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ON IDENTICALLY DISTRIBUTED LINEAR STATISTICS*

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1. Introduction

1.1. In [3], [4] Linnik gave the following characterization of the normal law; if $X_1 \not\equiv 0$ and X_2 are independent identically distributed random variables such that $a_1X_1+a_2X_2$ has the same distribution as X_1 where a_1 and a_2 are fixed non-zero real numbers satisfying $a_1^2+a_2^2=1$, then X_1 is normal with mean zero.

This result was extended by Shimizu [9] who gave a complete description of the characteristic functions of random variables with the following property:

(1) X_1 has the same distribution as

$$\sum_{i=1}^{p} a_iX_i - \sum_{i=p+1}^{n} a_iX_i$$

where $X_1, \cdots, X_n$ are independent identically distributed random variables and $a_1, \cdots, a_n$ are fixed real numbers satisfying $0<a_i<1$, $i=1, \cdots, n$.

If we denote the characteristic function of X_1 by φ then (1) is equivalent to the functional equation

$$\varphi(t)=\prod_{i=1}^{p} \varphi(a_it) \prod_{i=p+1}^{n} \varphi(-a_it) , \qquad -\infty<t<\infty . \tag{2}$$

The more general functional equation

$$\varphi(t)=\prod_{i=1}^{p} \varphi^{\gamma_i}(a_it) \prod_{i=p+1}^{n} \varphi^{\gamma_i}(-a_it) , \qquad -\infty<t<\infty , \tag{3}$$

where $\gamma_i>0$, $i=1, \cdots, n$ was considered by Ramachandran and Rao [7], [8] who gave a complete description of the characteristic functions which satisfy (3). In [8] they also considered the case of an infinite product

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(4) $$\varphi(t)=\prod_{i=1}^{\infty}\varphi^{\gamma_{2i}}(a_{2i}t)\prod_{i=1}^{\infty}\varphi^{\gamma_{2i-1}}(-a_{2i-1}t)\ , \qquad -\infty<t<\infty\ ,$$

where $\gamma_i \geqq 0$ and $0<a_i<1$. Their results in this case were not complete as they imposed the following further conditions on φ, $(\gamma_i)_1^\infty$ and $(a_i)_1^\infty$;

(5) $$\varphi(t) \text{ does not vanish}\ ,$$

(6) $$\lim_{i\to\infty} a_i=0$$

and

(7) $$1<\sum_{i=1}^{\infty}\gamma_i|a_i|^{\lambda}<\infty$$

for some $\lambda>0$.

1.2. The method of proof used in [3], [4], [8] and [9] consists of first proving that a characteristic function which satisfies (3) is infinitely divisible. The Lévy-Khintchine representation for infinitely divisible characteristic functions

(8) $$\log\varphi(t)=i\mu t-\frac{1}{2}\sigma^2t^2+\int_{(-\infty,0)}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)dM(u) +\int_{(0,\infty)}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)dN(u)$$

(see [5]) is then invoked which together with (3) implies that

(9) $$M(u)=\sum_{i=1}^{p}\gamma_iM(u/a_i)-\sum_{i=p+1}^{n}\gamma_iN(-u/a_i)\ , \qquad u<0\ ,$$

and

(10) $$N(u)=\sum_{i=1}^{p}\gamma_iN(u/a_i)-\sum_{1=p+1}^{n}\gamma_iM(-u/a_i)\ , \qquad u>0\ .$$

On writing $v(u)=M(-e^u)$ and $w(u)=-N(e^u)$ we obtain from (9) and (10)

(11) $$v(u)=\sum_{i=1}^{p}\gamma_iv(u+A_i)+\sum_{i=p+1}^{n}\gamma_iw(u+A_i)\ , \qquad -\infty<u<\infty\ ,$$

and

(12) $$w(u)=\sum_{i=1}^{p}\gamma_iw(u+A_i)+\sum_{i=p+1}^{n}\gamma_iv(u+A_i)\ , \qquad -\infty<u<\infty\ ,$$

where $A_i=-\log a_i$.

From the properties of M and N (see [5]) one may deduce that v and w are non-negative, non-increasing right continuous functions with $v(\infty)=w(\infty)=0$. The corresponding equations for the infinite case may be written as

(13) $$v(u)=\sum_{i=1}^{\infty}\gamma_{2i}v(u+A_{2i})+\sum_{i=1}^{\infty}\gamma_{2i-1}w(u+A_{2i-1}) , \qquad -\infty<u<\infty ,$$

and

(14) $$w(u)=\sum_{i=1}^{\infty}\gamma_{2i}w(u+A_{2i})+\sum_{i=1}^{\infty}\gamma_{2i-1}v(u+A_{2i-1}) , \qquad -\infty<u<\infty .$$

The problem is therefore reduced to solving (11) and (12) under the stated conditions. In [8], [9] this is done by adopting the method of Linnik [3], [4] to the more general problem. This involves locating the zeros of the Laplace transforms of v and w, applying some results in the theory of entire functions and then using the inversion formula for Laplace transforms (see [2], [8] and [9] for the details).

1.3. In this paper we give a complete and elementary solution for the finite case. The method of proof can be extended to the infinite case which is then solved under the additional assumptions (7) and

(15) $$\sup_{1\leqq i<\infty}|a_i|<1 .$$

This yields an extension of the results of Ramachandran and Rao [8] as we are thus able to dispense with (5). The greater generality of (15) in comparison to (6) is only apparent as a closer exmination of the proof given in [8] shows that use of (6) is only made to the extent that it implies (15).

Our solution consists of first solving the functional equation

(16) $$g(x)=\sum_{i=1}^{\infty}p_{2i}g(x+A_{2i})-\sum_{i=1}^{\infty}p_{2i-1}g(x+A_{2i-1}) , \qquad x\geqq x_0$$

where $A_i\geqq 0$, $p_i\geqq 0$ and $\sum_{i=1}^{\infty}p_i=1$ (plus other conditions when an infinite number of the p_i are non-zero). This is done in Section 2 where we also indicate how the general solutions of (13) and (14) may quickly be obtained from the general solution of (16). In Section 3 we apply the results of Section 2 to the functional equation (4). We first obtain the general form of $|\varphi(t)|$ and use this to obtain the general form of $\varphi(t)$. The only property of characteristic functions we use here is that if $1-|\varphi(t)|^2=O(|t|^\beta)$, $0<\beta<2$, then $1-i\mu t-\varphi(t)=O(|t|^\beta)$ for some μ. In this way we are able to dispense with the necessity of proving directly that $\varphi(t)$ is infinitely divisible and it is this which enables us to drop the condition (5).

Our solution of the functional equations (13) and (14) does not involve Laplace transforms or function theory. Furthermore our solution is not only elementary but also, in the finite case, simple. It is only Lemma 2, which is required for the infinite case, which presents any difficulty although even here the basic idea is very simple. In this sense our solution may be regarded as an extension of the methods used by Baxter [1] (a paper which seems to have been overlooked) and by Laha and Lukacs [5] in the normal case $\sum_{i=1}^{\infty} \gamma_i a_i^2 = 1$.

2. Solution of the functional equation (16)

2.1. Regarding (16), we shall assume without loss of generality that

$$\text{if } A_i = 0 \text{ then } p_i = 0 \ . \tag{17}$$

We require the following notation:

$$\mathcal{A} = \{\boldsymbol{A} = (A_1, A_2, \cdots): A_i \geqq 0,\ \sup A_i > 0,\ \inf_{A_i > 0} A_i > 0\}\ ,$$

$\mathcal{A}(0) = \{\boldsymbol{A}$: $\boldsymbol{A} \in \mathcal{A}$ and there does not exist a $\rho > 0$ such that A_i/ρ is a non-negative integer for all $i\}$,

$\mathcal{A}(\rho) = \{\boldsymbol{A}$: $\boldsymbol{A} \in \mathcal{A}$ and A_i/ρ is a non-negative integer for all i and the A_i/ρ have highest common factor $1\}$,

$\mathcal{B}(\rho) = \{\boldsymbol{A}$: $\boldsymbol{A} \in \mathcal{A}(\rho)$ and either (i) $A_{2i-1} = 0$ for all i or (ii) for at least one i, A_{2i-1}/ρ is positive and even or (iii) for at least one i, A_{2i}/ρ is positive and odd$\}$,

$C(\rho) = \{\boldsymbol{A}$: $\boldsymbol{A} \in \mathcal{A}(\rho)$ and either (i) $A_{2i} = 0$ for all i or (ii) A_{2i}/ρ is even for all i with $A_{2i} > 0$ and A_{2i-1}/ρ is odd for all i with $A_{2i-1} > 0\}$.

Note that we have $\mathcal{A}(\rho) = \mathcal{B}(\rho) \cup C(\rho)$ for any $\rho > 0$ and $\mathcal{A} = \bigcup_{\rho \geqq 0} \mathcal{A}(\rho)$. We can now state our fundamental theorem.

THEOREM 1. *Let $g(x)$ be a real valued function which satisfies the functional equation*

$$g(x) = \sum_{i=1}^{\infty} p_{2i} g(x + A_{2i}) - \sum_{i=1}^{\infty} p_{2i-1} g(x + A_{2i-1})\ , \qquad x \geqq x_0\ , \tag{18}$$

where $\boldsymbol{A} \in \mathcal{A}$, $p_i \geqq 0$, $\sum_{i=1}^{\infty} p_i = 1$ and (17) holds. Suppose further that g is such that

$$\sup_{x \geqq x_0} |g(x+y) - g(x)| < \infty \qquad \text{for all } y \geqq 0\ . \tag{19}$$

Then the following holds;

Case 1: If $\boldsymbol{A} \in \mathcal{A}(0)$ and $g(x)$ is continuous then $g(x) = c_1$ (constant). Further, if for some i, $p_{2i-1} > 0$, then $c_1 = 0$.

Case 2: *If* $\boldsymbol{A} \in \mathscr{B}(\rho)$ *then* $g(x)=\Delta(x)$ *where* $\Delta(x+\rho)=\Delta(x)$ *for all* $x \geqq x_0$. *Further, if for some* i, $p_{2i-1}>0$ *then* $\Delta(x)\equiv 0$.

Case 3: *If* $\boldsymbol{A} \in C(\rho)$ *then* $g(x)=\Pi(x)$ *where* $\Pi(x+\rho)=-\Pi(x)$ *for all* $x \geqq x_0$.

PROOF. Write $h(x, y)=g(x+y)-g(x)$. Then $h(x, y)$ for fixed y satisfies (18) and is bounded because of (19). Suppose $p_{2k}\neq 0$, set $h_{2k}(x, y)=h(x+A_{2k}, y)-h(x, y)$ and define

$$a_{2k}=\limsup_{x\to\infty} h_{2k}(x, y)\ , \qquad b_{2k}=\liminf_{x\to\infty} h_{2k}(x, y)\ . \tag{20}$$

Then $h_{2k}(x, y)$ satisfies (18) and hence

$$\begin{aligned} a_{2k} &\leqq \sum_{i=1}^{\infty} p_{2i} \limsup_{x\to\infty} h_{2k}(x+A_{2i}, y)-\sum_{i=1}^{\infty} p_{2i-1} \liminf_{x\to\infty} h_{2k}(x+A_{2i-1}, y) \\ &= a_{2k}\sum_{i=1}^{\infty} p_{2i}-b_{2k}\sum_{i=1}^{\infty} p_{2i-1}\ . \end{aligned} \tag{21}$$

Similarly

$$b_{2k}\geqq b_{2k}\sum_{i=1}^{\infty} p_{2i}-a_{2k}\sum_{i=1}^{\infty} p_{2i-1} \tag{22}$$

so that (21) and (22) become equalities ($p_i\geqq 0$, $\sum_{i=1}^{\infty} p_i=1$). This together with the assumption that $p_{2k}>0$ implies that for any sequence $(x_n)_1^\infty$ for which

$$\lim_{n\to\infty} h_{2k}(x_n, y)=a_{2k} \qquad \text{and} \qquad \lim_{n\to\infty} x_n=\infty$$

hold, $\lim_{n\to\infty} h_{2k}(x_n+A_{2k}, y)=a_{2k}$. On iteration we obtain for every positive integer L

$$\begin{aligned} &\lim_{n\to\infty} (h(x_n+LA_{2k}, y)-h(x_n, y)) \\ &\qquad =\lim_{n\to\infty} \left(\sum_{j=0}^{L-1} h_{2k}(x_n+jA_{2k}, y)\right)=La_{2k} \end{aligned}$$

and as $h(x, y)$ is bounded we conclude that $a_{2k}=0$. Similarly we can show that $b_{2k}=0$.

Now

$$|h_{2k}(x, y)|\leqq \sum_{i=1}^{\infty} p_i |h_{2k}(x+A_i, y)|$$

and as $p_i\geqq 0$ and $\sum_{i=1}^{\infty} p_i=1$ there exists an i with $A_i>0$ such that $|h_{2k}(x+A_i, y)|\geqq h_{2k}(x, y)$. As the positive A_i are bounded away from zero ($\boldsymbol{A} \in \mathscr{A}$) this implies

$$|h_{2k}(x, y)| \leqq \limsup_{z\to\infty} |h_{2k}(z, y)| = \max(|a_{2k}|, |b_{2k}|) = 0 .$$

Thus for all $x \geqq x_0$ and $y \geqq 0$ we have

$$h(x+A_{2k}, y) = h(x, y) . \tag{23}$$

Similarly by considering $h_{2k-1}(x, y) = h(x+A_{2k-1}, y) + h(x, y)$ we can obtain

$$\lim_{n\to\infty} h_{2k-1}(x_n + jA_{2k-1}, y) = (-1)^j a_{2k-1}$$

and on using

$$(-1)^{L-1} h(x_n + LA_{2k-1}, y) + h(x_n, y) = \sum_{j=0}^{L-1} (-1)^j h_{2k-1}(x_n + jA_{2k-1}, y)$$

we obtain

$$\lim_{n\to\infty} ((-1)^{L-1} h(x_n + LA_{2k-1}, y) + h(x_n, y)) = La_{2k-1} .$$

The boundedness of $h(x, y)$ yields $a_{2k-1} = 0$ and from this we may deduce as before

$$h(x+A_{2k-1}, y) = -h(x, y) \tag{24}$$

for all k with $A_{2k-1} > 0$.

We are now in a position to consider the three different cases.

Case 1: Suppose first that $\sum_{i=1}^{\infty} p_{2i-1} = 0$. On iterating (23) we obtain for any integers $l_1, \cdots, l_n$ (positive or negative) for which $x + \sum_{i=1}^{n} l_i A_{2i} \geqq x_0$, $h\left(x + \sum_{i=1}^{n} l_i A_{2i}, y\right) = h(x, y)$, provided $x \geqq x_0$. But as $A \in \mathcal{A}(0)$ the set

$$\left\{\sum_{i=1}^{n} l_i A_{2i} :\ l_1, \cdots, l_n \text{ integers, } n = 1, 2, \cdots\right\}$$

is dense in $(-\infty, \infty)$ and as $h(x, y)$ is continuous we can conclude that $h(z, y) = h(x, y)$ for all $x \geqq x_0$ and $z \geqq x_0$. We have therefore

$$\begin{aligned} h(x, y+y') &- h(x, y) - h(x, y') \\ &= \{g(x+y+y') - g(x)\} - \{g(x+y) - g(x)\} - \{g(x+y') - g(x)\} \\ &= \{g(x+y'+y) - g(x+y')\} - \{g(x+y) - g(x)\} \\ &= h(x+y', y) - h(x, y) = 0 \qquad \text{for } x \geqq x_0 \text{ and } y, y' \geqq 0 . \end{aligned}$$

Since $h(x, y)$ is as a function of y $(\geqq 0)$ continuous we may write $h(x, y) = h(x_0, y) = g(x_0 + y) - g(x_0) = cy$ for $x \geqq x_0$ and $y \geqq 0$, where c is a constant, or $g(x) = c_1 + cx$, $x \geqq x_0$, where $c_1 = g(x_0) - cx_0$ is a constant. Substituting this into (18) we obtain $c = 0$. Suppose now that $\sum_{i=1}^{\infty} p_{2i-1} > 0$ so that

$A_{2k-1}>0$ for some k. On iterating (23) and (24) we obtain for all integers $l_1, \cdots, l_n$ with $x+2\sum_{i=1}^{n} l_i A_i \geqq x_0$

$$h\left(x+A_{2k-1}+2\sum_{i=1}^{n} l_i A_i, y\right)=h(x+A_{2k-1}, y)=-h(x, y).$$

The set $\left\{2\sum_{i=1}^{n} l_i A_i : l_1, \cdots, l_n \text{ integers } n=1, 2, \cdots\right\}$ is dense in $(-\infty, \infty)$ and this together with the continuity of $h(x, y)$ yields $h(x, y)=-h(x, y)$ so that $g(x+y)=g(x)$ for all $x \geqq x_0$, $y \geqq 0$. Thus $g(x)=c_3$ for $x \geqq x_0$ and direct substitution into (18) coupled with the fact that $\sum_{i=1}^{\infty} p_{2i-1}>0$ gives immediately $c_3=0$.

Case 2: Suppose firstly that $\sum_{i=1}^{\infty} p_{2i-1}=0$. On iterating (23) we have as before

$$h\left(x+\sum_{i=1}^{n} l_i A_{2i}, y\right)=h(x, y)$$

for any integers $l_1, \cdots, l_n$ for which $x+\sum_{i=1}^{n} l_i A_{2i} \geqq x_0$. As $\boldsymbol{A} \in \mathscr{B}(\rho)$ there exist $l_1, \cdots, l_n$ such that $\sum_{i=1}^{n} l_i A_{2i}=\rho$ and hence $h(x+\rho, y)=h(x, y)$ for all $x \geqq x_0$. On rearranging this yields $g(x+y+\rho)-g(x+y)=g(x+\rho)-g(x)$ so that $g(x+\rho)-g(x)=c_4$. Iterating we obtain $g(x+r\rho)=g(x)+rc_4$ for every positive integer r. Substituting this into (18) yields

$$\begin{aligned} g(x) &= \sum_{i=1}^{\infty} p_{2i} g(x+A_{2i})=\sum_{i=1}^{\infty} p_{2i}(g(x)+c_4 A_{2i}/\rho) \\ &= g(x)+c_4\left(\sum_{i=1}^{\infty} p_{2i} A_{2i}/\rho\right). \end{aligned}$$

As $\sum_{i=1}^{\infty} p_{2i} A_{2i} \neq 0$ we conclude that $c_4=0$ and this completes the first part of Case 2.

Suppose now that $\sum_{i=1}^{\infty} p_{2i-1}>0$ so that $A_{2k-1}>0$ for some k. We consider first the case that A_{2k-1}/ρ is even. There then exist integers $l_1, \cdots, l_n$ such that $2\sum_{i=1}^{n} l_i A_i=A_{2k-1}$. The usual iteration of (23) and (24) yields

$$h(x, y)=h\left(x+2\sum_{i=1}^{n} l_i A_i, y\right)=h(x+A_{2k-1}, y)=-h(x, y)$$

so that $h(x, y)=0$. We conclude that $g(x)=c_5$ for all $x \geqq x_0$ and on substituting into (18) we obtain $c_5=0$.

Suppose now that A_{2k}/ρ is odd for some k. We may assume that

A_{2i-1}/ρ is odd for all i with $A_{2i-1}>0$ as otherwise we are back to the case just treated. It is therefore possible to choose integers j and l such that $(2j+1)A_{2k}/\rho-A_{2l-1}/\rho$ is a positive even integer. As the A_i have highest common factor 1 there exist integers $l_1,\cdots,l_n$ such that $(2j+1)A_{2k}-A_{2l-1}=\sum_{i=1}^{n} A_i l_i$. We obtain

$$\begin{aligned}h(x, y)&=h(x+(2j+1)A_{2k}, y)=h\left(x+A_{2l-1}+2\sum_{i=1}^{n} l_i A_i, y\right)\\&=h(x+A_{2l-1}, y)=-h(x, y)\end{aligned}$$

so that $h(x, y)=0$ and as before we conclude that $g(x)=0$ for all $x\geqq x_0$.
Case 3: Suppose that A_{2k}/ρ is positive and even and A_{2l-1}/ρ is positive and odd. We can choose integers j and m so that $(2j+1)A_{2l-1}-mA_{2k}=\rho$. This implies

$$\begin{aligned}h(x+\rho, y)&=h(x+(2j+1)A_{2l-1}-mA_{2k}, y)\\&=h(x+(2j+1)A_{2l-1}, y)\\&=-h(x, y)\end{aligned}$$

from which it follows that $g(x+\rho)+g(x)=c_6$. On iterating we obtain as $A\in C(\rho) g(x+A_{2i})=g(x)$ and $g(x+A_{2i-1})=c_6-g(x)$ if $A_{2i-1}>0$. Substituting this into (18) yields

$$g(x)=\sum_{i=1}^{\infty} p_{2i}g(x)-\sum_{i=1}^{\infty} p_{2i-1}(c_6-g(x))=g(x)-c_6\sum_{i=1}^{\infty} p_{2i-1}\,.$$

As $\sum_{i=1}^{\infty} p_{2i-1}\neq 0$ we have $c_6=0$ and this concludes the proof of the theorem.

2.2. In this section we prove two lemmas which enable us to conclude that certain functions which do not oscillate rapidly and which satisfy (18) with $\sum_{i=1}^{\infty} p_{2i-1}=0$ must be bounded and hence Theorem 1 may be applied to such functions. The proof in the finite case is much simpler than that in the infinite case and we therefore give it separately.

LEMMA 1. *Let $g(x)$ be a non-negative real valued function which satisfies the functional equation*

$$g(x)=\sum_{i=1}^{n} p_{2i}g(x+A_{2i})\,,\qquad x\geqq x_0 \tag{25}$$

where $A_{2i}>0$, $p_{2i}>0$ and $\sum_{i=1}^{n} p_{2i}=1$. Then if $e^{-\lambda x}g(x)$ is non-increasing for some $\lambda>0$ it follows that $g(x)$ is bounded.*

* The condition can be removed. In fact, it follows from (25) that $G(x)\equiv e^{-\lambda x}g(x)\geqq G(x+A_{2k})$ whenever $p_{2k}\exp(\lambda A_{2k})\geqq 1$. Then for a sufficiently large λ, $G(x)$ is non-increasing if $\rho=0$ and $G(x)\geqq G(x+\rho)$ if $\rho>0$. This is sufficient for the proof of Lemma 1.

PROOF. From (25) it follows that there exists an i such that $g(x_0+A_{2i})\leqq g(x_0)$. Similarly there exists a j such that $g(x_0+A_{2i}+A_{2j})\leqq g(x_0+A_{2i})\leqq g(x_0)$. In this manner we obtain a sequence $(A_{(2\nu)})_1^\infty$ with $A_{(0)}=0$ such that $0\leqq g\left(x_0+\sum_{\nu=1}^{m} A_{(2\nu)}\right)\leqq g(x_0)$ for all $m\geqq 0$ and where $A_{(2\nu)}=A_{2i_\nu}$ for some i_ν, $1\leqq i_\nu\leqq n$. As $\min_{1\leqq i\leqq n} A_{2i}>0$ we can find for each $x\geqq x_0$ an m such that

$$0\leqq x-x_0-\sum_{\nu=0}^{m} A_{(2\nu)}\leqq \max_{1\leqq i\leqq n} A_{2i}\ .$$

As $e^{-\lambda x}g(x)$ is non-increasing we have

$$\begin{aligned}0\leqq e^{-\lambda x}g(x)&\leqq \exp\left(-\lambda\left(x_0+\sum_{\nu=0}^{m} A_{(2\nu)}\right)\right)g\left(x_0+\sum_{\nu=0}^{m} A_{(2\nu)}\right)\\ &\leqq \exp\left(-\lambda\left(x_0+\sum_{\nu=0}^{m} A_{(2\nu)}\right)\right)g(x_0)\end{aligned}$$

and hence

$$0\leqq g(x)\leqq \exp\left(\lambda\left(x-x_0-\sum_{\nu=0}^{m} A_{(2\nu)}\right)\right)g(x_0)\leqq \exp\left(\lambda \max_{1\leqq i\leqq n} A_{2i}\right)g(x_0)$$

which proves the lemma.

The corresponding lemma for the infinite case is the following.

LEMMA 2. *Let $g(x)$ be a non-negative real valued function which satisfies the functional equation*

$$(26)\qquad g(x)=\sum_{i=1}^{\infty} p_{2i}g(x+A_{2i})\ ,\qquad x\geqq x_0\ ,$$

where $\boldsymbol{A}\in\mathcal{A}$, $p_{2i}\geqq 0$, $\sum_{i=1}^{\infty} p_{2i}=1$, and

$$(27)\qquad \sum_{i=1}^{\infty} p_{2i}\exp(\delta A_{2i})=c_7<\infty\qquad \textit{for some } \delta>0\ .$$

Then if $e^{-\lambda x}g(x)$ is non-increasing for some $\lambda>0$ it follows that $g(x)$ is bounded for $x\geqq x_0$.

PROOF. As $\boldsymbol{A}\in\mathcal{A}$ we may set $\inf_{A_{2i}>0} A_{2i}=1$ without loss of generality. For all positive integers n and m with $n<m$ we define

$$\mathcal{D}(n)=\{i:\ 1\leqq A_{2i}<n\}$$

$$\mathcal{D}(n,\ m)=\{i:\ n\leqq A_{2i}<m\}$$

$$\mathcal{D}(n,\ \infty)=\{i:\ n\leqq A_{2i}<\infty\}\ .$$

As $\sum_{i=1}^{\infty} p_{2i}=1$ we may choose n sufficiently large so that

$$\sum_{i \in \mathcal{D}(n)} p_{2i}=P(n)\geqq \exp(-\delta/2) \tag{28}$$

where δ is as in (27). We write

$$\tau_0=\inf\{y:\ y\geqq 1,\ g(x_0+y)\leqq g(x_0)\}$$

(it follows from (26) that τ_0 is well defined) and we choose μ_0 to satisfy

$$\tau_0\leqq \mu_0<\tau_0+1 \qquad \text{and} \qquad g(x_0+\mu_0)\leqq g(x_0)\ .$$

With n given by (28) we define

$$\nu_1=\min\{2i:\ i\in\mathcal{D}(n),\ g(x_0+A_{2i})\leqq \inf_{j\in\mathcal{D}(n)} g(x_0+A_{2j})+\exp(-\delta\tau_0)\}$$

and with $\nu_1,\cdots,\nu_k$ defined we define

$$\begin{aligned}\nu_{k+1}=\min\{2i:\ i\in\mathcal{D}(n),\ & g(x_0+S_k+A_{2i})\\ &\leqq \inf_{j\in\mathcal{D}(n)} g(x_0+S_k+A_{2j})+\exp(-\delta\tau_0)\}\end{aligned}$$

where $S_k=\sum_{i=1}^{k} A_{\nu_i}$. Finally we define the integer s_0 by

$$s_0=\begin{cases} 0 & \text{if } \tau_0<A_{\nu_1}+n+1 \\ \max\{r:\ S_r+n+1\leqq\tau_0\} & \text{if } \tau_0\geqq A_{\nu_1}+n+1\ . \end{cases}$$

The main part of the proof is concerned with obtaining the inequality

$$\begin{aligned}g(x_0+S_k)\leqq g(x_0)+\exp(-\delta\tau_0)\Big(\sum_{j=1}^{k} P(n)^{-j}(P(n) \\ +c_7\exp(\delta(S_{k-j}+1))g(x_0))\Big)\ , \qquad 1\leqq k\leqq s_0\ ,\end{aligned} \tag{29}$$

where c_7 is as in (27). We obtain (29) by induction.

Suppose first that $s_0\geqq 1$ and $k=1$ and set $n_1=[\tau_0]$ (the largest integer $m\leqq\tau_0$). As $s_0\geqq 1$ it follows that $n_1\geqq n+2$ and on using (26) we obtain

$$\begin{aligned}0=\sum_{i\in\mathcal{D}(n)} p_{2i}(g(x_0+A_{2i})-g(x_0))+\sum_{i\in\mathcal{D}(n,\,n_1)} p_{2i}(g(x_0+A_{2i})-g(x_0))\\ +\sum_{i\in\mathcal{D}(n_1,\,\infty)} p_{2i}(g(x_0+A_{2i})-g(x_0))\end{aligned}$$

where $\sum_{i\in\mathcal{D}}$ is to be interpreted as zero if $\mathcal{D}$ is empty. From the definitions of τ_0 and n_1 we have $g(x_0+A_{2i})>g(x_0)$ for $i\in\mathcal{D}(n,\,n_1)$ and hence

$$\sum_{i \in \mathcal{D}(n)} p_{2i}(g(x_0+A_{2i})-g(x_0)) \leqq \sum_{i \in \mathcal{D}(n_1, \infty)} p_{2i}(g(x_0)-g(x_0+A_{2i}))$$
$$\leqq g(x_0) \sum_{i \in \mathcal{D}(n_1, \infty)} p_{2i}$$

as $g(x)$ is non-negative. This yields

$$\inf_{i \in \mathcal{D}(n)} (g(x_0+A_{2i})-g(x_0)) \leqq P(n)^{-1}\Big(\sum_{i \in \mathcal{D}(n_1, \infty)} p_{2i}\Big)g(x_0)$$

so that

$$g(x_0+A_{\nu_1}) \leqq g(x_0)+\exp(-\delta\tau_0)+P(n)^{-1}g(x_0) \sum_{i \in \mathcal{D}(n_1, \infty)} p_{2i} \,. \tag{30}$$

Now

$$\sum_{i \in \mathcal{D}(n_1, \infty)} p_{2i} = \sum_{i \in \mathcal{D}(n_1, \infty)} p_{2i} \exp(\delta A_{2i}-\delta A_{2i})$$
$$\leqq \exp(-\delta n_1) \sum_{i \in \mathcal{D}(n_1, \infty)} p_{2i} \exp(\delta A_{2i})$$
$$\leqq c_7 \exp(-\delta n_1) \leqq c_7 \exp(-\delta(\tau_0-1))$$

and on substituting this into (30) we obtain

$$g(x_0+A_{\nu_1}) \leqq g(x_0)+\exp(-\delta\tau_0)(1+c_7 P(n)^{-1}\exp(\delta)g(x_0))$$

which is (29) with $k=1$.

Suppose now that $s_0 \geqq 2$, $1 \leqq k \leqq k+1 \leqq s_0$ and that (25) holds for k. On writing $n_k=[\tau_0-S_k]$ it follows as $s_0 \geqq k+1$ that $n+1 \leqq \tau_0+S_{k+1} \leqq \tau_0-S_k-1$ and hence $n_k \geqq n+2$. We may therefore write

$$g(x_0+S_k)-g(x_0) = \sum_{i \in \mathcal{D}(n)} p_{2i}(g(x_0+S_k+A_{2i})-g(x_0))$$
$$+ \sum_{i \in \mathcal{D}(n, n_k)} p_{2i}(g(x_0+S_k+A_{2i})-g(x_0))$$
$$+ \sum_{i \in \mathcal{D}(n_k, \infty)} p_{2i}(g(x_0+S_k+A_{2i})-g(x_0)) \,.$$

Now $n_k+S_k \leqq \tau_0-S_k+S_k=\tau_0$ so that $g(x_0+S_k+A_{2i})>g(x_0)$ if $i \in \mathcal{D}(n, n_k)$. Using this we obtain

$$g(x_0+S_k)-g(x_0) \geqq \sum_{i \in \mathcal{D}(n)} p_{2i}(g(x_0+S_k+A_{2i})-g(x_0))$$
$$+ \sum_{i \in \mathcal{D}(n_k, \infty)} p_{2i}(g(x_0+S_k+A_{2i})-g(x_0))$$

and arguing as in the case $k=1$ we finally obtain

$$g(x_0+S_{k+1}) \leqq g(x_0)+\exp(-\delta\tau_0)$$
$$+P(n)^{-1}(g(x_0+S_k)-g(x_0))$$
$$+c_7 P(n)^{-1}\exp(-\delta(\tau_0-S_k-1))g(x_0) \,.$$

On using (29) we obtain after elementary manipulations

$$g(x_0+S_{k+1})\leqq g(x_0)+\exp(-\delta\tau_0)\Big(\sum_{j=1}^{k+1}P(n)^{-j}(P(n)+c_7\exp(\delta(S_{k+1-j}+1))g(x_0))\Big)$$

which is (29) with $k+1$ in place of k. Thus (29) holds for k, $1\leqq k\leqq s_0$, and we now use this inequality to construct a sequence which corresponds to the sequence $(A_{(2\nu)})_0^\infty$ of Lemma 1.

From the definition of s_0 and τ_0 it follows that $S_{s_0}+n+1\leqq\tau_0$ which implies

$$\tau_0-S_{k-j}\geqq\sum_{i=k-j+1}^{s_0}A_{\nu_i}+n+1\geqq s_0-(k-j+1)+n+1\geqq j$$

as $k\leqq s_0$. Using this in (29) we obtain as $k\leqq s_0$

$$g(x_0+S_k)\leqq g(x_0)+\exp(-\delta\tau_0)P(n)^{-s_0}/(P(n)^{-1}-1)+c_7\Big(\sum_{i=1}^{s_0}\exp(-\delta j+\delta)P(n)^{-j}\Big)\,.$$

As $s_0\leqq\tau_0$ and $P(n)$ satisfies (29) this yields

$$g(x_0+S_k)\leqq g(x_0)+\exp(-\delta s_0/2)/(P(n)^{-1}-1)+c_7\exp(\delta/2)/(1-\exp(-\delta/2))\leqq c_8\,.$$

Here we have assumed that $P(n)<1$ but it is easily checked that this also holds if $P(n)=1$. In fact in this case the simpler proof of Lemma 1 works. The sequence $(g(x_0+S_k))_{k=1}^{s_0}$ is thus bounded by the constant c_8.

We are now in a position to construct the sequence $(A_{(2r)})_0^\infty$. We set $A_{(0)}=0$. If $s_0=0$ we set $A_{(2)}=\mu_0$. If $1\leqq k\leqq s_0$ we set $A_{(2k)}=A_{\nu_k}$ and $A_{(2s_0+2)}=\mu_0-S_{s_0}$. In all cases we have $1\leqq A_{(2k)}\leqq 2(n+1)$, $0\leqq g\Big(x_0+\sum_{r=0}^{k}A_{(2r)}\Big)\leqq c_8$, $k=1,\cdots,s_0+1$ and $g\Big(x_0+\sum_{r=0}^{s_0+1}A_{(2r)}\Big)\leqq g(x_0)$. We may now repeat this procedure this time starting with $x_1=x_0+\sum_{r=0}^{s_0+1}A_{(2r)}$ to obtain a sequence $(A_{(2s_0+2+2r)})_{r=1}^{s_1+1}$ which satisfies $1\leqq A_{(2s_0+2+2r)}\leqq 2(n+1)$,

$$g\Big(x_1+\sum_{r=1}^{k}A_{(2s_0+2+2r)}\Big)\leqq c_8\qquad\text{for } k=1,\cdots,s_1+1$$

and

$$g\Big(x_1+\sum_{r=1}^{s_1+1}A_{(2s_0+2+2r)}\Big)\leqq g(x_1)\leqq g(x_0)\,.$$

This procedure is once again repeated this time starting with $x_2=x_1+$

$\sum_{r=1}^{s_1+1} A_{(2s_0+2+2r)}$ and in this manner we finally obtain a sequence $(A_{(2r)})_0^\infty$ which satisfies $1 \leqq A_{(2r)} \leqq 2(n+1)$, $r=1, 2, \cdots$ and $g\left(x_0+\sum_{r=1}^{m} A_{(2r)}\right) \leqq c_8$, $m=1, 2, \cdots$. The remainder of the proof now follows the lines of the proof of Lemma 1 and this completes the proof of Lemma 2.

2.3. We now indicate how the functional equations (13) and (14) may be solved using Theorem 1 and Lemma 2. Adding (13) and (14) yields

$$v(u)+w(u)=\sum_{i=1}^{\infty} \gamma_i(v(u+A_i)+w(u+A_i)) .$$

If $\sum_{i=1}^{\infty} \gamma_i \leqq 1$ it is relatively simple to show (using the facts that $v(u)+w(u)$ is non-increasing and that $v(\infty)+w(\infty)=0$) that $v(u)+w(u)=0$. We may therefore suppose that

$$1<\sum_{i=1}^{\infty} \gamma_i \tag{31}$$

and following Ramachandran and Rao we assume in this case that

$$1<\sum_{i=1}^{\infty} \gamma_i \exp(-\eta A_i)<\infty \tag{32}$$

for some $\eta>0$. This implies that for some α, $0<\alpha<\eta$,

$$\sum_{i=1}^{\infty} \gamma_i \exp(-\alpha A_i)=1 . \tag{33}$$

We now set $p_i=\gamma_i \exp(-\alpha A_i)$ and if $\gamma_i=0$ we set $A_i=0$. Finally we set $g(u)=e^{\alpha u}v(u)$ and $h(u)=e^{\alpha u}w(u)$. Then g and h satisfy the functional equations

$$g(u)=\sum_{i=1}^{\infty} p_{2i}g(u+A_{2i})+\sum_{i=1}^{\infty} p_{2i-1}h(u+A_{2i-1})$$

and

$$h(u)=\sum_{i=1}^{\infty} p_{2i}h(u+A_{2i})+\sum_{i=1}^{\infty} p_{2i-1}g(u+A_{2i-1})$$

where $p_i \geqq 0$ and $\sum_{i=1}^{\infty} p_i=1$. Again following Ramachandran and Rao we now assume that $A \in \mathcal{A}$.

Adding the two equations above we see that $g+h$ satisfies the functional equation (26) (after renumbering). From (32) and (33) we conclude that $\sum_{i=1}^{\infty} p_i \exp(\delta A_i)<\infty$ for some $\delta>0$ and as $e^{-\alpha u}(g(u)+h(u))$ is non-increasing (this follows from the fact that $v(u)+w(u)$ is non-increasing) we may apply Lemma 2 and obtain that $g+h$ is bounded. Theorem 1 may now be applied as $g+h$ satisfies (18) (again after renumbering) with, in the notation of (18), $\sum_{i=1}^{\infty} p_{2i-1}=0$. We can therefore obtain the general form of $g+h$ (in Theorem 1, Case 1 we assume continuity but this can be easily replaced by the assumption that $e^{-\alpha x}g(x)$ is non-increasing as then g is continuous apart from jumps). In all cases $g+h$ is bounded and hence, as g and h are non-negative, $g-h$ is also bounded. The function $g-h$ satisfies (18) (after renumbering) and as it is bounded we may obtain the general form of $g-h$. From the general forms of $g+h$ and $g-h$ we may obtain the general forms of g and h and thus the general solutions of (13) and (14) under the stated conditions.

3. Determination of the characteristic function

3.1. φ satisfies the functional equation (4) and in order to replace the product by a sum we take logarithms. To this end we define t_0 by

$$t_0=\begin{cases} \inf\{t:\ \varphi(t)=0\} \\ \infty \qquad \text{if } \varphi(t)\neq 0 \text{ for all } t\,. \end{cases} \tag{34}$$

As $\varphi(t)$ is continuous and $\varphi(0)=1$ it is clear that $t_0>0$. On writing $\phi(t)=|\varphi(t)|^2$ and remembering that $\varphi(-t)=\overline{\varphi(t)}$ we deduce that

$$\phi(t)=\prod_{i=1}^{\infty}\phi^{\gamma_i}(a_i t)\,, \qquad -\infty<t<\infty\,, \tag{35}$$

and hence

$$\log\phi(t)=\sum_{i=1}^{\infty}\gamma_i\log\phi(a_i t)\,, \qquad |t|<t_0\,. \tag{36}$$

The proofs of the following assertions may be found in [8]. The proofs are elementary and do not rely on any of our previous results. Firstly, if $\sum_{i=1}^{\infty}\gamma_i\leqq 1$ then $\phi(t)\equiv 1$ and the characteristic function φ is degenerate. We may therefore assume that (31) holds. Secondly, the convergence of (4) implies that $\sum_{i=1}^{\infty}\gamma_i a_i^2\leqq 1$. If $\sum_{i=1}^{\infty}\gamma_i a_i^2=1$ then φ is the characteristic function of a normal random variable. We therefore assume that

$$\sum_{i=1}^{\infty}\gamma_i a_i^2<1\,. \tag{37}$$

Following Ramachandran and Rao we shall make the further assumption that

(38) $$1<\sum_{i=1}^{\infty}\gamma_i a_i^{\lambda}<\infty$$

for some $\lambda>0$. It follows from (37) and (38) that for some α, $0<\alpha<2$,

(39) $$\sum_{i=1}^{\infty}\gamma_i a_i^{\alpha}=1 .$$

Finally, we shall assume that

(40) $$0<\sup_{1\leq i\leq\infty} a_i<1 .$$

We note that if all but a finite number of the γ_i are zero then (38) is no restriction at all and the existence of an α which satisfies (39) follows immediately from (31) and (37).

We first obtain the general form of $\psi(t)$ under the conditions (38) and (40). We require the following notation

(41) $$A_i=\begin{cases} -\log a_i & \text{if } \gamma_i>0 \\ 0 & \text{if } \gamma_i=0 \end{cases}$$

and

(42) $$p_i=\gamma_i \exp(-\alpha A_i)$$

where α satisfies (39). Note that $\boldsymbol{A}\in\mathcal{A}$ because of (40). We can now prove

LEMMA 3. *Suppose that φ satisfies the functional equation*

(43) $$\varphi(t)=\prod_{i=1}^{\infty}\varphi^{\gamma_{2i}}(a_{2i}t)\prod_{i=1}^{\infty}\varphi^{\gamma_{2i-1}}(-a_{2i-1}t) , \qquad -\infty<t<\infty ,$$

where $\gamma_i\geqq 0$, $0<a_i<1$ and (38) *and* (40) *hold. Then $\psi(t)=|\varphi(t)|^2$ is of the form*

(44) $$\psi(t)=\exp(-2|t|^{\alpha}\Gamma(\log|t|)) , \qquad -\infty<t<\infty ,$$

where α is as in (39) *and where $\Gamma(x)\equiv$constant if $\boldsymbol{A}\in\mathcal{A}(0)$ whilst $\Gamma(x+\rho)=\Gamma(x)$ if $\boldsymbol{A}\in\mathcal{A}(\rho)$. Here the A_i are as defined by* (41).

PROOF. With α given by (39) we write

(45) $$g(x)=-e^{(\alpha+1)x}\int_0^{\exp(-x)}\log\psi(t)dt$$

for $x>x_0=-\log t_0$. Then g is non-negative and satisfies the functional equation

$$g(x)=\sum_{i=1}^{\infty}p_i g(x+A_i) , \qquad x\geqq x_0 .$$

From (38) and (39) we deduce that $\sum_{i=1}^{\infty} p_i \exp(\delta A_i) < \infty$ for some $\delta > 0$. An immediate consequence of (45) is that $e^{-(\alpha+1)x}g(x)$ is non-increasing and hence we conclude from Lemma 2 that $g(x)$ is bounded. We may therefore apply Theorem 1, Cases 1 and 2 to obtain $g(x) = \Delta(x)$ where $\Delta(x+\rho) = \Delta(x)$ if $A \in \mathcal{A}(\rho)$ and $\Delta(x) \equiv$ constant if $A \in \mathcal{A}(0)$. The corresponding form for $\psi(t)$ is (as $\psi(t) = \psi(-t)$)

$$\psi(t) = \exp(-2|t|^\alpha \Gamma(\log|t|)), \qquad |t| < t_0, \tag{46}$$

where $2\Gamma(x) = (\alpha+1)\Delta(x) + \Delta'(x)$. Thus Γ has the stated properties. Suppose now that $A \in \mathcal{A}(0)$. Then from the continuity of $\psi(t)$ we have

$$0 = \psi(t_0) = \exp(-c_9|t_0|^\alpha)$$

and we conclude that $t_0 = \infty$. If $A \in \mathcal{A}(\rho)$ and t_0 is not infinite we must have $\Gamma(\log|t_0|) = \infty$. This implies $\Gamma(\log|a_i t_0|) = \infty$ for all $a_i > 0$ and hence $\psi(a_i t_0) = 0$. As $|a_i t_0| < t_0$ we obtain a contradiction and thus also in this case we can conclude that $t_0 = \infty$. Therefore (44) holds for all t and this proves the lemma.

3.2. In order to obtain the general form of $\varphi(t)$ we require the following lemma concerning characteristic functions.

LEMMA 4. *Let $\chi(t)$ be a characteristic function such that at the origin*

$$1 - |\chi(t)|^2 = O(|t|^\beta), \qquad 0 < \beta < 2. \tag{47}$$

Then there exists a μ such that for all ε, $0 < \varepsilon < 1$,

$$\sup_{0<t<\infty} \left| \frac{1+i\mu t - \chi(t)}{|t|^\beta} - \frac{1+i\mu\varepsilon t - \chi(\varepsilon t)}{|\varepsilon t|^\beta} \right| < \infty \tag{48}$$

and

$$|1 + i\mu t - \chi(t)| = O(|t|^\gamma) \tag{49}$$

for any γ, $0 < \gamma < \beta$. Further, if $\beta \neq 1$ we may set $\gamma = \beta$.

PROOF. It is clear that for any $\delta > 0$ there exists a μ such that

$$\sup_{\delta<t<\infty} \left| \frac{1+i\mu t - \chi(t)}{|t|^\beta} - \frac{1+i\mu\varepsilon t - \chi(\varepsilon t)}{|\varepsilon t|^\beta} \right| < \infty$$

because if $0 < \beta < 1$ we may set $\mu = 0$. It is therefore sufficient to show that

$$\sup_{0<t<\delta} \left| \frac{1+i\mu t - \chi(t)}{|t|^\beta} - \frac{1+i\mu\varepsilon t - \chi(\varepsilon t)}{|\varepsilon t|^\beta} \right| < \infty$$

for some μ where $\mu=0$ if $0<\beta<1$. Let $\chi(t)$ be the characteristic function of the random variable X so that $|\chi(t)|^2$ is the characteristic function of the random variable $X-Y$ where X and Y are independent and identically distributed. Let F be the distribution function of X and F_s the distribution function of $X-Y$.

It follows from (47) (see for example [10]) that

$$F_s(-x)+1-F_s(x)=O(x^{-\beta}) , \qquad x\to\infty .$$

Let m be a median of F. Then

$$\begin{aligned}F_s(-x)+1-F_s(x-0)&\geqq \mathrm{P}\,(|X-Y|\geqq x)\\&\geqq \mathrm{P}\,(X\geqq x+m,\ Y\leqq m)+\mathrm{P}\,(X\leqq -x+m,\ Y\geqq m)\\&\geqq \frac{1}{2}\,\mathrm{P}\,(|X-m|\geqq x)\end{aligned}$$

and hence we have

$$F(-x)+1-F(x)=O(x^{-\beta}) , \qquad x\to\infty . \tag{50}$$

Let $G(x)=F(x)-F(-x)$ so that $1-G(x)=O(x^{-\beta})$, $x\to\infty$. Consider first the case $0<\beta<1$. Then

$$\begin{aligned}|1-\chi(t)|&=\left|\int_0^\infty (1-e^{ixt})dF(x)\right|\\&\leqq \int_{|x|\leqq 1/|t|} |1-e^{ixt}|\,dF(x)+2\int_{|x|>1/|t|} dF(x)\\&\leqq |t|\int_{|x|\leqq 1/|t|} |x|\,dF(x)+2(1-G(|t|^{-1}))\\&=|t|\int_0^{1/|t|} x\,dG(x)+2(1-G(|t|^{-1}))\\&\leqq |t|\int_0^{1/|t|} (1-G(x))dx+2(1-G(|t|^{-1}))\\&=O\left(|t|\int_0^{1/|t|} x^{-\beta}dx+|t|^{\beta}\right)\\&=O(|t|^{\beta}) .\end{aligned}$$

This implies $|1-\chi(t)|/|t|^{\beta}$ is bounded as $t\to 0$ and (48) and (49) follow immediately. A similar proof works in the case $1<\beta<2$ on setting $\mu=\int_{-\infty}^{\infty} x\,dF(x)$ which exists because of (47) (see [10]). If $\beta=1$ we cannot conclude that $|1-\chi(t)|=O(|t|)$ but only that $|1-\chi(t)|=O(|t\log|t||)$. We therefore have to treat this case separately. We have

$$\left|\frac{1-\chi(t)}{|t|}-\frac{1-\chi(\varepsilon t)}{|\varepsilon t|}\right|$$

$$\leqq \int_{|x|<1/|\varepsilon t|} \left| \frac{1-e^{ixt}}{t} - \frac{1-e^{ix\varepsilon t}}{\varepsilon t} \right| dF(x)$$

$$+\int_{|x|\geqq 1/|\varepsilon t|} 2(1+\varepsilon)/|\varepsilon t|\,dF(x)$$

$$=\int_{|x|<1/|\varepsilon t|} |x| \left| \frac{1+ixt-e^{ixt}}{tx} - \frac{1+i\varepsilon xt-e^{ix\varepsilon t}}{\varepsilon tx} \right| dF(x)$$

$$+2(1+\varepsilon)(1-G(1/|\varepsilon t|))/|\varepsilon t|$$

$$\leqq \int_{|x|\leqq 1/|\varepsilon t|} \frac{1}{2}|x|(|xt|+|xt|)dF(x)+O(1)$$

$$=O\left(|t|\int_0^{1/|\varepsilon t|} x^2 dG(x)\right)+O(1)$$

$$=O\left(|t|\int_0^{1/|\varepsilon t|} x(1-G(x))dx\right)+O(1)$$

$$=O(|t||\varepsilon t|^{-1})+O(1)=O(|\varepsilon|^{-1})$$

which completes the proof of (48). To prove (49) we note that in the case $\beta=1$, $1-|\chi(t)|^2=O(|t|^\gamma)$ for all γ, $0<\gamma<1$, and the desired result follows from the case $0<\beta<1$.

3.3. We can now prove our main theorem.

THEOREM 2. *Suppose the characteristic function* φ *satisfies the functional equation* (43) *where the* $(a_i)_1^\infty$ *and* $(\gamma_i)_1^\infty$ *satisfy* (38) *and* (40). *Then*

$$\varphi(t)=\exp(i\mu t-|t|^\alpha \Gamma(\log t)+i|t|^\alpha \operatorname{sign}(t)\Delta(\log|t|)) \tag{51}$$

where α, $0<\alpha<2$, *is the unique solution of* (39), $\mu=0$ *if* $0<\alpha\leqq 1$ *and where* Γ *and* Δ *are as follows with* $A=(A_1, A_2, \cdots)$ *and the* A_i *as defined by* (41).

Case 1: $A\in\mathcal{A}(0)$. *In this case* Γ *and* Δ *are both constant and* $\Delta\equiv 0$ *if* $a_{2i-1}\gamma_{2i-1}>0$ *for some* i.

Case 2: $A\in\mathcal{B}(\rho)$. *In this case* Γ *and* Δ *are both periodic functions with period* ρ ($\Gamma(x+\rho)=\Gamma(x)$, $\Delta(x+\rho)=\Delta(x)$) *and* $\Delta\equiv 0$ *if* $a_{2i-1}\gamma_{2i-1}>0$ *for some* i.

Case 3: $A\in C(\rho)$. *In this case* Γ *is a periodic function with period* ρ *and* Δ *satisfies* $\Delta(x+\rho)=-\Delta(x)$ *for all* x.

PROOF. It follows from Lemma 3 that $1-|\varphi(t)|^2=O(|t|^\alpha)$ and we can apply Lemma 4 to obtain

$$\sup_{0<t<\infty} \left| \frac{1+i\mu t-\varphi(t)}{|t|^\alpha} - \frac{1+i\mu\varepsilon t-\varphi(\varepsilon t)}{|\varepsilon t|^\alpha} \right| <\infty \tag{52}$$

for some μ ($\mu=0$ if $0<\alpha\leqq 1$). Again using Lemma 3 we may write

$$\varphi(t)=\exp(i\mu t-|t|^\alpha \Gamma(\log|t|)+i|t|^\alpha I(t))$$

where μ is as in (52) and $I(t)$ is an odd function of t. We note that because of (49) $|1+i\mu t-\varphi(t)|=O(|t|^{\gamma})$ for all γ, $0<\gamma<\alpha$, ($\gamma=\alpha$ if $\alpha\neq 1$) and hence

$$|1+i\mu t-e^{i\mu t}+e^{i\mu t}(1-e^{-i\mu t})\varphi(t)|=O(|t|^{\gamma}) ,$$

which implies

$$|1-\exp(-|t|^{\alpha}\Gamma(\log|t|)+i|t|^{\alpha}I(t))|-O(t^2)=O(|t|^{\gamma}) ,$$

As $0<\gamma<2$ we have

$$|t|^{\alpha}\Gamma(\log|t|)+|t|^{\alpha}I(t)=O(|t|^{\gamma})$$

and thus $I(t)=O(|t|^{-(\alpha-\gamma)})$ for all γ, $0<\gamma<\alpha$. This yields

(53) $$|I(t)|=O(|t|^{-\delta}) \qquad \text{for all } \delta>0$$

and if $\alpha\neq 1$ we may set $\gamma=\alpha$ to obtain

(54) $$|I(t)|=O(1) , \qquad \alpha\neq 1 .$$

This implies

$$1+i\mu t-\varphi(t)=|t|^{\alpha}\Gamma(\log|t|)-i|t|^{\alpha}I(t)+O(t^2+|t|^{2\alpha-\delta})$$

which on substituting into (52) gives

$$\sup_{0<t<\infty}|\Gamma(\log|t|)-\Gamma(\log|\varepsilon t|)-iI(t)+iI(\varepsilon t)+O(|\varepsilon t|^{2-\alpha}+|\varepsilon t|^{\alpha-\delta})|<\infty$$

and hence

(55) $$\sup_{0<t<\infty}|I(t)-I(\varepsilon t)|<\infty$$

for all $\varepsilon>0$. Suppose first that $0<\alpha\leqq 1$. Then as $\varphi(t)$ satisfies (43) we have ($I(t)$ is an odd function of t)

$$|t|^{\alpha}I(t)=\sum_{i=1}^{\infty}\gamma_{2i}|a_{2i}t|^{\alpha}I(a_{2i}t)-\sum_{i=1}^{\infty}\gamma_{2i-1}|a_{2i-1}t|^{\alpha}I(a_{2i-1}t) .$$

If we set $g(x)=I(e^{-x})$ and $p_i=\gamma_i a_i^{\alpha}$ we obtain

$$g(x)=\sum_{i=1}^{\infty}p_{2i}g(x+A_{2i})-\sum_{i=1}^{\infty}p_{2i-1}g(x+A_{2i-1})$$

where $p_i\geqq 0$, $\sum_{i=1}^{\infty}p_i=1$ and the A_i are as given by (41).

It follows from (55) that g satisfies (19) and hence we may apply Theorem 1 to obtain the following:

(i) if $A\in\mathcal{A}(0)$, $g(x)\equiv$constant and $g(x)\equiv 0$ if $A_{2i-1}>0$ for some i.

(ii) if $A\in\mathcal{B}(\rho)$, $g(x+\rho)=g(x)$ for all x and $g(x)\equiv 0$ if $A_{2i-1}>0$ for some i.

(iii) if $A \in \mathcal{C}(\rho)$, $g(x+\rho)=-g(x)$ for all x.
Cases 1, 2 and 3 follow immediately from this and the properties of the function $\Gamma(x)$ which are given in Lemma 3.

We now suppose that $\alpha>1$. Then again using (43) we obtain

$$\mu t+|t|^\alpha I(t)=\left(\sum_{i=1}^\infty \gamma_{2i}a_{2i}\mu-\sum_{i=1}^\infty \gamma_{2i}a_{2i-1}\mu\right)t+\sum_{i=1}^\infty \gamma_{2i}|a_{2i}t|^\alpha I(a_{2i}t)$$
$$-\sum_{i=1}^\infty \gamma_{2i-1}|a_{2i-1}t|^\alpha I(a_{2i-1}t) .$$

As $\alpha>1$ (54) implies that $I(t)$ is bounded and hence on dividing by t and letting t tend to zero we obtain

$$\mu=\sum_{i=1}^\infty \gamma_{2i}a_{2i}\mu-\sum_{i=1}^\infty \gamma_{2i-1}a_{2i-1}\mu$$

so that either $\mu=0$ or $\sum_{i=1}^\infty \gamma_{2i}a_{2i}-\sum_{i=1}^\infty \gamma_{2i-1}a_{2i-1}=1$. In either case we have

$$I(t)=\sum_{i=1}^\infty \gamma_{2i}a_{2i}^\alpha I(a_{2i}t)-\sum_{1=1}^\infty \gamma_{2i-1}a_{2i-1}^\alpha I(a_{2i-1}t)$$

and the remainder of the proof is as in the case $0<\alpha\leqq 1$. This completes the proof of the theorem.

3.4. It follows from Theorem 2 that $\varphi_0(t)=\varphi(t)\exp(-i\mu t)$ satisfies the functional equation $\varphi_0(t)=\varphi_0^\eta(at)$ where $a=\exp(-m\rho)$, $\eta=\exp(\alpha m\rho)$ and m is an even integer. Thus φ_0 is semi-stable in the sense of Lévy (see [2], p. 9) and is therefore infinitely divisible. This implies φ_0 has the representation (8) and the general form of the Poisson spectral functions M and N may then be obtained from (51). They can easily be shown to have the forms given by Shimizu in Theorems 2–4 in [9], with $\mathcal{A}_n(\rho)$ etc. being replaced by $\mathcal{A}(\rho)$ etc. (see also [8]).

Finally we remark that the proof of Theorem 2 is particularly simple if $\alpha\neq 1$ and the product in (43) is finite (all but a finite number of the γ_j's are zero). Firstly, the proof of Theorem 1 is simpler as we may assume $g(x)$ itself is bounded and it is then no longer necessary to consider the function $h(x, y)$. In fact the assumption (19) was introduced explicitly to cover the case $\alpha=1$. Secondly we only require the simple Lemma 1, the more difficult Lemma 2 being only required for the infinite case. Finally the proof of Lemma 4 is also simpler as we can prove directly that $|1+i\mu t-\varphi(t)|=O(|t|^\alpha)$, the annoying logarithm term being present only in the case $\alpha=1$. This in turn leads to a simplification of the proof of Theorem 2 itself.

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