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AN APPROXIMATION FORMULA $L_q \simeq \alpha \cdot \rho^\beta/(1-\rho)$

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1. Introduction

For $GI/G/1$ queueing systems, many analytical results have been published by several authors. From the practical point of view, however, the formula of a mean queue length, for example, is not near at hand in spite of its voluminous results. This is because the analysis of the system depends on the Wiener-Hopf type integral equation method. For the practical use, it is necessary to estimate the mean queue length easily by using numerical tables or still better by a simple formula though the estimation is not strict.

Recently, Page [5] has proposed a well-going formula for the mean waiting time using mean waiting times for $M/M/1$, $M/D/1$ and $D/M/1$ queueing systems. In this paper, we advance this approximation method and derive a new approximation formula by which one can estimate the mean queue length without using any mathematical tables.

For $GI/G/s$ queueing systems, analysis becomes much more complicated than $GI/G/1$ case and it is very difficult to evaluate exact value of a mean queue length for the given $GI/G/s$ queueing system. Recently, Mori [4] summed up inequalities on many subclasses of $GI/G/s$ queueing systems and they seem to give good characteristics for the mean waiting time. In this paper, we propose an approximation formula for the mean queue length and verify it not analytically but numerically for some examples. For $M/M/s$ queueing system, the mean queue length is expressed explicitly but its calculation is not easy. Here we derive the simple formula $L_q \simeq \rho^{\sqrt{2(s+1)}}/(1-\rho)$ from the careful observation of the ρ-L_q curves for various s values. Finally, we conjecture the approximation formula $L_q \simeq ((C_a^2+C_s^2)/2)(\rho^{\sqrt{2(s+1)}}/(1-\rho))$ from the analogy in the case of single server queueing systems.

All these formulae are examined numerically and are assured to be useful practically.

2. Approximation formulae

An approximation formula for the mean waiting time W proposed by Page [5] is as follows:

$$W \simeq C_a^2 C_s^2 W_{M/M} + C_a^2(1-C_s^2)W_{M/D} + C_s^2(1-C_a^2)W_{D/M} \tag{2.1}$$

where C_a and C_s are coefficients of variation (c.v.'s) of an interarrival time and a service time distribution functions (d.f.'s), respectively, and $W_{M/M}$, $W_{M/D}$ and $W_{D/M}$ are mean waiting times for $M/M/1$, $M/D/1$ and $D/M/1$ queueing systems, respectively. This formula is very successful for any $GI/G/1$ queueing systems in which c.v.'s of both an interarrival time and a service time d.f.'s are less than or equal to unity. If an arrival process is Poissonian, i.e. $M/G/1$ queueing system, the right-hand side of (2.1) becomes as follows:

$$C_s^2 W_{M/M} + (1-C_s^2)W_{M/D} = \frac{1+C_s^2}{2}\,\frac{\rho^2}{1-\rho}\,.$$

This is just the same as the formula given by Pollaczek Khinchin and Kendall, that is to say, the approximation formula (2.1) is exact for $M/G/1$ queueing systems.

For $D/M/1$ queueing system, the mean waiting time $W_{D/M}$ cannot be evaluated in a closed form, so we go this approximation step still further. $W_{D/M}$ and $W_{M/D}$ for various values are calculated in Table 2.1.

Table 2.1. Difference between $W_{D/M}$ and $W_{M/D}$ with $\mu=1$

ρ	$W_{D/M}$	$W_{M/D}$	ρ	$W_{D/M}$	$W_{M/D}$
0.35	0.076	0.269	0.70	0.876	1.167
0.40	0.120	0.333	0.75	1.203	1.500
0.45	0.179	0.409	0.80	1.695	2.000
0.50	0.255	0.500	0.85	2.521	2.833
0.55	0.353	0.611	0.90	4.185	4.500
0.60	0.479	0.750	0.95	9.204	9.500
0.65	0.647	0.929			

From this table, we can see that $W_{D/M}$ is well approximated by $W_{M/D} - 0.3/\mu$ in the wide range of $\rho<1$. We may use (2.2) in place of (2.1).

$$W \simeq C_a^2 C_s^2 W_{M/M} + C_a^2(1-C_s^2)W_{M/D} + C_s^2(1-C_a^2)(W_{M/D}-0.3/\mu)\,. \tag{2.2}$$

For the dimensionless property, we treat a mean queue length $L_q=\lambda W$ instead of W. Now we multiply λ to both sides of (2.2) and substitute $(1/2)W_{M/M}$ for $W_{M/D}$ and $\rho^2/(1-\rho)$ for $\lambda W_{M/M}$, we have

(2.3) $$L_q \simeq \frac{C_a^2+C_s^2}{2}\,\frac{\rho^2}{1-\rho}-0.3\rho C_s^2(1-C_a^2)\ .$$

The second term in (2.3) is less than 0.3 and we neglect this term for extreme simplicity, giving

(2.4) $$L_q \simeq \frac{C_a^2+C_s^2}{2}\,\frac{\rho^2}{1-\rho}\equiv \tilde{L}_q\ .$$

This is our approximation formula and this new formula approximates L_q fairly well for high values of ρ. From the very nature of things the approximation is not good for $D/M/1$ queueing system but (2.4) is still satisfactory in a practical sense.

In 1968, Marshall [3] proved some inequalities about the mean waiting time for some subclasses of $GI/G/1$ queueing systems. The result used in this paper is as follows:

DEFINITION. A random variable X is said to have IFR property iff a d.f. $F(t)$ of X has an increasing failure rate, i.e. for any dt, $(F(t+dt)-F(t))/(1-F(t))$ is an increasing function for all $t>0$ s.t. $F(t) \neq 1$.

LEMMA. *For a single server queueing system with IFR arrival process with* $\rho<1$,

(2.5) $$L_q \geqq \frac{C_a^2+\rho^2 C_s^2}{2(1-\rho)}-\frac{C_a^2+\rho}{2}\equiv \underline{L}_q\ .$$

Using this lemma, we prove our approximation formula to give good result for this subclass.

$$\begin{aligned}\tilde{L}_q-\underline{L}_q &= \frac{C_a^2+C_s^2}{2}\,\frac{\rho^2}{1-\rho}-\frac{C_a^2+\rho^2C_s^2}{2(1-\rho)}+\frac{C_a^2+\rho}{2}\\ &= \frac{-C_a^2(1-\rho^2)+C_a^2+\rho-\rho C_a^2-\rho^2}{2(1-\rho)}\\ &= \frac{\rho}{2}(1-C_a^2)\leqq \frac{1}{2}(1-C_a^2)\qquad (\because\ C_a\leqq 1)\ .\end{aligned}$$

On the other hand

$$\bar{L}_q-\tilde{L}_q=\frac{C_a^2+\rho^2C_s^2}{2(1-\rho)}-\frac{C_a^2+C_s^2}{2}\,\frac{\rho^2}{1-\rho}=\frac{C_a^2(1-\rho^2)}{2(1-\rho)}=\frac{1+\rho}{2}C_a^2\leqq C_a^2$$

where $\bar{L}_q$ is an upper bound of L_q for all $GI/G/1$ queueing systems, which was proved by Kingman [1]. These relations show that $\tilde{L}_q$ differs from the true value at most one and in reality this difference is less than 1/2 in many such systems. Since the assumption that an arrival process has IFR property is reasonable in many cases, our formula (2.4) may be said to have a generality.

For many server queueing systems, there is no analytical results to estimate the mean queue length in general. First, we calculate the mean queue length for $M/M/s$ queueing system using the elementary queueing theory, i.e.

$$L_q = L_q(\rho) = \frac{s^s \rho^{s+1}}{s!(1-\rho)^2} p_0 \tag{2.6}$$

where $p_0 = \left\{ \sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} \right\}^{-1}$ and $\rho = \lambda/s\mu$. From the analogy in the case of $M/M/1$ queueing system, we will approximate L_q by $\rho^\beta/(1-\rho)$ for some constant $\beta = \beta(s)$ which is a function of s. To determine β for some s, we calculate $L_q = L_q(\rho)$ for $\rho = 0.80(0.01)0.99$ and for each ρ, β is set to $\log(L_q(\rho)(1-\rho))/\log(\rho)$. Then calculate

$$A(\beta)^2 = \sum_{\rho=0.80(0.01)0.99} \left(L_q(\rho) - \frac{\rho^\beta}{1-\rho} \right)^2$$

for each β and let $\hat{\beta}$ be such β that minimizes $A(\beta)^2$. That is to say, $\hat{\beta}$ is a least square estimator in some sense. In the next section, some numerical results of these procedures will be shown. From these results, we can get $\sqrt{2(s+1)}$ as the approximation form of $\beta(s)$. Now, we establish the following approximation formula of the mean queue length for $M/M/s$ queueing system.

$$L_q \simeq \frac{\rho^{\sqrt{2(s+1)}}}{1-\rho} . \tag{2.7}$$

Considering the exact formula for the mean queue length, this simplicity is worthy of notes. This simple formula is easy to calculate without any numerical table. Testing the formula for many s and ρ values we conclude that our attempt is succeeded in that the difference between the approximated value and the true value is remarkably small.

For $M/G/s$ queueing systems, Lee and Longton [2] has derived the following approximation formula for the mean waiting time $W_{M/G/s}$.

$$W_{M/G/s} \simeq \frac{1+C_s^2}{2} W_{M/M/s} \tag{2.8}$$

where $W_{M/M/s}$ is a mean waiting time for $M/M/s$ queueing system. Combining two formula (2.7) and (2.8), we have the following formula for the mean queue length for $M/G/s$ queueing system which resembles to the single server case.

$$L_q \simeq \frac{1+C_s^2}{2} \frac{\rho^{\sqrt{2(s+1)}}}{1-\rho} . \tag{2.9}$$

The justifiability of this formula will be examined numerically in the next section at $\rho=0.9$ for $M/D/s$ queueing system.

Finally, we conjecture the following approximation formula for $GI/G/s$ queueing system where c.v.'s distributions included are all less than or equal to unity from the analogy in the case of single server queueing systems.

$$L_q \simeq \frac{C_a^2+C_s^2}{2} \frac{\rho^{\sqrt{2(s+1)}}}{1-\rho} . \tag{2.10}$$

Especially for $s=1$, (2.10) coinsides with (2.4). The correctness of this formula has not examined but for some special systems such as $D/M/s$, $M/D/s$ and $E_2/E_2/s$ as yet. But results of these comparisons will show good consequence which will be given in the next section.

3. Numerical examples

To demonstrate the superiority of our formula proposed in the preceding section, we give some numerical examples in this section.

For $GE_j/E_k/1$ queueing system, i.e. the interarrival time d.f. is the Pearson type III d.f. (generalization of the Erlang d.f.) where the square of the c.v. of the d.f. is $1/j$, the service time d.f. is the k-Erlang d.f. and there is a single service facility, the mean waiting time is evaluated numerically and its table for various j and k values are shown (see Page [5]). The mean queue length can be obtained from this table by multiplying by λ. Now we compare this exact values and approximated values derived from (2.4). Table 3.1 shows this comparison together with the approximated values by (2.1). According to this table,

Table 3.1. Mean queue length for $GE_j/E_k/1$ queueing system

	$1/j$ \ k	1	2	5	10
		1.538	0.780	0.313	0.212
	0.1	1.760	0.960	0.480	0.320
		1.540	0.850	0.436	0.298
		1.722	0.956	0.511	0.369
	0.2	1.920	1.120	0.640	0.480
		1.725	1.022	0.601	0.460
$\rho=0.8$		2.275	1.492	1.028	0.875
	0.5	2.400	1.600	1.120	0.960
		2.278	1.539	1.096	0.948
		2.830	2.036	1.560	1.402
	0.8	2.880	2.080	1.600	1.440
		2.831	2.056	1.590	1.435

Table 3.1. (Continued)

	$1/j$ \ k	1	2	5	10
$\rho=0.9$	0.1	4.194 4.455 4.200	2.214 2.430 2.302	1.051 1.215 1.164	0.674 0.810 0.785
	0.2	4.628 4.860 4.633	2.640 2.835 2.722	1.464 1.620 1.575	1.078 1.215 1.192
	0.5	5.929 6.075 5.933	3.923 4.050 3.979	2.726 2.835 2.807	2.328 2.430 2.413
	0.8	7.232 7.290 7.233	5.213 5.265 5.237	4.003 4.050 4.039	3.601 3.645 3.639

Three figures in each column show the true value, the approximated value by (2.4) and that by (2.1) from above.

Table 3.2. $\hat{\beta}$ and $\sqrt{2(s+1)}$

s	$\hat{\beta}$	$\sqrt{2(s+1)}$	s	$\hat{\beta}$	$\sqrt{2(s+1)}$
1	2.0	2.0	8	4.339	4.243
2	2.512	2.449	9	4.568	4.472
3	2.915	2.828	10	4.786	4.690
4	3.260	3.162	15	5.724	5.657
5	3.560	3.464	20	6.543	6.481
6	3.839	3.742	50	10.036	10.100
7	4.097	4.0	100	14.028	14.213

Table 3.3. L_q and $\tilde{L}_q$ for $M/M/s$ queueing system

s \ ρ		0.6	0.7	0.8	0.85	0.9	0.95
2	L_q	0.675	1.345	2.844	4.426	7.674	17.587
	$\tilde{L}_q$	0.715	1.391	2.895	4.477	7.725	17.638
4	L_q	0.431	1.000	2.386	3.906	7.090	16.937
	$\tilde{L}_q$	0.497	1.079	2.469	3.988	7.166	17.005
6	L_q	0.295	0.784	2.071	3.536	6.661	16.446
	$\tilde{L}_q$	0.370	0.878	2.170	3.629	6.742	16.507
10	L_q	0.152	0.517	1.637	3.002	6.018	15.686
	$\tilde{L}_q$	0.228	0.626	1.756	3.111	6.101	15.723
20	L_q	0.036	0.218	1.024	2.182	4.957	14.353
	$\tilde{L}_q$	0.091	0.330	1.177	2.325	5.052	14.344
50	L_q	0.001	0.026	0.348	1.075	3.275	11.953
	$\tilde{L}_q$	0.014	0.091	0.525	1.291	3.450	11.914

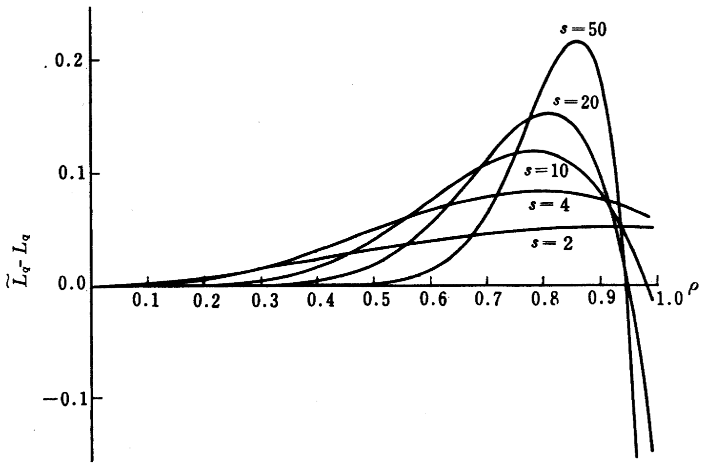

Fig. 3.1. The difference between L_q and $\tilde{L}_q$ for $M/M/s$.

the approximation formula (2.4) always gives larger value than the formula (2.1). This is because the formula (2.4) is obtained from (2.1) in replacing $W_{D/M}$ by $W_{M/D}$ which is larger than $W_{D/M}$. Considering that the difference is not so large and one can calculate the mean queue length easily without any table, we conclude that our new formula (2.4) is promising.

For $M/M/s$ queueing system, we show first, β and $\sqrt{2(s+1)}$ at Table 3.2. For almost all s values $\sqrt{2(s+1)}$ is less than β, which results that our formula (2.7) will estimate L_q slightly larger than the true value, but the difference between the estimated value $\tilde{L}_q$ and L_q is very small. For various s and ρ values, L_q and $\tilde{L}_q$ are calculated in Table 3.3 and

Table 3.4. $\tilde{L}_q$ and L_q for $GI/G/s$ queueing systems with $\rho=0.9$

s	$\tilde{L}_q$	L_q		
		$M/D/s$	$D/M/s$	$E_2/E_2/s$
2	3.86	3.86	3.48	3.69
3	3.71	3.72	3.27	3.51
4	3.58	3.60	3.10	3.36
5	3.47	3.50	2.96	3.24
6	3.37	3.40	2.83	3.13
7	3.28	3.32	2.72	3.03
8	3.20	3.24	2.61	2.94
9	3.12	3.17	2.52	2.85
10	3.05	3.10	2.43	2.77

$\tilde{L}_q$ values are common for all three systems.

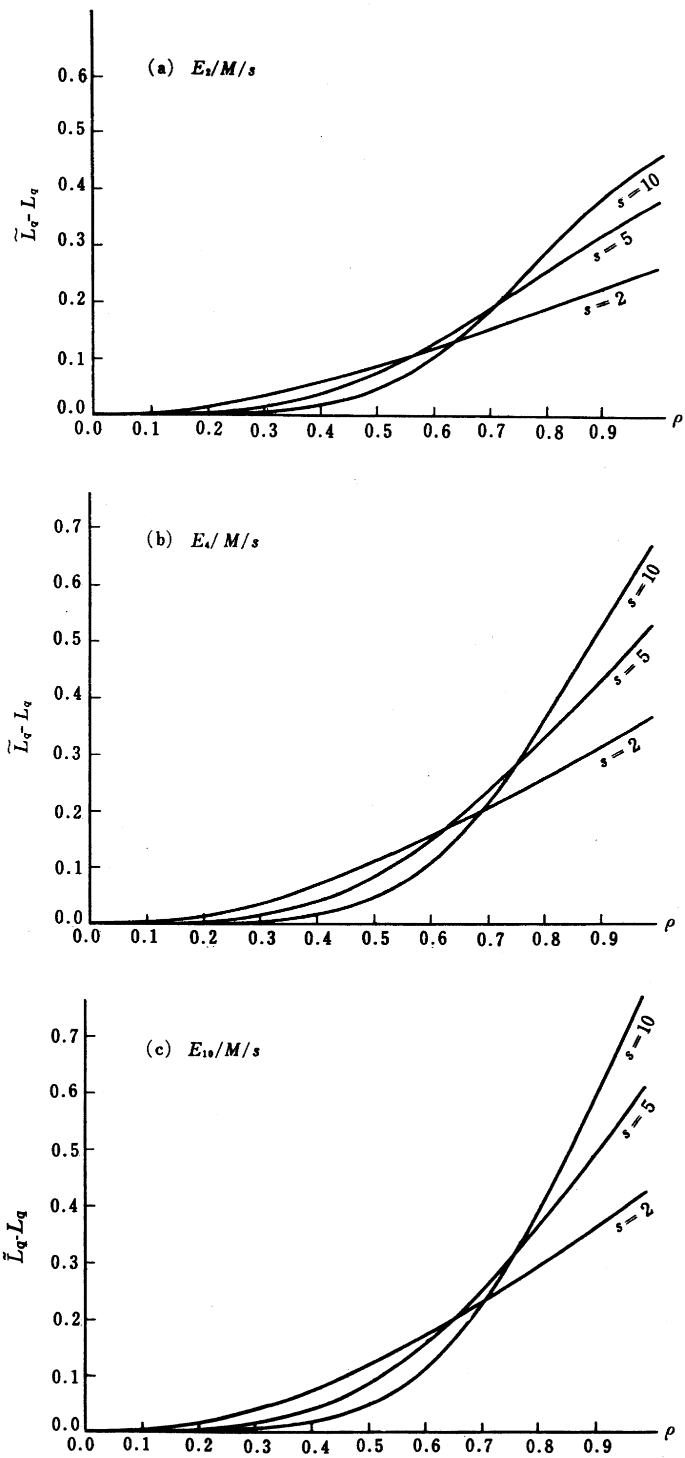

Fig. 3.2. The difference between L_q and $\tilde{L}_q$ for $E_k/M/s$.

the differences between L_q and $\tilde{L}_q$ are illustrated in Fig. 3.1. According to this figure, the approximation is surprisingly good.

For $M/D/s$, $D/M/s$ and $E_2/E_2/s$ queueing systems, we can calculate the mean queue length numerically using equilibrium equations method. For the practical use, we only compare L_q and $\tilde{L}_q$ for $\rho=0.9$ and these values are put together into Table 3.4. According to this table our conjectured formula (2.10) is not bad and is useful practically.

Finally, for $E_k/M/s$ queueing systems, the mean queue length L_q is calculated numerically for various k and s, and the differences between L_q and $\tilde{L}_q$ are illustrated in Fig. 3.2. One can deduce the same conclusion as above from these figures.

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