-\phi_\alpha(-\infty)\prec O(n^{-M})\qquad(\alpha=0,1,\cdots)$$

and that, from the asymptotic series, $1-\Phi(u)\cong\phi_0(-u)[u^{-1}-u^{-3}+1\cdot3u^{-5}-\cdots]$, valid for large $u$,

$$\Phi(-(k_1k_2n/2)^{1/2})-\Phi(-\infty)=1-\Phi((k_1k_2n/2)^{1/2})\prec O(n^{-M})\ ,$$

where the symbol $\prec$ means that the left-hand side is of order less than the right-hand side and $M$ is any sufficiently large positive number. Therefore, we can consider the integral domain of $x_i$ to be " $-\infty<x_i<+\infty$ " ($i=1,2,\cdots,k$) in our calculation. This kind of consideration has also been applied in the derivation of the following corollaries.

COROLLARY 1. *The joint density function of $x_{k+1},\cdots,x_m$ where $\omega_1>\cdots>\omega_k>\omega_{k+1}=\cdots=\omega_m$ $(=\omega>0)$, can be expanded for large $n$ as*

$$(2.10)\quad \prod_{i=k+1}^{m}[\exp(-x_i^2/2)/\sqrt{2}\,\Gamma((m-i+1)/2)]\prod_{\substack{k+1\\i<j}}^{m}(x_i-x_j)$$
$$\cdot\left\{1+(2/n)^{1/2}\sum_{i=k+1}^{m}R_{1i}(x_i)+(2/n)\left[\sum_{i=k+1}^{m}R_{2i}(x_i)\right.\right.$$
$$\left.\left.+\sum_{\substack{k+1\\i<j}}^{m}R_{1i}(x_i)R_{1j}(x_j)\right]+O(n^{-3/2})\right\}\ ,$$

*where* $R_{1i}(x_i)$ *and* $R_{2i}(x_i)$ *are given by* (2.6) *and* (2.8) *respectively.*

It is extremely difficult to obtain a general expansion for the marginal density function of $x_i$ ($i=k+1,\cdots,m$) for general $q$ ($=m-k$), except when $q$ is small. In the cases $q=2$ and $q=3$ expansions for the marginal density functions of the "extreme" variables $x_{k+1}$ and $x_m$ have been obtained. By integrating out the other variables in (2.10) we have

COROLLARY 2. *The marginal density function of each of the variables* $x=x_{k+1}$ *and* $y=x_m$, *where* $\omega_1>\cdots>\omega_k>\omega_{k+1}=\cdots=\omega_m$ $(>0)$ $(q=m-k)$, *can be expanded for* $q=2$ *and* $q=3$ *and for large* $n$ *as follows*:

(i) *when* $q=2$;

$$f(x)=f_0(x)+(2/n)^{1/2}f_1(x)+(2/n)f_2(x)+O(n^{-3/2}) \tag{2.11}$$

*and*

$$f(y)=f_0(-y)-(2/n)^{1/2}f_1(-y)+(2/n)f_2(-y)+O(n^{-3/2}) \,, \tag{2.12}$$

*where*

$$f_0(x)=\sqrt{\pi}\,\phi(x)[\phi(x)+x\Phi(x)] \,,$$

$$f_1(x)=\sqrt{\pi}\,\phi(x)[\gamma_1(x)\phi(x)+\gamma_2(x)\Phi(x)] \,,$$

$$f_2(x)=\sqrt{\pi}\,\phi(x)[\delta_1(x)\phi(x)+\delta_2(x)\Phi(x)] \,,$$

$$\gamma_1(x)=C_1x^3+(C_1+C_2)x \,,$$

$$\gamma_2(x)=C_1x^4+C_2x^2-3C_1-C_2 \,,$$

$$\delta_1(x)=D_1x^6+(3D_1+D_2)x^4+(C_1C_2+9D_1+D_2+D_3)x^2+48D_1+8D_2 \\ +2D_3+2D_4$$

*and*

$$\delta_2(x)=D_1x^7+D_2x^5+(-3C_1^2-C_1C_2+D_3)x^3+(-3C_1C_2-C_2^2+15D_1 \\ +3D_2+D_3+2D_4)x \,.$$

(ii) *when* $q=3$;

$$f(x)=g_0(x)+(2/n)^{1/2}g_1(x)+(2/n)g_2(x)+O(n^{-3/2}) \tag{2.13}$$

*and*

$$f(y)=g_0(-y)-(2/n)^{1/2}g_1(-y)+(2/n)g_2(-y)+O(n^{-3/2}) \,, \tag{2.14}$$

*where*

$$g_0(x)=\sqrt{2}\,\phi(x)[(2\pi)^{1/2}\phi(x)\Phi(x)+x\phi(x\sqrt{2})+2^{-1/2}(2x^2-1)\Phi(x\sqrt{2})] \,,$$

$$g_1(x)=\sqrt{2}\,\phi(x)[\xi_1(x)\phi(x)\Phi(x)+\xi_2(x)\phi(x\sqrt{2})+\xi_3(x)\Phi(x\sqrt{2})] \,,$$

$$g_2(x)=\sqrt{2}\,\phi(x)\,[\mu_1(x)\phi(x)\Phi(x)+\mu_2(x)\phi(x\sqrt{2})+\mu_3(x)\Phi(x\sqrt{2})]\ ,$$

$$\xi_1(x)=2(2\pi)^{1/2}[C_1x^3+(3C_1+C_2)x]\ ,$$

$$\xi_2(x)=C_1x^4+(C_1+C_2)x^2-4C_1-C_2\ ,$$

$$\xi_3(x)=2^{-1/2}[2C_1x^5+(2C_2-C_1)x^3-(18C_1+5C_2)x]\ ,$$

$$\mu_1(x)=(2\pi)^{1/2}[4D_1x^6+2(C_1C_2+12D_1+D_2)x^4+(C_2^2+4C_1C_2+54D_1+8D_2+2D_3)x^2-2C_2^2-14C_1C_2+105D_1+23D_2+5D_3+3D_4]\ ,$$

$$\mu_2(x)=(1/8)\,[8D_1x^7+8(3D_1+D_2)x^5+8(3D_1+D_2+D_3)x^3+(-12C_2^2-68C_1C_2+828D_1+148D_2+32D_3+24D_4)x]$$

*and*

$$\mu_3(x)=(1/8\sqrt{2})\,[16D_1x^8+8(2D_2-D_1)x^6+8(-18C_1^2-4C_1C_2-D_2+2D_3)x^4+2(-20C_2^2-92C_1C_2+492D_1+92D_2+20D_3+24D_4)x^2+28C_2^2+196C_1C_2-420D_1-172D_2-40D_3-24D_4]\ .$$

*Here $C_1$ and $C_2$ are the coefficients of $x^3$ and $x$ respectively in $R_{1i}(x)$ given by* (2.6), *and $D_1$, $D_2$, $D_3$ and $D_4$ are the coefficients of $x^6$, $x^4$, $x^2$ and $x^0$ respectively in $R_{2i}(x)$ given by* (2.8) *namely*;

(2.15) $$C_1=[1/3(k_1k_2)^{1/2}]\,(1+k_1)\ ,$$

$$C_2=-[1/2(k_1k_2)^{1/2}]\,(1+q-kk_1-A_0)\ ,$$

$$D_1=(1/18k_1k_2)\,(1+k_1)^2\ ,$$

$$D_2=-(1/12k_1k_2)\,[5+2q+(5-2m+4q)k_1+(3-2k)k_1^2-2(1+k_1)A_0]\ ,$$

$$D_3=(1/8k_1k_2)\,[(1+q)(3+q)-2k(1+q)k_1-k(2-k)k_1^2-2(3+q-kk_1)A_0+A_0^2-2B_0]$$

*and*

$$D_4=(1/48k_1k_2)\,[1-3q-2q^2-(1-3q-2q^2)k_1+(1+3q-6m-2q^2-6mk)k_1^2+12(1+q-k_1-mk_1)A_0-6A_0^2+6(1+q)B_0]\ ,$$

*where $A_0$ and $B_0$ are given by* (2.9).

Putting $k=0$ ($q=m$) in the expansions given in Corollaries 1 and 2, we can obtain expansions for the joint density function of $x_1,\cdots,x_m$ and for the marginal density functions of $x_1$ and $x_m$, in the case when $\omega_1=\cdots=\omega_m\,(>0)$. Thus

COROLLARY 3. *The joint density function of $x_1,\cdots,x_m$, when $\omega_1=\cdots=\omega_m\,(>0)$, has the expansion* (2.10), *with* 1 *replacing $k+1$ and $R_{1i}(x_i)$*

*and* $R_{2i}(x_i)$ *given by* (2.6) *and* (2.8), *with* $m$ *replacing* $q$ *and* $A_0=B_0=0$.

COROLLARY 4. *When* $m=2$ *and* $m=3$, *the marginal density functions of* $x_1$ *and* $x_m$, *where* $\omega_1=\cdots=\omega_m\,(>0)$, *have the expansions* (2.11), (2.12), (2.13) *and* (2.14) *with the constants* $C_1$, $C_2$, $D_1$, $D_2$, $D_3$ *and* $D_4$ *given by* (2.15), *with* $m$ *replacing* $q$, $k=0$, *and* $A_0=B_0=0$.

By integrating out the variables $x_{k+1},\cdots,x_m$ in the expansion (2.4) an expansion for the joint density function of $x_1,\cdots,x_k$ is easily obtained.

COROLLARY 5. *The joint density function of* $x_1,\cdots,x_k$, *where* $\omega_1>\cdots>\omega_k>\omega_{k+1}=\cdots=\omega_m\,(>0)$, *can be expanded for large* $n$ *as*

$$(2.16)\quad \prod_{i=1}^{k}\phi(x_i)\cdot\Big\{1+(2/n)^{1/2}\sum_{i=1}^{k}R_{1i}(x_i)+(2/n)\Big[\sum_{i=1}^{k}R_{2i}(x_i)$$
$$+\sum_{i<j=2}^{k}R_{1i}(x_i)R_{1j}(x_j)+\sum_{i<j=2}^{k}x_ix_j\,\omega_i\omega_j/2k_1k_2(\omega_i-\omega_j)^2\Big]+O(n^{-3/2})\Big\}\ ,$$

*where* $R_{1i}(x_i)$ *and* $R_{2i}(x_i)$ *are given by* (2.5) *and* (2.7) *respectively.*

It is clear that an expansion like (2.16) can be obtained for the joint density function of any subset of the variables $x_1,\cdots,x_k$, where the corresponding population roots $\omega_1,\cdots,\omega_k$ are all simple. The form of the general expansion is obvious. In particular we have

COROLLARY 6. *The marginal density function of* $x_i$, *when* $\omega_i$ *is a simple root of* $\Sigma_1\Sigma_2^{-1}$, *may be expanded for large* $n$ *as*

$$(2.17)\qquad \phi(x_i)\,[1+(2/n)^{1/2}R_{1i}(x_i)+(2/n)R_{2i}(x_i)+O(n^{-3/2})]\ ,$$

*where* $R_{1i}(x_i)$ *and* $R_{2i}(x_i)$ *are given by* (2.5) *and* (2.7) *respectively.*

When $k=m$, (2.4) reduces to the expansion for the joint density function of $x_1,\cdots,x_m$ in the case when all the latent roots of $\Sigma_1\Sigma_2^{-1}$ are distinct ($\omega_1>\cdots>\omega_m>0$).

It is noted that Chikuse [6] investigated asymptotic distributions of the latent roots of the sample covariance matrix, and it may be interesting to compare the results given in this paper with those derived in [6].

Asymptotic moments of $b_i$, when $\omega_i$ is a simple root, can be obtained from (2.17); in particular we obtain

$$(2.18)\quad \mathrm{E}\,(b_i)=\omega_i+[(1+m)k_1+A_i]\omega_i/k_1k_2n+O(n^{-2})\ ,$$

$$(2.19)\quad \mathrm{Var}\,(b_i)=2\omega_i^2/k_1k_2n+2[2(2+m)k_1+(1+m)k_1^2+2k_1A_i-B_i]\omega_i^2/(k_1k_2n)^2+O(n^{-3})\ ,$$

$$(2.20)\quad \kappa_3(b_i)=8(1+k_1)\omega_i^3/(k_1k_2n)^2+O(n^{-3})$$

and

(2.21) $\kappa_4(b_i)=48(1+3k_1+k_1^2)\omega_i^4/(k_1k_2n)^3+O(n^{-4})$

where $\kappa_3(b_i)$ and $\kappa_4(b_i)$ are the third and fourth cumulants of $b_i$, and $A_i$ and $B_i$ are given by (2.9). From the expansion for the joint density function of $x_i$ and $x_j$, where $\omega_i$ and $\omega_j$ are simple, we obtain

$$\text{Cov}\,(b_i, b_j)=2\omega_i^2\omega_j^2/(\omega_i-\omega_j)^2(k_1k_2n)^2+O(n^{-3})\ . \tag{2.22}$$

We note from (2.18) that, as an estimate of $\omega_i$, $b_i$ has a bias term of order $n^{-1}$. A "better" estimate of $\omega_i$ having a bias of order $n^{-2}$ is

$$\check{\omega}_i=b_i-b_i\left[\sum_{\substack{j=1\\ j\neq i}}^{m} b_j/(b_i-b_j)-k_1(m+1)\right]\Big/k_1k_2n\ .$$

It is easily seen that the expansion (2.17) is the Edgeworth expansion obtained by substituting the expressions (2.18)–(2.21) for the first four moments of $b_i$ in the general Edgeworth expansion form given in Kendall and Stuart [11], p. 164.

## 3. Distributions of the extreme latent roots

In this section we derive exact expressions for the marginal distribution functions of the extreme roots $b_1$ and $b_m$ of $S_1S_2^{-1}$, valid when the corresponding population roots $\omega_1$ and $\omega_m$ of $\Sigma_1\Sigma_2^{-1}$ are simple. An alternative approach is then suggested, valid for deriving asymptotic expansions for the distribution functions of the extreme roots when the corresponding population extreme roots are simple. Now $A_1=n_1S_1$ and $A_2=n_2S_2$ have independent Wishart distributions $W_m(n_1, \Sigma_1)$ and $W_m(n_2, \Sigma_2)$ respectively.

The largest latent root $a_1$ of $A_1A_2^{-1}$ is considered first. Since the events "$a_1<y$", "$0<A_1A_2^{-1}<yI_m$" and "$0<A_1<yA_2$, $A_2>0$" are equivalent, we have

$$\begin{aligned}\text{P}\,(a_1<y)=&\prod_{j=1}^{2}\left[\Gamma_m\left(\frac{1}{2}n_j\right)\det\,(2\Sigma_j)^{n_j/2}\right]^{-1}\\ &\cdot\int_{A_2>0}\text{etr}\left(-\frac{1}{2}\Sigma_2^{-1}A_2\right)\det A_2^{n_2/2-p}\\ &\cdot\int_{0<A_1<yA_2}\text{etr}\left(-\frac{1}{2}\Sigma_1^{-1}A_1\right)\det A_1^{n_1/2-p}dA_1dA_2\ ,\end{aligned} \tag{3.1}$$

where, throughout this section, $p=(m+1)/2$. Put

$$I_1=\int_{0<A_1<yA_2}\text{etr}\left(-\frac{1}{2}\Sigma_1^{-1}A_1\right)\det A_1^{n_1/2-p}dA_1\ . \tag{3.2}$$

Making the transformation $T=(yA_2)^{-1}A_1$ it is easily seen that (3.2) becomes

$$(3.3)\qquad I_1=\Gamma_m\left(\frac{1}{2}n_1\right)\Gamma_m(p)\left[\Gamma_m\left(\frac{1}{2}n_1+p\right)\right]^{-1}\det(yA_2)^{n_1/2}$$
$$\cdot\ {}_1F_1\left(\frac{1}{2}n_1;\frac{1}{2}n_1+p;-\frac{1}{2}y\Sigma_1^{-1}A_2\right)$$

where ${}_1F_1$ is a confluent hypergeometric function of matrix argument (see Herz [9], Constantine [7]). From (3.1), (3.2), (3.3) and the general system via integrals, defined by Herz [9], of hypergeometric functions ${}_qF_r$, it is shown that

$$(3.4)\qquad \mathrm{P}(a_1<y)=\Gamma_m(p)\Gamma_m\left(\frac{1}{2}(n_1+n_2)\right)\left[\Gamma_m\left(\frac{1}{2}n_2\right)\Gamma_m\left(\frac{1}{2}n_1+p\right)\right]^{-1}$$
$$\cdot\det[y(\Sigma_1\Sigma_2^{-1})^{-1}]^{n_1/2}{}_2F_1\left(\frac{1}{2}(n_1+n_2),\frac{1}{2}n_1;\frac{1}{2}n_1+p;\right.$$
$$\left.-y(\Sigma_1\Sigma_2^{-1})^{-1}\right).$$

Since the event "$b_1<y$" is equivalent to the event "$a_1<n_1y/n_2$", we have

LEMMA 3.1. *The distribution function of the largest root $b_1$ of $S_1S_2^{-1}$ is given by*

$$(3.5)\qquad \mathrm{P}(b_1<y)=\Gamma_m(p)\Gamma_m\left(\frac{1}{2}(n_1+n_2)\right)\left[\Gamma_m\left(\frac{1}{2}n_2\right)\Gamma_m\left(\frac{1}{2}n_1+p\right)\right]^{-1}$$
$$\cdot\det[(n_1y/n_2)(\Sigma_1\Sigma_2^{-1})^{-1}]^{n_1/2}{}_2F_1\left(\frac{1}{2}(n_1+n_2),\right.$$
$$\left.\frac{1}{2}n_1;\frac{1}{2}n_1+p;-(n_1y/n_2)(\Sigma_1\Sigma_2^{-1})^{-1}\right).$$

We now consider the distribution of the smallest root $a_m$ of $A_1A_2^{-1}$. The events "$a_m>y$", "$A_1A_2^{-1}>yI_m$" and "$0<A_2<y^{-1}A_1$, $A_1>0$" are equivalent. It follows that $\mathrm{P}(a_m>y)$ can be obtained from $\mathrm{P}(a_1<y)$ given by (3.4) by replacing $n_1$, $n_2$, $y$, $\Sigma_1$ and $\Sigma_2$ by $n_2$, $n_1$, $y^{-1}$, $\Sigma_2$ and $\Sigma_1$ respectively. Noting that the event "$b_m>y$" is equivalent to the event "$a_m>n_1y/n_2$", we have

LEMMA 3.2. *The distribution function of the smallest root $b_m$ of $S_1S_2^{-1}$ is given by*

$$(3.6)\qquad \mathrm{P}(b_m>y)=\Gamma_m(p)\Gamma_m\left(\frac{1}{2}(n_1+n_2)\right)\left[\Gamma_m\left(\frac{1}{2}n_1\right)\Gamma_m\left(\frac{1}{2}n_2+p\right)\right]^{-1}$$

$$\cdot \det [(n_2/n_1 y)\Sigma_1\Sigma_2^{-1}]^{n_2/2} {}_2F_1\Big(\frac{1}{2}(n_1+n_2),$$
$$\frac{1}{2}n_2; \frac{1}{2}n_2+p; -(n_2/n_1 y)\Sigma_1\Sigma_2^{-1}\Big) .$$

A system of partial differential equations satisfied by the ${}_2F_1$ function has been given by Muirhead [14]. We now assume that the respective population extreme roots $\omega_1$ and $\omega_m$ are simple and write $n=n_1+n_2$, $n_1=k_1n$, $n_2=k_2n$ $(k_1+k_2=1)$. Then, starting from the above system, we can derive asymptotic expansions for large $n$ for the distribution functions of the "standardized" extreme roots $x_i=(k_1k_2n/2)^{1/2}(b_i/\omega_i-1)$ $(i=1, m)$. The detailed calculation is found in Chikuse [5] and is omitted here. The resulting expansions agree with the expansions for the marginal density functions of $x_1$ and $x_m$, given by (2.17).

## 4. Numerical comparison

To examine the expansions obtained in the previous sections, we compute approximate powers of the 0.05-level test of $\Sigma_1=\Sigma_2$ based on the largest root $b_1$ in the bivariate case, $m=2$. We follow previous notation. The expansion for the distribution function of $x_1=(k_1k_2n/2)^{1/2}\cdot(b_1/\omega_1-1)$ can be obtained from (2.17) for the case, $\omega_1>\omega_2$, and from (2.11), in connection with Corollary 4, for the case, $\omega_1=\omega_2$. These are respectively given as follows:

$$\text{(4.1)}\quad \begin{aligned} \mathrm{P}(x_1<x)=&\Phi(x)-(18k_1k_2n)^{-1/2}\phi(x)\{2(1+k_1)x^2-2+7k_1\\ &+3\omega_2/(\omega_1-\omega_2)\}-(36k_1k_2n)^{-1}\phi(x)\{4(1+k_1)^2x^5\\ &-2[11-5k_1-7k_1^2-6(1+k_1)\omega_1/(\omega_1-\omega_2)]x^3\\ &+3[2-2k_1+11k_1^2-6(2-3k_1)\omega_1/(\omega_1-\omega_2)\\ &-3\omega_1^2/(\omega_1-\omega_2)^2]x\}+O(n^{-3/2})\end{aligned}$$

and

$$\text{(4.2)}\quad \begin{aligned} \mathrm{P}(x_1<x)=&-\sqrt{\pi}\,\phi(x)\Phi(x)+\Phi(x\sqrt{2})-(36k_1k_2n)^{-1/2}\\ &\cdot[(2\pi)^{1/2}x\phi(x)\Phi(x)+\phi(x\sqrt{2})][2(1+k_1)x^2+3(-1+2k_1)]\\ &-(36k_1k_2n)^{-1}\{\sqrt{\pi}\,\phi(x)\Phi(x)[4(1+k_1)^2x^6+6(-5-k_1+k_1^2)x^4\\ &+9(3-4k_1)x^2+3(-2+11k_1-11k_1^2)]+2^{-1/2}\phi(x\sqrt{2})\\ &\cdot[4(1+k_1)^2x^5+2(-13+k_1+5k_1^2)x^3+3(5-8k_1+8k_1^2)x]\}\\ &+O(n^{-3/2}) .\end{aligned}$$

The approximate powers are computed from these expansions and compared in Table I with exact powers given by Pillai and Al-Ani [17]. Here upper 5% points of the distribution of $b_1$ were obtained from tables in Pillai [16]. The agreement is seen to be quite good, except in the case when $\omega_1$ is close to $\omega_2$ with $\omega_1 \neq \omega_2$. A decrease in accuracy in such situations is to be expected since this is near the case when $\omega_1$ is not a simple root and the limiting distribution is non-normal.

Table I Powers of the 0.05-level test of $\Sigma_1=\Sigma_2$ based on the largest root against several alternatives
(Columns (1), (2) and (3) are values of limiting term, $n^{-1/2}$ term and $n^{-1}$ term in expansions, and columns (4) and (5) are (1)+(2)+(3) and exact values.)

| $n_1$ | $n_2$ | $\omega_1$ | $\omega_2$ | (1) | (2) | (3) | (4) | (5) |
|---|---|---|---|---|---|---|---|---|
| 5 | 33 | 1.0 | 1.0 | 0.001 | 0.009 | 0.036 | 0.046 | 0.050 |
| | | 1.05 | 1.05 | 0.002 | 0.017 | 0.052 | 0.071 | 0.061 |
| | | 1.5 | 1.0 | 0.024 | 0.087 | 0.013 | 0.124 | 0.125 |
| | | 1.5 | 1.333 | 0.024 | 0.199 | −0.262 | −0.039 | 0.175 |
| | | 4.0 | 1.0 | 0.570 | 0.000 | 0.002 | 0.572 | 0.585 |
| 7 | 33 | 1.0 | 1.0 | 0.001 | 0.009 | 0.032 | 0.042 | 0.050 |
| | | 1.05 | 1.05 | 0.002 | 0.017 | 0.047 | 0.066 | 0.062 |
| | | 1.5 | 1.0 | 0.029 | 0.089 | 0.011 | 0.129 | 0.137 |
| | | 1.5 | 1.333 | 0.029 | 0.207 | −0.229 | 0.007 | 0.193 |
| | | 4.0 | 1.0 | 0.638 | 0.019 | 0.004 | 0.661 | 0.684 |

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## REFERENCES

[1] Anderson, T. W. (1951). The asymptotic distribution of certain characteristic roots and vectors, *Proc. 2nd Berkeley Symp. Math. Statist. Prob.*, 103-130.

[2] Anderson, T. W. (1963). Asymptotic theory for principal component analysis, *Ann. Math. Statist.*, **34**, 122-148.

[3] Chang, T. C. (1970). On an asymptotic representation of the distribution of the characteristic roots of $S_1S_2^{-1}$ when roots are not all distinct, *Ann. Math. Statist.*, **41**, 440-445.

[4] Chattopadhyay, A. K. and Pillai, K. C. S. (1973). Asymptotic expansions for the distributions of characteristic roots when the parameter matrix has several multiple roots, *Multivariate Analysis* III (P. R. Krishnaiah, ed.), Academic Press, New York, 117-127.

[5] Chikuse, Y. (1974). Asymptotic expansions for the distributions of the latent roots of two matrices in multivariate analysis, Ph.D. thesis, Yale University.

[6] Chikuse, Y. (1976). Asymptotic distributions of the latent roots of the covariance matrix with multiple population roots, *J. Multivariate Analysis*, **6**, 237-249.

[7] Constantine, A. G. (1963). Some noncentral distribution problems in multivariate analysis, *Ann. Math. Statist.*, **34**, 1270-1285.

[8] Girshick, M. A. (1939). On the sampling theory of roots of determinantal equations, *Ann. Math. Statist.*, **10**, 203-224.

[9] Herz, C. S. (1955). Bessel functions of matrix argument, *Ann. Math.*, **61**, 474-523.

[10] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples, *Ann. Math. Statist.*, **35**, 475-501.

[11] Kendall, M. G. and Stuart, A. (1969). *The Advanced Theory of Statistics*, Vol. 1, Hafner, New York.

[12] Khatri, C. G. (1967). Some distribution problems connected with the characteristic roots of $S_1S_2^{-1}$, *Ann. Math. Statist.*, **38**, 944-948.

[13] Li, H. C., Pillai, K. C. S. and Chang, T. C. (1970). Asymptotic expansions for distributions of the roots of two matrices from classical and complex Gaussian populations, *Ann. Math. Statist.*, **41**, 1541-1556.

[14] Muirhead, R. J. (1970). Systems of partial differential equations for hypergeometric functions of matrix argument, *Ann. Math. Statist.*, **41**, 991-1001.

[15] Pillai, K. C. S. (1955). Some new test criteria in multivariate analysis, *Ann. Math. Statist.*, **26**, 117-121.

[16] Pillai, K. C. S. (1956). On the distribution of the largest or the smallest root of a matrix in multivariate analysis, *Biometrika*, **43**, 122-127.

[17] Pillai, K. C. S. and Al-Ani, S. (1970). Power comparisons of tests of equality of two covariance matrices based on individual characteristic roots, *J. Amer. Statist. Ass.*, **65**, 438-446.